DEMONSTRATIO MATHEMATICA
Vol. XL No 3 2007

G. S. Jeong, B. E. Rhoades

MORE MAPS FOR WHICH $F(T) = F(T^n)$

Abstract. We continue our investigation of situations in which the fixed point sets for maps and their iterates are the same.

In a previous paper [21] we demonstrated a number of situations in which the fixed point set of a map is the same as the fixed point set of each iterate of the map. We also investigated the same phenomenon for pairs of maps, with respect to common fixed points. In this paper we continue that study.

In Section 1 we examine some contractive conditions involving pairs of commuting maps. In section 2 we examine some contractive conditions involving multivalued maps for which the δ-metric is used. In section 3 we examine some contractive conditions on 2-metric spaces.

Let $F(T)$ denote the fixed point set of a map T. As in [21] we shall say that a map T has property P if $F(T) = F(T^n)$ for each $n \in \mathbb{N}$. We shall say that a pair of maps S and T have property Q if $F(S) \cap F(T) = F(S^n) \cap F(T^n)$ for each $n \in \mathbb{N}$. If T_i is a collections of maps, then, if $\cap F(T_i) = \cap F(T_i^n)$ we shall say that this collection has property R.

An important consequence of this study is that none of these maps has any nontrivial periodic points.

1. Commuting maps

THEOREM 1.1. *Let f and g be commuting selfmaps of a complete metric space (X, d), f continuous. Let $\Phi : \mathbb{R}_5^+ \to \mathbb{R}^+$, Φ nondecreasing in each variable and $\phi(t) := \Phi(t,t,t,t,t)$, ϕ satisfying* $\lim_n \phi^n(t) = 0$ *for each $t > 0$,*

Key words and phrases: commuting maps, D-metric spaces, fixed points, multivalued maps, 2-metric spaces.
1991 *Mathematics Subject Classification*: 47H10.

and $\lim_{t\to\infty}(t-\phi(t))=\infty$. *If* $gX \subset fX$ *and* f *and* g *satisfy*

$$d(gx,gy) \le \Phi(d(fx,fy),d(fx,gx),d(fy,gy)\\ d(fx,gy),d(fy,gx)\},$$

then f *and* g *have property* Q.

Proof. From Theorem 1 of [6] f and g have a unique common fixed point. The same conclusion, with Φ upper semicontinuous and satisfying $\Phi(t,t,t,t,t)<t$ for each $t>0$, and g continuous, appears in [24].

Let $u \in F(f^n)\cap F(g^n)$. Then, for any positive integers i,ℓ,r,t satisfying $1\le i,\ell,r,t\le n$,

$$\begin{aligned}(1.1)\qquad d(g^i f^r u, g^\ell f^t u) &\le \Phi(d(fg^{i-1}f^r u, fg^{\ell-1}f^t u),\\ &\qquad d(fg^{i-1}f^r u, g^i f^r u), d(fg^{\ell-1}f^t u, g^\ell f^t u),\\ &\qquad d(fg^{i-1}f^r u, g^\ell f^t u), d(fg^{\ell-1}f^t u, g^i f^r u))\\ &\le \Phi(t,t,t,t,t),\end{aligned}$$

where

$$\begin{aligned}t := \max\{&d(f^{r+1}g^{i-1}u, f^{t+1}g^{\ell-1}u),\\ &d(f^{r+1}g^{i-1}u, f^r g^i u), d(f^{t+1}g^{\ell-1}u, f^t g^\ell u),\\ &d(f^{r+1}g^{i-1}u, f^t g^\ell u), d(f^{t+1}g^{\ell-1}u, f^r g^i u)\}.\end{aligned}$$

Define

$$\delta = \max_{0\le i,\ell,r,t\le n} d(g^i f^r u, g^\ell f^t u).$$

Assuming that $\delta>0$, it then follows from (1.1) that

$$\delta \le \Phi(\delta,\delta,\delta,\delta,\delta) < \delta,$$

a contradiction. Therefore $\delta=0$, which implies that

$$d(gu,u)=d(fu,u)=0.$$

Hence $u=fu=gu$. ∎

Special cases of Theorem 1.1 are the contractive conditions appearing in Theorem 2.1 of [3], [27], [26], [36], [43], [22], [30], [8] ,[16], and Theorem 3.1 of [34].

