DEMONSTRATIO MATHEMATICA
Vol. XL No 3 2007

Janusz Januszewski

TRANSLATIVE COVERING OF A SQUARE BY A SEQUENCE OF ARBITRARY-ORIENTED SQUARES

Abstract. Given a collection of squares in the plane whose side lengths are not larger than 1 and whose total area is at least 3. Then the unit square can be covered by translates of these squares.

1. Introduction

More than forty years ago Leo Moser posed the following question (see, for instance, Problem LM 5 in [5]): *"Can any set of rectangles of largest edge 1 and total area 3 be used to cover a unit square (No rotations, please)?"*.

In the present paper we study such coverings of the unit square I by squares. Let (S_i) be a sequence of squares in the plane (each square has an arbitrary specified orientation). We say that (S_i) *permits a covering* of I if there exist rigid motions σ_i such that $I \subset \bigcup \sigma_i S_i$. The covering is *translative* if all the motions are translations. We say that squares $S_1, S_2, \ldots$ are *packed* in a rectangle R if the squares are subsets of R and if they have pairwise disjoint interiors.

Moon and Moser [4] presented a covering method which permits a translative covering of I by any sequence of squares whose total area is 3 in the case when each square from the sequence has sides parallel to the sides of I. In [3] it was shown that I can be covered by any sequence of squares whose total area is not smaller than 2 (obviously, translations and rotations are used for the covering). In this paper we show that *any sequence of squares of side lengths not greater than 1 whose total area is greater than or equal to 3 permits a translative covering of I.* Consequently, we give the positive answer to the Moser's problem in case of a covering by squares. The bound of 3 cannot be reduced. The reason is that I cannot be translatively covered by any three squares of side lengths smaller than 1 and with sides parallel

Key words and phrases: translative covering, square
2000 *Mathematics Subject Classification*: 52C15.

to the sides of I. The reader can find further results concerning coverings by convex bodies in [1] and [2].

2. Preliminary lemmas

By $a \times h$ we mean a rectangle such that one of its sides, of length a, is parallel to the first coordinate axis and the other side has length h. Such a rectangle is *proper*, h is called the *height* and a is called the *width* of this rectangle. A trapezoid is *proper* if it is right and if its bases are parallel to the first coordinate axis. Moreover, let $[c_1, c_2] \times [d_1, d_2] = \{(x, y);\ c_1 \le x \le c_2,\ d_1 \le y \le d_2\}$. There is no loss of generality in assuming that $I = [0,1] \times [0,1]$. The area of C is denoted by $|C|$.

In covering methods presented in Section 3 we will use for the covering "large" proper rectangles or "large" proper trapezoids contained in squares $S_1, S_2, \dots$. The following two lemmas say how large proper rectangle and how large proper trapezoid is contained in each square.

LEMMA 1. *Every square of side length s contains a proper rectangle $\sqrt{s^2 - h^2} \times h$, for any $0 < h < s$.*

Proof. In Fig. 1 we have $b \geq s$ and therefore $a = \sqrt{b^2 - h^2} \geq \sqrt{s^2 - h^2}$. ∎

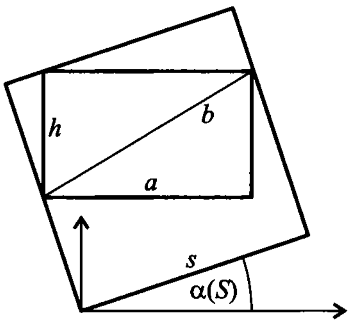

Fig. 1.

Let S be a square and let τ be a translation such that $(0,0)$ is one of the vertices of τS, that each point of τS has a non-negative second coordinate and that no point $(c, 0)$ belongs to τS, for any $c < 0$. Denote by v_1, v_2, v_3, v_4, where $v_1 = (0,0)$, the vertices of τS taken counter-clockwise. Denote by $\alpha(S)$, where $0^\circ \leq \alpha(S) < 90^\circ$, the angle between the segment $v_1 v_2$ and the first coordinate axis (see Fig. 1).

LEMMA 2. *Every square S of side length not greater than $\sqrt{2}$ contains a proper rectangle $P = a \times h$, where $\frac{1}{2}a < h \leq 2a$, such that $|P| = \frac{2}{5}|S|$ and that $h = 2^{-i}$, where $i \in \{0, 1, 2, \dots\}$. Moreover, if S does not contain $\frac{1}{4} \times \frac{1}{2}$*

and if $P = a \times \frac{1}{4}$, *where* $a \geq \frac{1}{5}$, *then* S *contains a proper trapezoid* $T \supset P$ *with bases* a *and* $a + \frac{1}{20}$.

Proof. We can assume that $\alpha = \alpha(S) \leq 45°$. Denote by s the side length of S.

Observe that if $0 < d \leq \frac{s}{\cos\alpha + \sin\alpha}$, then S contains a rectangle $c \times d$, where

$$(1) \qquad c = \frac{1}{\cos\alpha}(s - d\sin\alpha).$$

Indeed, we have $c = \frac{s-w}{\cos\alpha}$ and $w = d\sin\alpha$ in Fig. 2. Obviously, if $d = \frac{s}{\cos\alpha+\sin\alpha}$, then $c = d$ and $c \times d$ is inscribed in S.

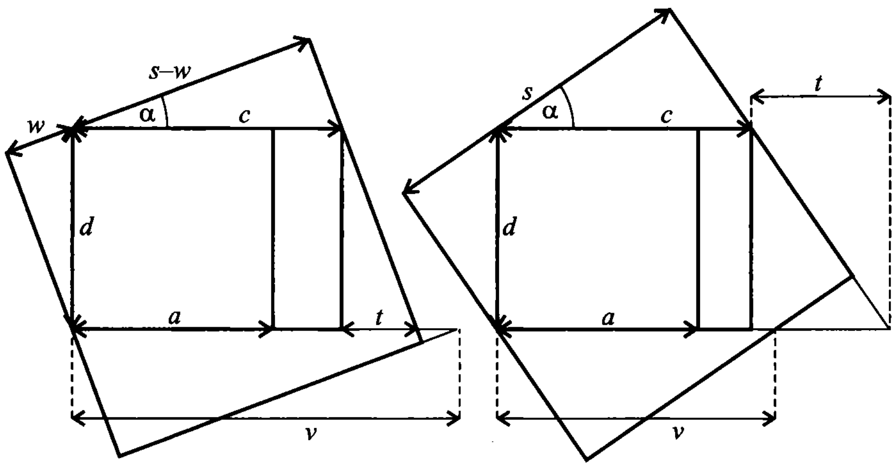

Fig. 2.

Put d equal to $\lambda = \frac{s}{\sin\alpha + 2\cos\alpha}$ in (1). This implies that $c = 2\lambda$. It is easy to check that

$$\sin\alpha + 2\cos\alpha \leq \sin\left(\arctan\frac{1}{2}\right) + 2\cos\left(\arctan\frac{1}{2}\right) \leq \sqrt{5}.$$

Consequently, the area of $2\lambda \times \lambda \subset S$ is not smaller than $\frac{2}{5}|S|$. Moreover, $S \supset \lambda \times 2\lambda$. Finally observe that there exists a non-negative integer i such that $\lambda < 2^{-i} \leq 2\lambda$. Let $h = 2^{-i}$ and let R be the proper rectangle of maximum area and height h contained in S. Obviously, $|R| \geq 2\lambda^2 \geq \frac{2}{5}|S|$. This implies that there exists a proper rectangle $P = a \times h \subset S$ with $|P| = \frac{2}{5}|S|$.

To prove the second part of Lemma 2 assume that $P = a \times d$, where $d = \frac{1}{4}$ and $a \geq \frac{1}{5}$.

By $|P| = \frac{1}{4}a = \frac{2}{5}|S|$ we have $s^2 = \frac{5}{8}a$. Thus $4s = 4\sqrt{\frac{5}{8}a} \geq \sqrt{2}$. Moreover, $\cos\alpha + \sin\alpha \leq \cos 45^\circ + \sin 45^\circ \leq \sqrt{2}$. Consequently, $\cos\alpha + \sin\alpha \leq 4s$ and $\frac{1}{4} \leq \frac{s}{\cos\alpha+\sin\alpha}$. This implies that $S \supset c \times \frac{1}{4}$, where c is described by (1).

To finish the proof it is sufficient to show that $v \geq a + \frac{1}{20}$ and that $c + t \geq a + \frac{1}{20}$, where v and t are marked in Fig. 2 (we assume that $d = \frac{1}{4}$ in this figure). We have $t = \frac{1}{4}\tan\alpha$. Thus

$$c + t - a - \frac{1}{20} = \frac{1}{\cos\alpha}\sqrt{\frac{5}{8}a} - a - \frac{1}{20} \geq \sqrt{\frac{5}{8}a} - a - \frac{1}{20} > 0.$$

Observe that $|S| < (\frac{1}{4}\sin\alpha + \frac{1}{2}\cos\alpha)^2$, because S does not contain $\frac{1}{4} \times \frac{1}{2}$. Thus $s = \sqrt{\frac{5}{8}a} < \frac{1}{4}\sin\alpha + \frac{1}{2}\cos\alpha$. A computation shows that

$$v = \frac{1}{\sin\alpha}\left(\sqrt{\frac{5}{8}a} - \frac{1}{4}\cos\alpha\right) \geq a + \frac{1}{20},$$

provided $\frac{1}{5} \leq a \leq \frac{8}{5}(\frac{1}{4}\sin\alpha + \frac{1}{2}\cos\alpha)^2$. ∎

Assume that $S \supset P = a \times \frac{1}{4}$, where $|P| = \frac{2}{5}|S|$, and $\frac{1}{5} \leq a < \frac{1}{2}$ and assume that S does not contain $\frac{1}{4} \times \frac{1}{2}$. Observe that S contains the trapezoid $T \supset P$ described in Lemma 2, and that it contains a homothetic copy of T with the ratio -1. These two proper trapezoids are said to be *generated by* P.

The proof of Theorem will be divided into two parts depending on the size of the three largest squares in the sequence. The following lemma implies, that if in the sequence (S_i) the squares S_1, S_2, S_3 are "large" and $\sum |S_i| \geq 3$, then there exists a proper square Q such that $\sum_{i\geq 4} |S_i| > 3|Q|$ and $I \setminus Q$ can be covered by S_1, S_2, S_3 (since $\sum_{i\geq 4} |S_i| \geq 3 - (|S_1| + |S_2| + |S_3|)$ and (2), it follows that $\sum_{i\geq 4} |S_i| > 3 - (3 - 3|Q|) = 3|Q|$).

LEMMA 3. *Assume that S_1, S_2, S_3 are squares and that $s_1 \geq s_2 \geq s_3$, where s_i denotes the side length of S_i for $i = 1, 2, 3$. Moreover, assume that $|S_1| + |S_2| + |S_3| \geq 2.25$ and that $s_3 \geq 0.25\sqrt{5}$. If I cannot be translatively covered by S_1, S_2 and S_3, then S_1, S_2 and S_3 permit a translative covering of a part of I such that the uncovered part of I is contained in a proper square Q and*

$$|Q| < 1 - \frac{1}{3}(|S_1| + |S_2| + |S_3|). \tag{2}$$

Proof. Consider three cases.

Case 1, when $s_3 \leq 0.99$.

By $s_1^2 + s_2^2 + s_3^2 \geq 2.25$, $s_1 \leq 1$ and $s_3 \leq s_2$ we see that $s_2^2 \geq 0.625$. Thus $s_2 + 0.5\sqrt{2}s_3 > 1$. Obviously, S_3 contains a proper square of side length

$0.5\sqrt{2}s_3$. By Lemma 1 we see that S_2 (and S_1 as well) contains a rectangle $(1-0.5\sqrt{2}s_3)\times b_2$, where $b_2=[s_2^2-(1-0.5\sqrt{2}s_3)^2]^{\frac{1}{2}}$. The translations are defined as follows:

$$\sigma_1 S_1 \supset [0.5\sqrt{2}s_3, 1]\times[1-b_2,1],\ \sigma_2 S_2\supset[0,b_2]\times[0,1-0.5\sqrt{2}s_3],$$

$$\sigma_3 S_3 \supset [0,0.5\sqrt{2}s_3]\times[1-0.5\sqrt{2}s_3,1].$$

Consequently, we can take $Q=[b_2,1]\times[0,1-b_2]$. A computation shows that (2) holds.

Case 2, when $s_3>0.99$ *and when at least one of* S_1,S_2,S_3 *does not contain any segment of length* 1 *parallel to the first coordinate axis.*

Denote such a square by S_m. There is no loss of generality in assuming that $\alpha_m=\alpha(S_m)\le 45°$. Denote by s the side length of S_m and denote by r_1 and r_2, where $r_1\ge r_2$, the side lengths of the remaining two squares. Obviously, $s<\cos\alpha_m$. By the proof of Lemma 2 we see that S_m contains a square of side length

$$d=\frac{s}{\sin\alpha_m+\cos\alpha_m}>\frac{s}{s+\sqrt{1-s^2}}.$$

Put $b_3=\sqrt{r_2^2-(1-d)^2}$. By Lemma 1 we see that each S_i, for $i\in\{1,2,3\}$ and $i\ne m$, contains $b_3\times(1-d)$ as well as $(1-d)\times b_3$. The translations are defined so that $\sigma_m S_m\supset[0,d]\times[1-d,1]$ and that the remaining two squares cover $([0,b_3]\times[0,1-d])\cup([d,1]\times[1-b_3,1])$. Consequently, we can take $Q=[b_3,1]\times[0,1-b_3]$. A computation shows that (2) holds.

Case 3, when $s_3>0.99$ *and when each square from the set* $\{S_1,S_2,S_3\}$ *contains a segment of length* 1 *parallel to the first coordinate axis.*

Subcase 3a, when at least one of the angles $\alpha_1=\alpha(S_1)$, $\alpha_2=\alpha(S_2)$, $\alpha_3=\alpha(S_3)$ *is between* $21°$ *and* $69°$.

Denote such an angle by α_l. Without loss of generality we can assume that $21°\le\alpha_l\le 45°$. By Lemma 1 and by $s_1>0.99$, $s_2>0.99$ we conclude that each square from the set $\{S_1,S_2,S_3\}$ contains a rectangle 0.5×0.85. It is easy to see that S_l contains a rectangle $1\times u$, where

$$u=\frac{s_l-\cos\alpha_l}{\sin\alpha_l}>\frac{0.99-\cos 21°}{\sin 21°}>0.15.$$

Consequently, I can be covered by S_1,S_2 and S_3.

Subcase 3b, when either $\alpha_n\le 21°$ *or* $\alpha_n\ge 69°$ *for* $n=1,2,3$.

Denote by β_0 the smallest angle among the following:

$$\alpha_1,\ \alpha_2,\ \alpha_3,\ 90°-\alpha_1,\ 90°-\alpha_2,\ 90°-\alpha_3.$$

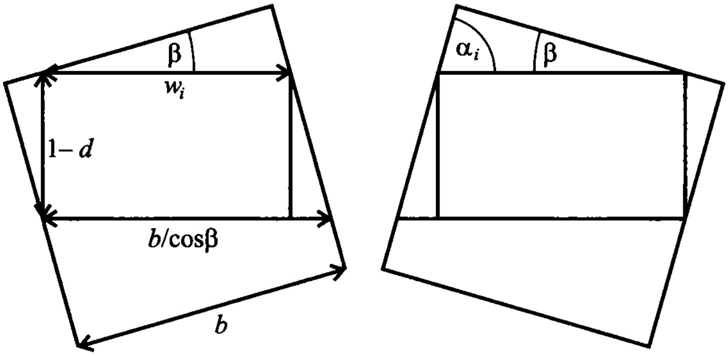

Fig. 3.