THEOREM 1.2. *Let* T,I,J *be selfmaps of a complete metric space* (X,d) *with* $TI=IT, TJ=JT$ *and satisfying*

(1.2) $$d(Tx, Ty) \le c \max\{d(Ix, Jy), d(Ix, Tx), d(Jy, Ty), d(Ix, Ty), d(Jy, Tx)\}$$

for each $x, y \in X, 0 \le c < 1$. If, for each $x \in X$ there exists a $y \in X$ such that $Tx = Iy = Jy$ and if one of the maps T, I, J is continuous, then T, I, and J have have property R.

Proof. From Theorem 1 of [11] T, I, and J have a unique common fixed point. Let $u \in F(T^n) \cap F(I^n) \cap F(J^n)$. The proof is completed by using the technique of Theorem 1.1. ■

Special cases of Theorem 1.2 are Theorem 2 of [13], [40], [33], [29], Theorem 2 of [49], Theorem 2 of [1], Theorem 1 of [17], and Theorem 1 of [50].

PROPOSITION 1.1. *Let P, Q, S, T be selfmaps of a complete metric space (X, d) with $PS = SP, PT = TP, QS = SQ, QT = TQ, ST = TS$ satisfying*

(1.3) $$d(Px, Qy) \le \phi(d(Sx, Ty), d(Sx, Qy), d(Px, Ty), d(Px, Sx), d(Ty, Qy))$$

for all $x, y \in X$, where $\phi : \mathbb{R}_+^5 \to \mathbb{R}_+$ is upper semicontiuous and nondecreasing in each coordinate variable and such that $\phi(t, t, t, t, t) < t$ for each $t > 0$. If S and T are continuous and there exists a sequence $\{x_n\} \subset X$, defined by $PTx_{2n} = TSx_{2n+1}, QSx_{2n+1} = TSx_{2n+2}$ such that $\sup\{d(TSx_i, TSx_j)\} < \infty$, then P, Q, S, and T have property R.

Proof. From Theorem 1 of [45] P, Q, S, and T have a unique common fixed point. Let $u \in F(P^n) \cap F(Q^n) \cap F(S^n) \cap F(T^n)$. Use the technique of Theorem 1.1 with

$$\delta := \max_{0 \le i,j,k,\ell \le n} d(R^i S^j u, Q^k T^\ell u).$$ ■

PROPOSITION 1.2. *Let S, T, I, J be selfmaps of a complete metric space (X, d) satisfying $SI = IS, TJ = JT, SX \subset IX, TX \subset JX, I$ and J are continuous and satisfy*

(1.4) $$d(Sx, Ty) \le c \max\{d(Ix, Jy), d(Ix, Ty), d(Sx, Jy)\}$$

for all $x, y \in X$, where $0 < c < 1$. then S, T, I, and J have property R.

Proof. From Theorem 1 of [10] S, T, I, and J have a unique common fixed point. Let $u \in F(S^n) \cap F(T^n) \cap F(I^n) \cap F(J^n)$. Use the technique of

Theorem 1.1 with

$$\delta := \max_{0 \le i,j,k,\ell \le n} d(S^i I^j u, T^k J^\ell u). \qquad \blacksquare$$

2. Multivalued maps using the δ metric

Let A and B be arbitrary sets. Then the δ-metric for multivalued maps is defined by $\delta(A,B) = \sup_{a \in A, b \in B} d(a,b)$.

Let $B(X)$ denote the class of all nonempty bounded subsets of X.

PROPOSITION 2.1. *Let F be a selfmap of $B(X)$ satisfying*

$$\delta(Fx,Fy) \le c \max\{\delta(x,Fx), \delta(y,Fy), \delta(x,Fy), \delta(y,Fx), d(x,y)\} \tag{2.1}$$

for all $x,y \in X, 0 \le c < 1$. Then F has property P.

Proof. From [9] F has a unique fixed point z and $Fz = \{z\}$. Let $u \in F(F^n)$. Then use the standard technique with

$$\delta := \max_{0 \le i,j, \le n} \delta(F^i u, F^j i). \qquad \blacksquare$$

PROPOSITION 2.2. *Let $F, G : X \to BC(X)$ satisfying*

$$\delta(Fx, Gy) \le c \max\{\delta(x,Gy), \delta(y,Fx), d(x,y)\}$$

for all $x,y \in X, 0 \le c < 1$. Then F and G have property Q.

Proof. From Theorem 2 of [12] F and G have a unique common fixed point z and $Fz = Gz = \{z\}$. Let $u \in F(F^n) \cap F(G^n)$. Then use the standard technique with

$$\delta = \max_{0 \le i,j \le n} \delta(F^i u, G^j u). \qquad \blacksquare$$

3. Maps on 2-metric spaces

A 2-metric space is a space X in which, for each triple of points a,b,c there exists a real valued nonnegative function satisfying:

(1a) for each pair of points $a, b, a \ne b$ of X there exists a point $c \in X$ such that $d(a,b,c) \ne 0$,
(1b) $d(a,b,c) = 0$ whenever at least two of the points are equal,
(2) $d(a,b,c) = d(a,c,b) = d(b,c,a)$, and
(3) $d(a,b,c) \le d(a,b,d) + d(a,d,c) + d(d,b,c)$.

Further, it is assumed that d is continuous in each coordinate.

PROPOSITION 3.1. *Let X be a complete 2-metric space. f a selfmap of X satisfying: there exists a number $0 \le h < 1$ such that, for each $x, y, a \in X$,*

$$d(fx, fy, a) \le h \max\{d(x, y, a), d(x, fx, a), d(y, fy, a) \\ d(x, fy, a), d(y, fx, a)\}.$$

Then f has property P.

Proof. That f has a unique fixed point follows from Theorem 1 of [39].

Let $n \in \mathbb{N}$ and suppose that $u \in F(f^n)$. For any $a \in X$, any integers i, j satisfying $0 < i, j \ne n$,

$$(3.1) \quad d(f^i u, f^j u, a) \le h \max\{d(f^{i-1}u, f^{j-1}u, a), d(f^{i-1}u, f^i u, a) \\ d(f^{j-1}u, f^j u, a), d(f^{i-1}u, f^j u, a), d(f^{j-1}u, f^i u, a)\} \\ \le h\delta,$$

where

$$\delta := \max_{0 \le i,j < n} d(f^i u, f^j u, a).$$

Since inequality (3.1) is true for each i and j satisfying $0 \le i, j < n$, it follows that $\delta \le h\delta$, which implies that $\delta = 0$. In particular, $d(u, fu, a) = 0$ for each $a \in X$, which implies, from (1b), that $u = fu$. ∎

Theorem 1 of [25] has the same contractive condition as Proposition 3.1, so the map of that theorem also has property P.

Other contractive conditions on 2-metric spaces which satisfy property P are Theorem 1 of [41] and Theorem 1 of [42].

PROPOSITION 3.2. *Let S, T be selfmaps of a complete 2-metric space X such that there exist numbers $k, k' \ge 0$ with $0 < k + 2k' < 1$ satisfying*

$$(3.2) \quad d(Sx, Ty, a) \le k \max\{d(x, y, a), d(x, Sx, a), d(y, Ty, a), \\ + k'[d(x, Ty, a) + d(y, Sx, a)]$$

for each $x, y \in X$. Then S and T have property Q.

Proof. From Theorem 1 of [43] S and T have a unique common fixed point. Let $u \in F(S^n) \cap F(T^n)$. Use the technique of Proposition 3.1 with

$$\delta = \max_{0 \le i,j, \le n} d(S^i u, T^j u, a).$$ ∎

Other maps satisfying 2-metric contractive conditions for which property Q is true are Theorem 3.1 of [47], [20], and Theorem 3.1 of [23].

PROPOSITION 3.3. *Let (X,d) be a complete 2-metric space, S,T selfmaps of X such that $S(X)\subset T(X)$ and $ST=TS$. If T is continuous and there exists an $h, 0<h<1$ such that*

$$d(Sx,Sy,a)\le h\max\{d(Tx,Ty,a),d(Sx,Tx,a),d(Sy,Ty,a),\\ [d(Sx,Ty,a)+d(Sy,Tx,a)]/2\}$$

for each $x,y,a\in X$, then S and T have property Q.