Let $k \in \{1,2,3\}$ be an integer such that either $\beta_0 = \alpha_k$ or $\beta_0 = 90° - \alpha_k$. The side length of S_k is denoted by b_0. Denote by S_i and S_j the squares from the set $\{S_1, S_2, S_3\}$ that are not S_k. Put $b = s_i$ and $b_1 = s_j$. Let $\beta = \alpha_i$ provided $\alpha_i \le 21°$, and $\beta = 90° - \alpha_i$ provided $\alpha_i \ge 69°$. Moreover, let $\beta_1 = \alpha_j$ provided $\alpha_j \le 21°$, and $\beta_1 = 90° - \alpha_j$ provided $\alpha_j \ge 69°$. Without loss of generality we can assume that $\beta \le \beta_1$. By the assumption of Case 3 we see that $b_0 \ge \cos\beta_0$, $b \ge \cos\beta$ and $b_1 \ge \cos\beta_1$. We can assume that $\beta_0 > 0°$, $\beta > 0°$ and $\beta_1 > 0°$, because $\alpha_n = 0°$ for $n \in \{1,2,3\}$ implies that $s_n = 1$ and, consequently, that I can be covered by S_n.

Observe that $S_k \supset d \times d$, where $d = \frac{b_0}{\sin\beta_0 + \cos\beta_0}$ (as in Case 2). We can assume that $d < 1$, because otherwise I can be covered by S_k. Moreover, it is easy to check that $S_j \supset (1-d) \times w_j$, where $w_j = \frac{b_1 - (1-d)\sin\beta_1}{\cos\beta_1}$, and that S_i contains a proper trapezoid of height $1-d$ and of bases of length $w_i = \frac{b-(1-d)\sin\beta}{\cos\beta}$ and $\frac{b}{\cos\beta}$ (see Fig. 3; we have $\beta = \alpha_i$ in the left-hand picture

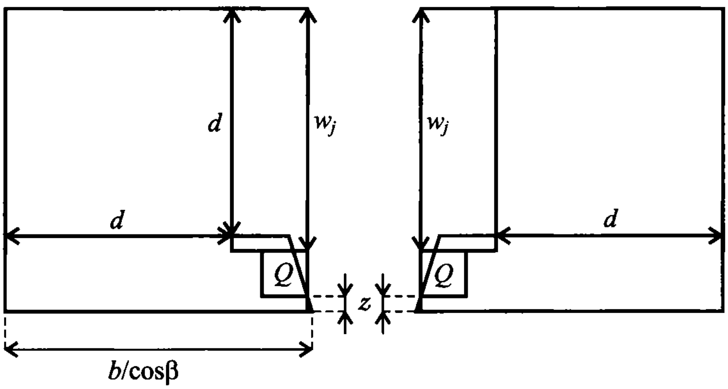

Fig. 4.

and $\beta = 90^\circ - \alpha_i$ in the right-hand picture). Let $z = \frac{\frac{b}{\cos\beta}-1}{\tan\beta} = \frac{b-\cos\beta}{\sin\beta}$ (see Fig. 4, where $\beta = \alpha_i$ in the left-hand picture and $\beta = 90^\circ - \alpha_i$ in the right-hand picture).

If $z + w_j \geq 1$, then I can be covered. Otherwise, it is possible to cover by S_1, S_2, S_3 a part of I so that the uncovered part of I is contained in a proper square Q of side length $1 - z - w_j$ (see Fig. 4). A computation shows that

$$b_0^2 + b^2 + b_1^2 + 3(1 - z - w_j)^2 < 3$$

provided $\cos\beta_0 \leq b_0 \leq 1$, $\cos\beta \leq b \leq 1$, $\cos\beta_1 \leq b_1 \leq 1$, $0^\circ < \beta_0 \leq \beta \leq \beta_1 \leq 21^\circ$ and $1 - z - w_j \geq 0$. ∎

3. Two covering methods

Let $S_1, S_2, \ldots$ be a sequence of squares of total area not smaller than 3, and let $1 \geq s_1 \geq s_2 \geq \ldots$, where s_i denotes the side length of S_i.

Now we present a preliminary covering method (the so-called p-method).

For each positive integer i we determine the rectangle $P_i = a_i \times h_i$ in the following way. Denote by $P_{max}(S_i)$ the proper rectangle of maximum area and height from the set $\{1, \frac{1}{2}, \frac{1}{4}, \ldots\}$ that is contained in S_i; if there are two such rectangles (for example $\frac{1}{4} \times \frac{1}{2}$ and $\frac{1}{2} \times \frac{1}{4}$), then we take as P_{max} the one with the largest height. By Lemma 2 we deduce that $|P_{max}(S_i)| \geq \frac{2}{5}|S_i|$. If the height of $P_{max}(S_i)$ equals either $\frac{1}{2}$ or 1, then let $P_i = P_{max}(S_i)$. Otherwise, let $P_i = a_i \times h_i$ be the rectangle contained in S_i with the largest possible height from the set $\{1, \frac{1}{2}, \frac{1}{4}, \ldots\}$ such that $|P_i| = \frac{2}{5}|S_i|$ and that $\frac{1}{2}a_i < h_i \leq 2a_i$. Obviously, $|S_i| \leq 2.5|P_i|$ for each positive integer i.

Assume that P_j and P_n have height $\frac{1}{4}$ and width greater than $\frac{1}{5}$. Let T_j be a trapezoid generated by P_j and let T_n be a trapezoid generated by P_n. If it is possible to cover $(a_j + a_n + 0.05) \times \frac{1}{4}$ by T_j and T_n, then we say that T_j and T_n are *of the same type*. For example, trapezoids denoted in the right-hand picture in Fig. 7 by the integers 1 and 3 are of the same type, but trapezoids denoted by 1 and 2 are not of the same type.

We change the order of squares $S_i \supset P_i$ in the sequence so that $h_i \geq h_{i+1}$ and that $a_i \geq a_{i+1}$ provided $h_i = h_{i+1}$, for $i = 1, 2, \ldots$.

By the *$l - th$ h-layer*, where $l \in \{1, 2, \ldots, h^{-1}\}$ we mean $[0,1] \times [(l-1)h, lh]$.

We place P_1 in the first h_1-layer so far to the left as it is possible, i.e., the translation σ_1 is defined so that $\sigma_1 P_1$ is packed in I and that $(0,0)$ is one of the vertices of $\sigma_1 P_1$ (see Fig. 5; in Figures 5, 6, 7, 8 each rectangle $\sigma_i P_i$ is denoted by the integer i, for short). Assume that $i > 1$ and that $P_1, \ldots, P_{i-1}$ have been placed. Denote by $z_i(l)$ the greatest number not larger than 1

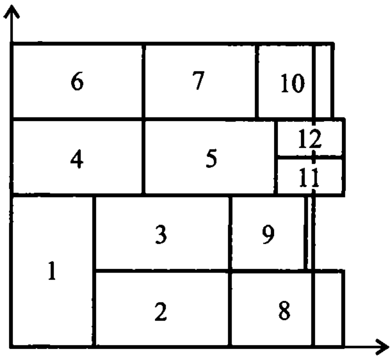

Fig. 5.

such that each point of the l-th h_i-layer with the first coordinate not greater than $z_i(l)$ is covered by a placed rectangle.

If it is possible to pack P_i translatively in I, then denote by j the smallest integer such that $z_i(j) + a_i \le 1$ (i.e., the j-th layer is the possibly lowest layer in which we can pack P_i). We pack P_i in the j-th layer so far to the left as it is possible, i.e.,

$$\sigma_i P_i = [z_i(j), z_i(j) + a_i] \times [(j-1)h_i, jh_i] \tag{3}$$

(see $\sigma_2 P_2, \dots, \sigma_7 P_7$ and $\sigma_9 P_9$ in Fig. 5).

Otherwise, denote by j the smallest integer such that $z_i(j) \le z_i(l)$ for each $l \in \{1, \dots, h_i^{-1}\}$ (i.e., the j-th layer is the least filled one; if there is a number of such equally filled layers, then we choose the possibly lowest one). The translation σ_i is defined by condition (3) (see $\sigma_8 P_8$ and $\sigma_{10} P_{10}, \sigma_{11} P_{11}, \sigma_{12} P_{12}$ in Fig. 5).

Obviously, we always cover I by squares and σ_i is defined for $S_i \supset P_i$. We stop the covering process immediately when I is covered.

A placed rectangle is called *right* if it contains a point (x, y) with $x > 1$. If P is of the form $a \times h$, then denote by P^v a *vertical* rectangle $h \times a$; we say also that P is *horizontal.*

Observe that all rectangles placed by the p-method have pairwise disjoint interiors. If all rectangles placed by the p-method are contained in $C_1 = [0, 1.2] \times [0, 1]$ and if I is not covered, then $\sum |S_i| \le 2.5 \sum |P_i| < 2.5 \cdot 1.2 = 3$, which is a contradiction. Unfortunately, it is possible that some right rectangles of height greater than or equal to $\frac{1}{8}$ are not contained in C_1.

Assume that $k \in \{0, \dots, 4\}$. Moreover, assume that (S_i) is a sequence of squares chosen so that if we use the p-method for the covering of I, then no right rectangle has height greater than $\frac{1}{4}$. By the t_k-method for covering I by (S_i) we mean the following method.

If either $k = 0$ or $k > 0$ and there are at most $2k$ rectangles of height $\frac{1}{4}$ and of width greater than $\frac{1}{5}$, then first we use for the covering the p-method as long as there is a point of $C_2 = [0, \frac{3}{4}] \times [0, 1]$ not covered by any placed rectangle. Otherwise, denote by $P_\eta, P_{\eta+1}, \dots, P_{\eta+\nu}$ all rectangles of height $\frac{1}{4}$. Obviously, $\nu \geq 2k$ and $a_{\eta+2k} > \frac{1}{5}$. We find k pairs T_{n_i}, T_{m_i} $(i = 1, \dots, k)$ of trapezoids of the same type generated by rectangles $P_\eta, \dots, P_{\eta+2k}$. Trapezoids T_{n_i}, T_{m_i} can be used for the translative covering of $P_i^{new} = [a_{n_i} + a_{m_i} + 0.05] \times 0.25$. From now on we take each pair P_{n_i}, P_{m_i} for $i = 1, \dots, k$ as a rectangle P_i^{new}, and we use the p-method for the covering as long as there is a point of C_2 not covered by any placed rectangle.

Figure 7 illustrates the case when there are 10 rectangles of width $\frac{1}{4}$ and of height greater than $\frac{1}{5}$ in the sequence. By the t_4-method we find four pairs of trapezoids of the same type generated by rectangles $P_1, \dots, P_9$. In this figure trapezoids generated by P_1 and P_3, P_2 and P_6, P_4 and P_5, P_7 and P_9 are of the same type. We take these four pairs as four "new" rectangles and we place them by the p-method.

Let m be the smallest integer such that each point of C_2 has been covered by a placed rectangle preceding P_m (rectangles P_i^{new} included, of course). The rectangle P_m is called the *boundary rectangle*. Observe that $h_m \leq \frac{1}{4}$, because otherwise, if we use the p-method for the covering by (S_i), then at least one right rectangle has height greater than $\frac{1}{4}$.

We change the position of some placed rectangles preceding P_m (i.e., we move them by translations), if necessary, so that the part of I not covered by placed rectangles is a connected set, that placed rectangles have still pairwise disjoint interiors and that C_2 is covered. The left-hand pictures in Fig. 6 and Fig. 7 illustrate the p-method, and the right-hand pictures

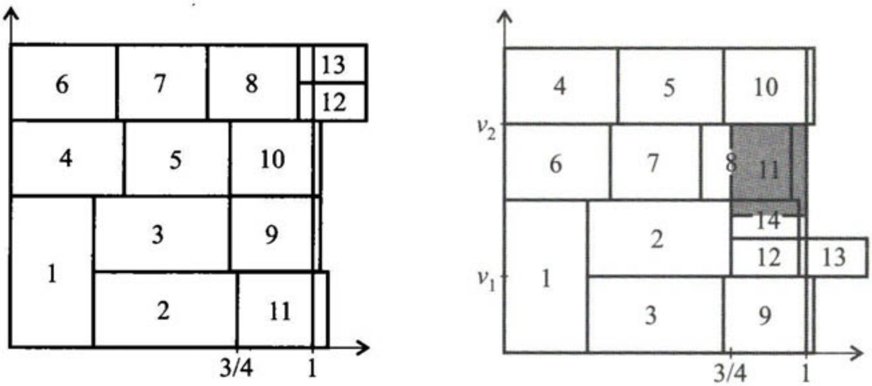

Fig. 6.

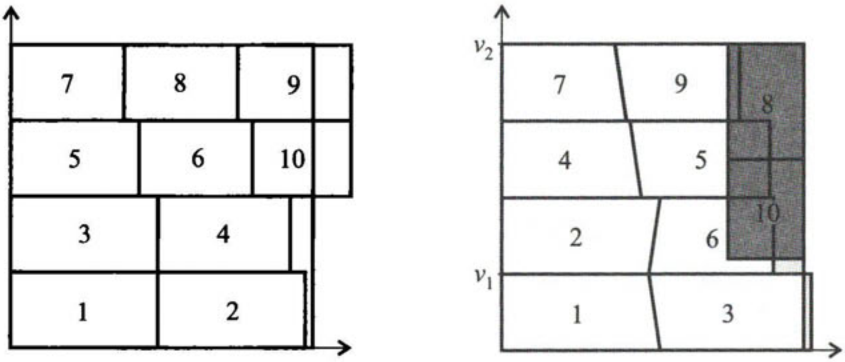

Fig. 7.

illustrate the t_0-method and the t_4-method, respectively. In Fig. 7 the part of I not covered by four "new" rectangles is a connected set, and therefore we do not change the position of these "new" rectangles. In Fig. 6 we place rectangles $P_1, \dots, P_{10}$ by the p-method (here $m = 11$) and we change the position of placed rectangles, because $I \setminus \bigcup_{i=1}^{10} \sigma_i P_i$ is not a connected set.

Denote by v_1 the largest number and by v_2 the smallest number such that the uncovered part of I is contained in $[\frac{3}{4}, 1] \times [v_1, v_2]$. Each square S_i for $i \geq m$ is placed as follows.

First consider the case when $h_m = \frac{1}{4}$ and $a_m > \frac{1}{5}$ (we say then that the boundary rectangle P_m is *big*).

If $h_i = \frac{1}{4}$, then we use $P_i^v = h_i \times a_i \supset S_i$ for the covering. Let $\mu_i = 1$ provided the point $(1, 1)$ is not covered by any placed rectangle preceding P_i. Otherwise, let μ_i denote the smallest number such that each point of $[\frac{3}{4}, 1] \times [\mu_i, 1]$ is covered by a placed rectangle preceding P_i (vertical rectangles included). We place S_i so that $\sigma_i P_i^v = [\frac{3}{4}, 1] \times [\mu_i - a_i, \mu_i]$.

In Fig. 6 we have $m = 11$, $h_m = \frac{1}{4}$, $a_m > \frac{1}{5}$, and therefore we use for the covering vertical rectangle P_{11}^v (obviously, here $\mu_m = v_2$).

If $h_i \leq \frac{1}{8}$, then the translation σ_i is defined by condition (3), where j is the smallest integer such that $z_i(j) < 1$ (see P_{12}, P_{13} and P_{14} in Fig. 6). Let us remind that $z_i(l)$ denotes the greatest number not larger than 1 such that each point of the l-th h_i-layer with the first coordinate not greater than $z_i(l)$ is covered by a placed rectangle (vertical rectangles included).