Proof. From Theorem 1 of [19] S and T have a unique common fixed point.
Let $u\in F(S^n)\cap F(T^n), 0\le i,r,j,\ell\le n$. Then

$$\begin{aligned} d(S^iT^ju,S^rT^\ell u)\le h\max\{&d(S^{i-1}T^{j+1}u,S^{r-1}T^{\ell+1}u,a),\\ &d(S^iT^ju,S^{i-1}T^{j+1}u,a),d(S^rT^\ell u,S^{r-1}T^{\ell+1}u,a),\\ &[d(S^iT^ju,S^{r-1}T^{\ell+1}u,a)+d(S^rT^\ell u,S^{i-1}T^j,a]/2\}\\ \le h\delta,& \end{aligned}$$

where

$$\delta:=\max_{0\le i,r,j,\ell\le n} d(S^iT^ju,S^rT^\ell u,a),$$

since, if i or $\ell=n$, then $S^{i+1}=S$ and $T^{\ell+1}=T$.

We then have

$$d(S^iT^ju,S^rT^\ell u,a)\le h\delta$$

for each $0\le i,j,r,\ell\le n$, which implies that $\delta\le h\delta$, and $\delta=0$.

In particular, $d(u,Su,a)=0$ and $d(u,Tu,a)=0$ for each $a\in X$, and $u=Su=Tu$. ∎

PROPOSITION 3.4. *Let S and T be continuous selfmaps of a complete 2-metric space, $A:X\to SX\cap TX, A$ continuous, $AS=SA, AT=TA$, and satisfying*

$$\text{(3.3)}\qquad d(Ax,Ay,a)\le c\max\{d(Ax,Sx,a),d(Sx,Ty,a),d(Ay,Ty,a)\}$$

for all $x,y,a\in X, 0<c<1$. Then A,S, and T satisfy property R.

Proof. From Theorem 2 of [15] A,S, and T have a unique common fixed point. Let $u\in F(A^n)\cap F(S^n)\cap F(T^n)$. Use the technique of Proposition 3.3 with

$$\delta:=\max_{0\le i,j,k,\ell\le n} d(A^iS^ju,A^kT^\ell u,a).$$ ∎

PROPOSITION 3.5. *Let P,Q,T be selfmaps of a complete 2-metric space X,T continuous, $PT=TP, QT=TQ$, and $PX\cup QX\subset TX$. If there exists an*

h, $0 < h < 1$ such that, for all $x, y, a \in X$,

$$d(Px, Qy, a) \leq h \max\{d(Tx, Ty, a), d(Px, Tx, a), d(Qy, Ty, a), [d(Px, Ty, a) + d(Qy, Tx, a)]/2\}. \tag{3.4}$$

Then P, Q, and T have property R.

Proof. From Theorem 1 of [44] P, Q, and T have a unique common fixed point. Let $u \in F(P^n) \cap F(Q^n) \cap F(T^n)$. Use the technique of Proposition 3.3 with

$$\delta := \max_{0 \leq i,j,k,\ell \leq n} d(P^i T^j u, Q^k T^\ell u, a).$$ ∎

PROPOSITION 3.6. *Let P, Q, S, T be selfmaps of a complete 2-metric space with $PT = TP, PS = SP, QT = TQ, QS = SQ, TS = ST, S$ and T continuous, and satisfying*

$$d(Px, Qy, a) \leq \phi(d(Sx, Ty, a), d(Px, Sx, a), d(Qy, Ty, a), d(Px, Ty, a), d(Qy, Sx, a)) \tag{3.5}$$

for all $x, y, a \in X$, where $\phi : \mathbb{R}_+^5 \to \mathbb{R}_+, \phi$ upper semicontinuous and increasing in each coordinate, with $\phi(t, t, t, t, t) < t$ for each $t > 0$. If there exists a sequence $\{x_n\}$ such that $PTx_{2n} = STx_{2n+1}, QSx_{2n+1} = STx_{2n+2}$, and, for infinitely many n,

$$\sup\{d(STx_i, STx_j, a) : i, j \geq n, a \in X, i, j \text{ not of the same parity}\} = \sup\{d(STx_i, STx_j, a) : i, j \geq n, a \in X\},$$

then P, Q, S, and T have property R.