Finally consider the case when either $h_m \leq \frac{1}{8}$ or $h_m = \frac{1}{4}$ and $a_m \leq \frac{1}{5}$ (we say then that the boundary rectangle is *small*). The right-hand pictures in Figures 6 and 7 illustrate the case when the boundary rectangle is big. Fig. 8 illustrates the t_1-method (here trapezoids generated by P_2 and P_3 are of the same type) when the boundary rectangle P_6 is small ($h_6 = \frac{1}{4}$ and $a_6 \leq \frac{1}{5}$).

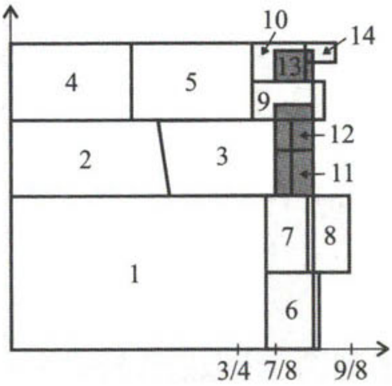

Fig. 8.

If $h_i = \frac{1}{4}$ (obviously, $a_i \leq \frac{1}{5}$), then the translation σ_i is defined by condition (3), where j is the smallest integer such that $z_i(j) < 1$ (see P_6, P_7 and P_8 in Fig. 8).

If either $h_i < \frac{1}{8}$ or $h_i = \frac{1}{8}$ and there is a point of $C_3 = [0, \frac{7}{8}] \times [0, 1]$ not covered by any placed rectangle preceding P_i (vertical rectangles included), then we move P_i in the place described by the p-method. This means that σ_i is defined by (3), where j is chosen in the following way. If there is an integer $l \in \{1, 2, \ldots, h_i^{-1}\}$ such that $z_i(l) + a_i \leq 1$, then j is the smallest integer such that $z_i(j) + a_i \leq 1$ (see P_{10} in Fig. 8). If $z_i(l) + a_i > 1$ for each $l \in \{1, 2, \ldots, h_i^{-1}\}$, then j is the smallest integer such that $z_i(j) \leq z_i(l)$ for each $l \in \{1, 2, \ldots, h_i^{-1}\}$ (see P_9 and P_{14} in Fig. 8).

If $h_i = \frac{1}{8}$ and if each point of C_3 is covered by a placed rectangle, then we use $P_i^v = \frac{1}{8} \times a_i \supset S_i$ for the covering. Let $\lambda_i = 0$ provided the point $(1, 0)$ is not covered by any placed rectangle preceding P_i. Otherwise, denote by λ_i the greatest number such that each point of $[\frac{7}{8}, 1] \times [0, \lambda_i]$ is covered by a placed rectangle preceding P_i (vertical rectangles included). The translation σ_i is defined so that $\sigma_i P_i^v = [\frac{7}{8}, 1] \times [\lambda_i, \lambda_i + a_i]$ (see P_{11}, P_{12} and P_{13} in Fig. 8, here $\lambda_{11} = \frac{1}{2}$, $\lambda_{12} = \frac{1}{2} + a_{12}$, $\lambda_{13} = \frac{7}{8}$) .

4. Main result

THEOREM 1. *The unit square can be translatively covered by any (finite or infinite) sequence of squares of side lengths not greater than 1 whose total area is not smaller than* 3.

P r o o f. Let $S_1, S_2, \ldots$ be a sequence of squares of side lengths not greater than 1 whose total area is not smaller than 3. Without loss of generality we can assume that $s_1 \geq s_2 \geq \ldots$, where s_i denotes the side length of S_i.

We can assume that there are at least three squares in the sequence. If there is only one square, then $|S_1| \geq 3$. Consequently, I can be covered by S_1, because we know by Lemma 1 that $S_1 \supset 1 \times 1$. If there are only two squares, then $|S_1| + |S_2| \geq 3$. If $s_1^2 \geq 2$, then $S_1 \supset 1 \times 1$ and S_1 permits a covering of I. If $s_1^2 < 2$, then $s_2^2 > 1$. By Lemma 1 we know that $S_1 \supset 1 \times \sqrt{s_1^2 - 1}$ and that $S_2 \supset 1 \times \sqrt{s_2^2 - 1}$. It is easy to check that $\sqrt{s_1^2 - 1} + \sqrt{s_2^2 - 1} \geq 1$. Consequently, I can be covered by S_1 and S_2.

Part I. Assume that it is impossible to cover by S_1, S_2 and S_3 a part of I so that the uncovered part of I is contained in a proper square Q and that (2) holds.

Consider five cases depending on the size of the first square in the sequence. In all cases we present a covering method for a covering of I by squares $S_1, S_2, \ldots$, and we show that if I is not covered by using this method, then the sum of areas of squares is smaller than 3, which again is a contradiction.

Case 1, when $h_1 \leq \frac{1}{8}$.

We use the t_0-method (obviously, the boundary rectangle is small).

Observe that each right rectangle of height $\frac{1}{8}$ covers a point of C_3 (see, for example, P_9 in Fig. 8). Each rectangle of height $\frac{1}{8}$ has width smaller than $\frac{1}{4}$. Thus each placed horizontal rectangle of height $\frac{1}{8}$ is contained in $C_4 = [0, \frac{9}{8}] \times [0, 1]$. Also each placed rectangle of height smaller than $\frac{1}{8}$ is contained in C_4 (see, for example, P_{14} in Fig. 8). It is possible that a placed vertical rectangle of height $\frac{1}{8}$ covers points of $I \setminus C_3$ covered earlier by horizontal rectangles of height $\frac{1}{8}$ (see P_{12} and P_9 or P_{13} and P_{10} in Fig. 8), and it is possible that a placed rectangle of height smaller than $\frac{1}{8}$ covers points covered earlier by a vertical rectangle of height $\frac{1}{8}$ (see P_{14} and P_{13} in Fig. 8). But the total area of all vertical rectangles of height $\frac{1}{8}$ and the parts of right rectangles lying outside I is smaller than $\frac{1}{8} + \frac{1}{2} \cdot \frac{1}{8} = \frac{3}{16}$ (by a part of $\sigma_i P_i$ lying *outside* I we mean $\sigma_i P_i \setminus I$). Consequently, if I is not covered, then

$$\sum |S_i| = 2.5 \sum |P_i| < 2.5\left(1 + \frac{3}{16}\right) < 2.5 \cdot 1.2 = 3.$$

Case 2, when $h_1 = \frac{1}{4}$.

We use the t_4-method.

If the boundary rectangle P_m is small, then all placed rectangles of height $\frac{1}{4}$ are contained in C_1. Denote by ξ_1 the greatest number and by ξ_2 the smallest number such that no point of the segment $\{(x, y);\ x = 1,\ \xi_1 < y < \xi_2\}$ is covered by a placed rectangle of height $\frac{1}{4}$ (if $h_m < \frac{1}{4}$, then

$\xi_1 = v_1$, $\xi_2 = v_2$). Let $\xi = \xi_2 - \xi_1$. If I is not covered, then (cf. Case 1)

$$\sum |S_i| = 2.5 \sum |P_i| < 2.5\left[1 + 0.2(1-\xi) + \frac{3}{16}\xi\right] < 2.5 \cdot 1.2 = 3.$$

If the boundary rectangle P_m is big, then we use for the covering four pairs of trapezoids generated by rectangles of height $\frac{1}{4}$ (see Fig. 7). Obviously, the area of a part of C_1 covered by four pairs of trapezoids generated by eight rectangles equals the area of these eight rectangles plus 0.05. Assume that I is not covered. Each placed horizontal rectangle of height $\frac{1}{4}$, as well as each placed horizontal rectangle of height not greater than $\frac{1}{8}$, is contained in $[0, 1.25] \times [0, 1]$. Moreover, placed horizontal rectangles have pairwise disjoint interiors. Consequently, $\sum |S_i| = 2.5 \sum |P_i| < 2.5(1.25 - 0.05) = 3$.

Case 3, when $h_1 = 1$.

We can assume that $h_2 \le \frac{1}{2}$, because otherwise I can be covered by S_1 and S_2. We use P_1^v for the covering: we cover by P_1^v, and consequently by S_1, the second $\frac{1}{2}$-layer ($|S_1| \le 1$, of course). The rectangles $P_2, P_3, \ldots$ are used for the further covering by the t_0-method. If I is not covered, then $\sum |S_i| < 1 + 2.5 \cdot 0.5 \cdot 1.25 < 3$.

Case 4, when $h_1 = \frac{1}{2}$ *and* $a_1 \ge 0.35$.

Subcase 4a, when $h_3 = \frac{1}{2}$ *and* $a_2 + a_3 \ge 1$.

Observe that $a_1 \ge a_2 \ge \frac{1}{2}$. This implies that S_1 permits a covering of 0.5×0.5 and that S_2 and S_3 permit a covering of 0.5×1. We have $|S_1| + |S_2| + |S_3| \ge 2.25$. The reason is that it is possible to cover a part of I by S_1, S_2 and S_3 so that the uncovered part is contained in $Q = 0.5 \times 0.5$. Thus $|S_1| + |S_2| + |S_3| < 2.25$ implies $|Q| < 1 - \frac{1}{3}(|S_1| + |S_2| + |S_3|)$, which again is a contradiction (see the assumption of Part I). By Lemma 3 and by the assumption of Part I of the proof we conclude that $s_3 < 0.25\sqrt{5}$.

We cover $[0,1] \times [0.5, 1]$ by S_2 and S_3 and we use the t_0-method for the covering of $[0,1] \times [0, 0.5]$ by $P_1, P_4, P_5, \ldots$. If I is not covered, then

$$\sum |S_i| < 1 + (0.25\sqrt{5})^2 + 2.5 \cdot 0.5 \cdot 1.25 < 3.$$

Subcase 4b, when it is possible to cover 1×0.5 *by the rectangles from the set* $\{P_2, P_3, P_4, P_5\}$ *that have height* $\frac{1}{2}$.

First assume that $h_4 = \frac{1}{2}$, $a_2 + a_3 < 1$ and $a_2 + a_3 + a_4 \ge 1$. We cover the second $\frac{1}{2}$-layer by S_2, S_3, S_4. We show that $|S_2| + |S_3| + |S_4| \le 1.5$. Obviously, $P_i = P_{max}(S_i)$ and, by Lemma 1, $s_i^2 \le a_i^2 + h_i^2$ for $i = 2, 3, 4$. If $a_2 \ge \frac{1}{2}$, then

$|S_2|+|S_3|+|S_4| \le a_2^2+0.5^2+a_3^2+0.5^2+a_4^2+0.5^2 \le 0.75+a_2^2+2(1-a_2)^2 \le 1.5$. If $a_2 < \frac{1}{2}$, then $|S_2| + |S_3| + |S_4| \le \frac{3}{4} + 3a_2^2 < 1.5$.

Now assume that $h_5 = \frac{1}{2}$ and $a_2 + a_3 + a_4 < 1$. Obviously, $a_2 + a_3 + a_4 + a_5 \ge 1$. We cover the second $\frac{1}{2}$-layer by $S_2, \dots, S_5$. A computation shows that $|S_2| + \dots + |S_5| \le 1.5$.

For the covering of the first $\frac{1}{2}$-layer by the remaining rectangles we use the t_0-method.

An easy computation shows that $a_1^2 + \frac{1}{4} < \frac{5}{4}a_1 - \frac{1}{16}$ provided $0.35 \le a_1 \le 0.9$. Moreover, $|S_1| \le 1 < 2.5 \cdot 0.5 \cdot 0.9 - \frac{1}{16} < 2.5|P_1| - \frac{1}{16}$ provided $a_1 > 0.9$. Thus

$$|S_1| < 2.5|P_1| - \frac{1}{16}. \tag{4}$$

As a consequence, if I is not covered, then $\sum |S_i| < 1.5 + 2.5 \cdot 0.5 \cdot 1.25 - \frac{1}{16} = 3$.

Subcase 4c, when it is impossible to cover 1×0.5 *by the rectangles from the set* $\{P_2, P_3, P_4, P_5\}$ *that have height* $\frac{1}{2}$. This implies that $h_5 \le \frac{1}{4}$.

First assume that $h_2 \le \frac{1}{4}$. We use the t_2-method. If I is not covered, then by (4) we deduce that

$$\sum |S_i| < 2.5(0.5 \cdot 1.25 + 0.5 \cdot 1.2) - \frac{1}{16} = 3.$$

Now assume that $h_2 = \frac{1}{2}$ and $a_2 \ge 0.35$. We use the t_0-method. By $|S_2| < 2.5|P_2| - \frac{1}{16}$ we conclude that if I is not covered, then $\sum |S_i| < 2.5 \cdot 1.25 - \frac{2}{16} = 3$.

Finally assume that $h_2 = \frac{1}{2}$ and $a_2 < 0.35$. If $a_1 + a_2 \ge 1$, then we cover by S_1 and S_2 the second $\frac{1}{2}$-layer (obviously, $|S_1| + |S_2| < 1 + 0.35^2 + 0.5^2$) and we use the t_0-method for the covering of the first $\frac{1}{2}$-layer by $P_3, P_4, \dots$. If I is not covered, then

$$\sum |S_i| < 1 + 0.35^2 + 0.5^2 + 2.5 \cdot 0.5 \cdot 1.25 < 3.$$

If $a_1 + a_2 < 1$ and if $h_3 = \frac{1}{2}$, then we cover by $\frac{1}{2} \times \frac{1}{4} \supset S_2$ and by $\frac{1}{2} \times \frac{1}{4} \supset S_3$ the fourth $\frac{1}{4}$-layer. We use $P_1, P_4, P_5, \dots$ for the covering by the t_1-method (obviously, $a_1 + a_4 < 1$ and $h_5 \le \frac{1}{4}$). By (4) we deduce that if I is not covered, then

$$\sum |S_i| < 2(0.35^2 + 0.5^2) + 2.5[0.25 \cdot (1.25 - 0.05) + 0.5 \cdot 1.25] - \frac{1}{16} < 3.$$

If $a_1 + a_2 < 1$ and if $h_3 \le \frac{1}{4}$, then we use the t_2-method. If I is not covered, then $\sum |S_i| < 2.5[0.5 \cdot (1.25 - 0.05) + 0.5 \cdot 1.25] - \frac{1}{16} < 3$.

Case 5, when $h_1 = \frac{1}{2}$ and $a_1 < 0.35$.

Let $n = 0$ provided $h_2 \leq \frac{1}{4}$, and let n be the largest even integer such that $h_n = \frac{1}{2}$ provided $h_2 = \frac{1}{2}$. Observe that squares S_i, S_{i+1}, where $i = 1, 3, \dots, n-1$, permit a covering of a $\frac{1}{4}$-layer, because each such square contains $\frac{1}{2} \times \frac{1}{4}$.

If $n = 8$, then I can be covered by $S_1, \dots, S_8$. Otherwise, consider three possibilities.

Subcase 5a, when $n \geq 2$ and $h_{n+1} \leq \frac{1}{4}$ (i.e., when there is an even number of rectangles of height $\frac{1}{2}$).

We cover $[0,1] \times [1 - \frac{1}{8}n, 1]$ by $S_1, \dots, S_n$. The rectangles $P_{n+1}, P_{n+2}, \dots$ are used for the covering of $[0,1] \times [0, 1 - \frac{1}{8}n]$ by the $t_{4-0.5n}$-method. If I is not covered, then

$$\sum |S_i| < n(0.35^2 + 0.5^2) + 2.5(1 - \frac{1}{8}n) \cdot 1.2 < 3.$$

Subcase 5b, when $n \geq 2$ and $h_{n+1} = \frac{1}{2}$.