Proof. From Theorem 1 of [46] P, Q, S, and T have a unique common fixed point. Let $u \in F(P^n) \cap F(Q^n) \cap F(S^n) \cap F(T^n)$. Use the technique of Proposition 3.3 with

$$\delta := \max_{0 \leq i,j,k,\ell \leq n} d(P^i S^j u, Q^k T^\ell u, a).$$ ∎

Dhage [4] introduced the notion of a D-metric space and claimed that D-metric convergence defines a Hausdorff topology and that the D-metric is (sequentially) continuous in all three variables. In two recent dissertations [37] and [32], examples are provided to show that : (1) in a D-metric space D-metric convergence need not always define a topology; (2) even when the D-metric convergence defines a topology it need not be Hausdorff; and (3) even when the D-metric convergence defines a metrizable topology the D-metric need not be continuous even in a single variable.

Consequently, there is no point in considering Properties P, Q, or R for maps defined on a D-metric space.

References

[1] C. C. Chang, *On a fixed point theorem of contractive type*, Comment. Math. Univ. St. Pauli 32 (1983), 15–19.

[2] A. K. Chatterjee and M. R. Singh, *Common fixed point for four maps on a metric space*, Bull. Calcutta Math. Soc. 81 (1989), 466A–466B.

[3] K. M. Das and K. Naik, *Common fixed point theorems for commuting maps on a metric space*, Proc. Amer. Math.Soc. 77 (1979), 369–373.

[4] B. C. Dhage, *A study of some fixed point theorems*, Ph. D. Thesis, Marathwads University, Aurangabad, India (1984).

[5] B. C. Dhage, *Some results on common fixed points—I*, Indian J. Pure Appl. Math. 30 (1999), 827–837.

[6] X. P. Ding, *New results on common fixed points*, Math. Sem. Notes 10 (1982), 623–631.

[7] B. Fisher, *On a conjecture on common fixed points*, Math. Sem. Notes 6 (1978), 153–156.

[8] B. Fisher, *A common fixed point theorem for commuting maps*, Math. Sem. Notes 7 (1979), 297–300.

[9] B. Fisher, *Set-valued mappings on metric spaces*, Fund. Math. 112 (1981), 141–145.

[10] B. Fisher, *Four mappings with a common fixed point*, J. Univ. Kuwait 8 (1981), 131–140.

[11] B. Fisher, *Three mappings with a common fixed point*, Math. Sem. Notes 10 (1982), 293–302.

[12] B. Fisher, *Common fixed points for set-valued mappings*, Indian J Math. 25 (1983), 265–270.

[13] B. Fisher, *A common fixed point theorem for three mappings in a compact metric space*, Punjab Univ. J. Math. 16 (1983), 5–8.

[14] B. Fisher and K. Iseki, *A generalization of a common fixed point theorem*, Math. Japonica 35 (1990), 1013–1017.

[15] B. Fisher and M. S. Khan, *Some fixed point theorems for commuting mappings*, Math. Nachr. 106 (1982), 323–326.

[16] B. Fisher and S. Sessa, *A pair of commuting mappings with a common fixed point*, Rev. of Research, Faculty of Science, Univ. of Novi Sad 16 (1986), 89–99.

[17] A. Ganguly, *An extension of the fixed point theorem of Kasahara*, Math. Sem. Notes 10 (1982), 307–310.

[18] D. K. Ganguly and D. Bandyopadhyay, *Some results on common fixed point theorems in metric spaces*, Bull. Calcutta Math. Soc. 83 (1991), 137–145.

[19] V. K. Gupta, S. P. Singh, and B. M. L. Tiwari, *Common fixed points of commuting mappings in 2-metric spaces and an application*, Math. Nachr. 95 (1980), 293–297.

[20] K. Iseki, P. L. Sharma, and B. K. Sharma, *Contraction type mapping on 2-metric space*, Math. Japonica 21 (1976), 67–70.

[21] G. S. Jeong and B. E. Rhoades, *A comparison between the fixed point sets of T and T^n*, to appear in Fixed Point Theory and Applications, 6.

[22] G. Jungck, *Commuting maps and fixed points*, Amer. Math. Monthly 83 (1976), 261–263.

[23] S. M. Kang, S. S. Chang, and J. W. Yu, *Common fixed points of expansive mappings*, Math. Japonica 34 (1989), 373–379.

[24] S. Kasahara, *On some recent results on fixed points*, Math. Sem. Notes 6 (1978), 373–382.

[25] M. S. Khan, *On the convergence of sequences of fixed points in 2-metric spaces*, Indian J. Pure Appl. Math. 9 (1979), 1062–1067.
[26] M. S. Khan, *Remarks on some fixed-point theorems*, Comptes Rendue l'Acad. Bulgar. des Sci. 33 (1980), 1581–1583.
[27] M. S. Khan, *Remarks on some fixed point theorems*, Demonstratio Math. 15 (1982), 375–379.
[28] T. Kubiak, *Common fixed points of pairwise commuting maps*, Math. Nachr. 118 (1984), 123–127.
[29] Z.-Q. Liu, *A note on unique common fixed point*, Bull. Calcutta Math. Soc. 85 (1993), 469–472.
[30] M. Maiti and M. K. Ghosh, *A fixed point theorem for commuting maps*, Indian J. Pure Appl. Math. 15 (1984), 121–122.
[31] S. N. Mishra, *On fixed points of orbitally continuous maps*, Nanta Math. 12 (1979), 83–90.
[32] Z. Y. Mustafa, *A new approach to generalization of metric spaces*, Ph. D. Thesis, University of Newcastle, Australia (2004).
[33] R. K. Pande and P. K. Dubey, *On common fixed point theorems of three mappings*, Math. Student 61 (1992), 97–100.
[34] S. Park, *Fixed points and periodic points of contractive pairs of maps*, Proc. College of Nat. Sci., Seoul Nat. Univ. 5 (1980), 228–230.
[35] A. Rajput and S. Arya, *Fixed point theorems for ϕ-contractive mappings in D-metric spaces*, Jnanabha 31/32 (2002), 113–120.
[36] S. Ranganathan, *A fixed point theorem for commuting mappings*, Math. Sem. Notes 6 (1978), 351–357.
[37] N. Srinasa Rao, *A study of the topological properties of D-metric spaces and fixed point theorems in metric and D-metric spaces*, Ph. D. Thesis Acharya Nagarjuna University, (2004).
[38] B. K. Ray, *On common fixed points in metric spaces*, Indian J. Pure Appl. Math. 19 (1988), 960–962.
[39] B. E. Rhoades, *Contraction type mappings on a 2-metric space*, Math. Nachr. 91 (1979), 151–155.
[40] S. B. Sengupta and S. K. Dutta, *Common fixed point of operators*, Math. Student 56 (1988), 85–88.
[41] A. K.Sharma, *On fixed points in 2-metric spaces*, Math. Sem. Notes 6 (1978), 467–474.
[42] A. K. Sharma, *A note on fixed points in 2-metric spaces*, Indian J. Pure Appl. Math. 11 (1980), 1580–1583.
[43] S. L. Singh, *On common fixed points of commuting mappings*, Math. Sem. Notes 5 (1977), 131–134.
[44] S. L. Singh, *Fixed point theorems for commuting mappings*, Indian J. Math. 24 (1982), 25–26.
[45] S. L. Singh and S. Kasahara, *On some recent results on common fixed points*, Indian J. Pure Appl. Math. 13 (1982), 757–761.
[46] S. L. Singh and C. W. Norris, *Common fixed point theorems in 2-metric spaces*, Indian J. Math. 25 (1982), 165–170.
[47] T. Som and S. Das, *Some fixed point theorems on one-valued L-spaces with one-valued 2-metric*, Ganita 51 (2000), 127–134.
[48] R. Srivastava, *Common fixed point theorem for continuous mappings*, Ganita 42 (1991), 71–74.

[49] C.-C. Yeh, *On common fixed point of continuous mappings*, Math. Sem. Notes 6 (1978), 115–126.
[50] C.-C. Yeh, *On common fixed point theorems of continuous mappings*, Indian J. Pure Appl. Math. 10 (1979), 415–420.

G. S. Jeong
DEPARTMENT OF MATHEMATICS EDUCATION
SUNCHON NATIONAL UNIVERSITY
SUNCHON, 540 742, SOUTH KOREA
E-mail: gsjeong@sunchon.ac.kr

B. E. Rhoades
DEPARTMENT OF MATHEMATICS
INDIANA UNIVERSITY
BLOOMINGTON, IN 47405-7106, U.S.A.
E-mail: rhoades@indiana.edu

Received August 7, 2006; revised version April 10, 2007.