The rectangles P_1, P_2, P_3 are used for the covering of $[0, a_1 + a_2 + a_3] \times [0.5, 1]$. Moreover, if $n \in \{4, 6\}$, then we cover the first $\frac{1}{4}$-layer by S_4 and S_5. If $n = 6$, then we cover the second $\frac{1}{4}$-layer by S_6 and S_7. The rectangles $P_{n+2}, P_{n+3}, \dots$ are used for the further covering by the $t_{3-0.5n}$-method.

Assume that I is not covered. Let $\lambda = a_1 + a_2 + a_3$.

If $\lambda \geq 1$, then

$$\sum |S_i| < 1.5 + 3(0.35^2 + 0.5^2) < 3.$$

Assume that $\lambda < 1$. Obviously, $\lambda \geq 0.75$. If no vertical rectangle of width $\frac{1}{4}$ has been used for the covering, then $\sum |S_i| < 2.5 \cdot 1.2 = 3$. Otherwise, let ζ be the largest number such that each point of $\{(x,y);\ x = 1,\ 1 - \zeta \leq y \leq 1\}$ is covered by a placed vertical rectangle of width $\frac{1}{4}$. Obviously, $\zeta > \frac{1}{5}$.

If $\frac{1}{5} < \zeta < \frac{1}{2}$, then denote by l the smallest integer such that each point of $[\lambda, 1] \times [\frac{1}{8}l, 1]$ is covered by a placed vertical rectangle of width $\frac{1}{4}$. Observe that some points of $[\lambda, 1] \times [\frac{1}{8}(l-1), \frac{1}{8}l]$ can be covered by vertical rectangles of width $\frac{1}{4}$ as well as by rectangles of height not greater than $\frac{1}{8}$. The area of the part of I covered by vertical rectangles of height $\frac{1}{4}$, as well as by rectangles of height not greater than $\frac{1}{8}$, does not exceed $(1 - \lambda)(\zeta - \frac{1}{8})$ provided $\zeta < \frac{1}{4}$ and it does not exceed $\frac{1}{8}(1 - \lambda)$ provided $\zeta \geq \frac{1}{4}$.

A standard computation shows that

$$a_1^2 + a_2^2 + a_3^2 \leq (\lambda - 0.5)^2 + 0.25^2 + 0.25^2 = \lambda^2 - \lambda + \frac{3}{8}.$$

Thus $|S_1| + |S_2| + |S_3| \leq \lambda^2 - \lambda + \frac{9}{8}$.

Denote by ξ_1 the greatest number and by ξ_2 the smallest number such that no point of the segment $\{(x,y);\ x=1,\ \xi_1<y<\xi_2\}$ is covered either by a placed horizontal rectangle of height $\frac{1}{4}$ or by a placed vertical rectangle of width $\frac{1}{4}$. Let $\xi=\xi_2-\xi_1$.

If $\zeta<\frac{1}{4}$, then

$$\sum|S_i|<1.5+\lambda^2-\lambda+\frac{9}{8}+2.5\left[0.25\zeta+(1.25-\lambda)(0.5-\zeta)+(1-\lambda)\left(\zeta-\frac{1}{8}\right)\right]<3.$$

If $\frac{1}{4}\le\zeta<\frac{1}{2}$, then

$$\sum|S_i|<1.5+\lambda^2-\lambda+\frac{9}{8}+2.5\left[0.25\zeta+(1.25-\lambda)(0.5-\zeta)+\frac{1}{8}(1-\lambda)\right]<3.$$

If $\zeta\ge\frac{1}{2}$, then $\sum|S_i|<1.5+\lambda^2-\lambda+\frac{9}{8}+2.5\cdot0.5\cdot0.25<3$.

Subcase 5c, when $n=0$.

If either $h_8\le\frac{1}{8}$ or $h_8=\frac{1}{4}$ and $a_8\le0.2$, then we use the t_0-method and we argue as in Case 1 (the boundary rectangle is small). Assume that $h_8=\frac{1}{4}$ and that $a_8>0.2$.

If there is a pair of trapezoids of the same type generated by rectangles of height $\frac{1}{4}$ and width greater than $\frac{1}{5}$ such that this pair permits a covering of $1\times\frac{1}{4}$, then we cover $[0,1]\times[0.75,1]$ by these trapezoids. The remaining rectangles are used for the covering by the t_0-method. If I is not covered, then

$$\sum|S_i|<2.5(0.25+0.75\cdot1.25)<3.$$

Assume that it is impossible to cover $1\times\frac{1}{4}$ by any pair of trapezoids of the same type generated by rectangles of height $\frac{1}{4}$ and width greater than $\frac{1}{5}$.

First assume that $a_3\ge0.75-a_1$. The first three translations are defined as follows:

$$\sigma_1P_1=[0,a_1]\times[0.5,1],\ \sigma_2P_2=[a_1,a_1+a_2]\times[0.75,1],$$
$$\sigma_3P_3=[a_1,a_1+a_3]\times[0.5,0.75].$$

The remaining rectangles are used for the covering of I by the t_2-method. Assume that I is not covered.

If no vertical rectangle of width $\frac{1}{4}$ has been used for the covering, then $\sum|S_i|<2.5\cdot1.2=3$. Otherwise, denote by ζ the largest number such that each point of $\{(x,y);\ x=1,\ 1-\zeta\le y\le1\}$ is covered by a placed vertical rectangle of width $\frac{1}{4}$. Obviously, $\zeta>0.2$. Moreover, $0.75\le a_1+a_3<0.85$ and $0.75\le a_1+a_2<0.85$.

If $\zeta\ge\frac{1}{2}$, then

$$\sum|S_i|<2.5[0.5(0.85+0.25)+0.5\cdot1.2]<3.$$

If $\zeta < \frac{1}{2}$, then we change the position of all squares $\sigma_i S_i$ for $i \geq m$ (where P_m is the boundary rectangle). We do not use any vertical rectangle of width $\frac{1}{4}$ for the covering. Let u be the greatest integer such that $h_u = \frac{1}{4}$. Obviously, $a_m + \cdots + a_u = \zeta < \frac{1}{2}$. We place $P_m, \ldots, P_u$ so that

$$\sigma_m P_m \cup \cdots \cup \sigma_u P_u = [a_1 + a_2,\ a_1 + a_2 + a_m + \cdots + a_u] \times [0.75, 1].$$

The rectangles $P_{u+1}, P_{u+2}, \ldots$ (of height not greater than $\frac{1}{8}$) are used for the further covering as in the t_2-method when the boundary rectangle is small (this means that if $h_i = \frac{1}{8}$, then we use P_i for the covering provided there is a point of C_3 not covered by any placed rectangle preceding P_i, and we use P_i^v otherwise). Assume that I is not covered. If there is no right rectangle of height $\frac{1}{4}$ either in the first $\frac{1}{4}$-layer or in the second $\frac{1}{4}$-layer, then

$$\begin{aligned}\sum |S_i| &< a_1^2 + 0.5^2 \\ &+ 2.5\left[0.25(0.5+0.5) + 0.25(1-a_1) + 0.5(1-0.05) + 0.75 \cdot \frac{3}{16}\right] < 3.\end{aligned}$$

If there is a right rectangle of height $\frac{1}{4}$ in the first $\frac{1}{4}$-layer (obviously, the total area of rectangles used for the covering of this layer is smaller than $0.25(1.25 - 0.05)$) and there is no right rectangle in the second $\frac{1}{4}$-layer, then

$$\begin{aligned}\sum |S_i| &< a_1^2 + 0.5^2 \\ &+ 2.5\left[0.25 + 0.25(1-a_1) + 0.25(1-0.05) + 0.25 \cdot 1.2 + 0.5 \cdot \frac{3}{16}\right] < 3.\end{aligned}$$

If there is a right rectangle of height $\frac{1}{4}$ in the second $\frac{1}{4}$-layer, then the total area of rectangles used for the covering of the first two $\frac{1}{4}$-layers is smaller than

$$0.25(1.25 - 0.05) + 0.25\left[0.75 - 0.05 + \frac{1}{2}(0.75 - 0.05)\right] = 0.5625,$$

because each right rectangle of height $\frac{1}{4}$ covers a point of C_2. Consequently,

$$\sum |S_i| < a_1^2 + 0.5^2 + 2.5\left[0.25 + 0.25(1-a_1) + 0.5625 + 0.25 \cdot \frac{3}{16}\right] < 3.$$

Assume that $a_3 < 0.75 - a_1$ and that $a_1 \geq 0.3$. We use the t_4-method. Observe that $a_1 + 2a_3 < a_1 + 2(0.75 - a_1) \leq 1.2$. If I is not covered, then $\sum |S_i| < 2.5 \cdot 1.2 = 3$.

Finally assume that $a_3 < 0.75 - a_1$ and that $a_1 < 0.3$.

If $a_6 \geq 0.46$, then we use $S_2, \ldots, S_6$ for the covering of the first $\frac{1}{2}$-layer: we cover $[0, 0.96] \times [0.5, 1]$ by two pairs of trapezoids of the same type, and the remaining square that contains 0.46×0.25 is used for the covering of $[0.96, 1] \times [0.5, 1]$. It is possible, because we deduce by Lemma 2 that this square contains a trapezoid of height 0.25 and of bases of length 0.46 and 0.51 parallel to the second coordinate axis and, consequently, it contains 0.04×0.5. The sum of areas of five squares used for the covering of the second $\frac{1}{2}$-layer is smaller than $2.5(2 \cdot 0.25 \cdot 0.95 + 0.25 \cdot 0.5) = 1.5$. The rectangles $P_1, P_7, P_8, \ldots$ are used for the covering of the uncovered part of I by the t_2-method. If I is not covered, then

$$\sum_{i=1}^{6} |S_i| + \sum_{i \geq 7} |S_i| < a_1^2 + 0.5^2 + 1.5 + 2.5[0.25 \cdot 0.95 + 0.25(1.2 - a_1)] \leq 3.$$

If $a_6 < 0.46$, then we place S_1 so that $\sigma_1 P_1 = [0, a_1] \times [0.5, 1]$ and use the t_4-method for the further covering with an additional condition: if $h_{10} = \frac{1}{4}$ and $a_{10} > 0.2$, then one pair of trapezoids (from four ones) of the same type is chosen from the trapezoids generated by P_2, P_3 and P_4. We have $a_4 < 0.75 - a_1$ (otherwise $a_3 \geq a_4 \geq 0.75 - a_1$) and $a_4 < 0.475$ (otherwise $a_2 \geq a_3 \geq 0.475$ and we can cover $1 \times \frac{1}{4}$ by a pair of trapezoids generated by P_2, P_3, P_4). This implies that even if two pairs of "large" trapezoids of the same type together with P_1 are used for covering the second $\frac{1}{2}$-layer, then the total area of the corresponding five squares used for covering this layer does not exceed

$$a_1^2 + 0.5^2 + 2.5 \cdot 2 \cdot 0.25(0.75 - a_1 + 0.46) < 1.5$$

provided $a_1 > 0.275$, and does not exceed

$$a_1^2 + 0.5^2 + 2.5 \cdot 2 \cdot 0.25(0.475 + 0.46) < 1.5$$

provided $a_1 \leq 0.275$. It is easy to check that if I is not covered, then $\sum |S_i| < 3$.

Part II. Assume that it is possible to cover translatively a part of I by S_1, S_2, S_3 so that the uncovered part is contained in a proper square Q_1 of area smaller than

$$1 - \frac{1}{3}(|S_1| + |S_2| + |S_3|) \leq \frac{1}{3} \sum_{i>3} |S_i|.$$

We cover a part of I in this way. By $\sum_{i>3} |S_i| > 3|Q_1|$ we conclude that there exists an integer $z \geq 4$ such that $\sum_{i=4}^{z} |S_i| > 3|Q_1|$.

There are three possibilities:

(i) there exists an index $j \in \{4, 5, 6\}$ such that Q_1 can be translatively covered by $S_4, \ldots, S_j$;

(*ii*) it is possible to cover translatively a part of Q_1 by S_4, S_5, S_6 so that the uncovered part is contained in a proper square Q_2 of area smaller than

$$|Q_1| - \frac{1}{3}(|S_4| + |S_5| + |S_6|) \le \frac{1}{3}\sum_{i=7}^{z} |S_i|;$$

(*iii*) it is impossible to cover translatively a part of Q_1 by S_4, S_5, S_6 so that the uncovered part of Q_1 is contained in a proper square Q and that

$$|Q| < |Q_1| - \frac{1}{3}(|S_4| + |S_5| + |S_6|).$$

In case (*ii*) we continue this covering process, i.e., we cover a part of Q_1 by S_4, S_5, S_6. By $\sum_{i=4}^{z} |S_i| > 3|Q_1|$ we conclude that there are two possibilities:

(*a*) on a stage of this covering process I has been covered;

(*b*) there exists an integer τ and a proper square Q_τ whose area does not exceed $\frac{1}{3}\sum_{i=3\tau+1}^{z} |S_i|$ such that the following two conditions are fulfilled:

(*b*1) $I \setminus Q_\tau$ has been translatively covered by squares preceding $S_{3\tau+1}$;

(*b*2) it is impossible to cover translatively a part of Q_τ by $S_{3\tau+1}, S_{3\tau+2}$, $S_{3\tau+3}$ so that the uncovered part of Q_τ is contained in a proper square Q of area

$$|Q| < |Q_\tau| - \frac{1}{3}(|S_{3\tau+1}| + |S_{3\tau+2}| + |S_{3\tau+3}|).$$

Observe that in case (*b*) we have $z \ge 3\tau + 3$. The reason is that if $z < 3\tau + 3$ and $|Q_\tau| \le \frac{1}{3}\sum_{i=3\tau+1}^{z} |S_i|$, then arguing as at the beginning of the proof of Theorem we see that Q_τ can be translatively covered either by $S_{3\tau+1}$ or by $S_{3\tau+1}$ and $S_{3\tau+2}$.

In case (*b*) let $\mathcal{T}$ be an affine transformation of E^2 such that $\mathcal{T}(Q_\tau) = I$. By Part I of the proof we conclude that $\mathcal{T}(Q_\tau)$ can be translatively covered by $\mathcal{T}(S_{3\tau+1}), \mathcal{T}(S_{3\tau+2}), \ldots$.

Consequently, I can be translatively covered by $S_1, S_2, \ldots$. ∎

References

[1] G. Fejes Tóth and W. Kuperberg, *Packing and Covering with Convex Sets*, in: Handbook of Convex Geometry, Vol. B, North-Holland, Amsterdam, 1993, 799–860.

[2] H. Groemer, *Covering and packing by sequences of convex bodies*, in: Discrete Geometry and Convexity, Ann. New York Acad. Sci., Vol. 440 (1985), 262–278.

[3] J. Januszewski, *Covering the unit square by squares*, Beitr. Algebra Geom. 43 (2) (2002), 411–422.

[4] J. W. Moon and L. Moser, *Some packing and covering theorems*, Colloq. Math. 17 (1967), 103–110.

[5] W. Moser and J. Pach, *Research problems in discrete geometry*, Privately published collection of problems, 1994.

INSTITUTE OF MATHEMATICS AND PHYSICS
UNIVERSITY OF TECHNOLOGY AND NATURAL SCIENCES
Ul. Kaliskiego 7
85-796 BYDGOSZCZ, POLAND
E-mail: januszew@utp.edu.pl

Received June 23, 2006.