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FIXED POINT THEOREMS FOR SET AND SINGLE VALUED MAPS WITHOUT CONTINUITY AND COMPATIBILITY

Abstract. The new concept of weak commutativity of type (KB) is used to prove some fixed point theorems for set and single-valued mappings. We show that continuity of any mapping is not necessary for the existence of common fixed point. We also show that completeness of the whole space can be replaced by a weaker condition.

1. Introduction

Sessa [15] introduced the concept of weakly commuting maps. Jungck [4] defined the notion of compatible maps in order to generalize the concept of weak commutativity and showed that weakly commuting mappings are compatible but the converse is not true. Jungck and Rhoades [5], [6] defined δ-compatibility and weak compatibility between a set valued mapping and a single-valued mapping and generalized the weak commutativity defined in [3].

Fixed point theorems for set valued and single-valued mappings provide technique for soling variety of applied problems in mathematical sciences and engineering. (e.g. Krzyśka and Kubiaczyk [8], Sessa and Khan [16]).

Number of these theorems are very useful but their hypothesis are very difficult to satisfy as they require continuity and compatibility of involved mappings. There are so many functions which are not continuous but have a fixed point. For example the function f defined on R by

$$f(x) = 0,\ x \leq 0, \qquad f(x) = 1,\ x > 0.$$

This function f is not continuous at 0 but has 0 as a fixed point.

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Another example is the Dirichlet function defined on R by

$$f(x) = 1 \quad \text{if } x \text{ is rational,}$$

$$f(x) = 0 \quad \text{if } x \text{ is irrational.}$$

The Dirichlet function is not continuous at any point but has 1 as a fixed point.

These observations motivated several authors of the field to prove fixed point theorems for noncompatible, discontinuous mappings.

Pant [9]–[12] initiated the study of noncompatible maps and introduced pointwise R-weak commutativity of mappings in [9]. He also showed that pointwise R-weak commutativity is a necessary, hence minimal condition for the existence of a common fixed point of contractive type maps [10].

Pathak, Cho and Kang [13] introduced the concept of R- weakly commuting mappings of type A and showed that they are not compatible.

Recently, I. Kubiaczyk and Bhavana Deshpande [7] extended the concept of R- weakly commutativity of type A for single-valued mappings to set valued mappings and introduced weak commutativity of type (KB).

In this paper, we prove some common fixed point theorems by using concept of weak commutativity of type (KB). We show that continuity of any mapping is not necessary for the existence of common fixed point. We also show that completeness of the whole space can be replaced by a weaker condition. We improve and generalize the results of Tas, Telki and Fisher [17], Fisher [2] and Rashwan and Ahmed [14].

2. Preliminaries

In the sequel (X, d) denotes a metric space and B(X) is the set of all non empty bounded subsets of X. As in [1], [3] we define

$$\delta(A, B) = \sup\{d(a, b) : a \in A,\ b \in B\},$$
$$D(A, B) = \inf\{d(a, b) : a \in A,\ b \in B\},$$
$$H(A, B) = \inf\{r > 0 : A_r \supset B,\ B_r \supset A\},$$

for all A, B in $B(X)$, where

$$A_r = \{x \in X : d(x, a) < r \text{ for some } a \in A\},$$
$$B_r = \{y \in X : d(y, b) < r \text{ for some } b \in B\}.$$

If $A = \{a\}$ for some $a \in$ A, we denote $\delta(a, B)$, $D(a, B)$ and $H(a, B)$ for $\delta(A, B)$, $D(A, B)$ and $H(A, B)$, respectively. Also, if $B = \{b\}$ and $A = \{a\}$, one can deduce that $\delta(A, B) = D(A, B) = H(A, B) = d(a, b)$.

It follows immediately from the definition of δ(A, B) that

$$\delta(A,B)=\delta(B,A)\geq 0,\ \delta(A,B)\leq\delta(A,C)+\delta(C,B)$$
$$\delta(A,B)=0\quad\text{iff } A=B=\{a\},\ \delta(A,A)=\operatorname{diam}A,$$

for all $A,B,C\in B(X)$.

DEFINITION 2.1 ([3]). A sequence $\{A_n\}$ of nonempty subsets of X is said to be convergent to a subset A of X if

(i) Each point a in A is the limit of a sequence $\{a_n\}$, where a_n is in A_n for all n $\in$ N.

(ii) For arbitrary $\epsilon>0$, there exists an integer m such that $A_n\subseteq A_\epsilon$ for $n>m$, where $A_\epsilon=\{x\in X:\exists a\in A,\ a$ depending on x and $d(x,a)<\epsilon\}$. A is said to be the limit of the sequence $\{A_n\}$.

LEMMA 2.1 ([3]). *If $\{A_n\}$ and $\{B_n\}$ are sequences in $B(X)$ converging to A and B in $B(X)$, respectively, then the sequence $\{\delta(A_n,B_n)\}$ converges to $\delta(A,B)$.*

LEMMA 2.2 ([3]). *Let $\{A_n\}$ be a sequence in $B(X)$ and y be a point in X such that $\delta(A_n,y)\to 0$. Then the sequence $\{A_n\}$ converges to the set $\{y\}$ in $B(X)$.*

DEFINITION 2.2 ([3]). The mappings $F:X\to B(X)$ and $f:X\to X$ are said to be weakly commuting if $fFx\in B(X)$ and

$$\delta(Ffx,fFx)\leq\max\{\delta(fx,Fx),\operatorname{diam}fFx\}\qquad\text{for all } x\in X.$$

Note that if F is single-valued mapping then the set $\{fFx\}$ consists of a single point. Therefore, $\operatorname{diam}fFx=0$ for all $x\in X$ and above inequality reduces to the well known condition given by Sessa [15]; that is

$$d(Ffx,fFx)\leq d(fx,Fx)\qquad\text{for all } x \text{ in } X.$$

Two commuting mappings F and f are weakly commuting but the converse is not true as shown in [3].

DEFINITION 2.3 ([5]). The mappings $F:X\to B(X)$ and $f:X\to X$ are δ-compatible if $\lim_{n\to\infty}\delta(Ffx_n,fFx_n)=0$ whenever $\{x_n\}$ is a sequence in X such that $fFx_n\in B(X)$, $Fx_n\to\{t\}$, $fx_n\to t$ for some t in X.

DEFINITION 2.4 ([9]). The mappings $f,g:X\to X$ will be called R-weakly commuting, provided there exists some positive real number R such that

$$d(fgx,gfx)\leq Rd(fx,gx)$$

for each x in X. f and g will be called R-weakly commuting at a point x if $d(fgx,gfx)\leq Rd(fx,gx)$ for some $R>0$.

Definition 2.5 ([13]). The mappings $f, g : X \to X$ are said to be R-weakly commuting of type (A_f) if there exists a positive real number R such that

$$d(fgx, ggx) \leq Rd(fx, gx) \quad \text{for all x} \in X.$$

Definition 2.6 ([13]). The mappings $f, g : X \to X$ are said to be R-weakly commuting of type (A_g) if there exists a positive real number R such that

$$d(gfx, ffx) \leq Rd(fx, gx) \quad \text{for all } x \in X.$$

Remark 2.1 ([13]). (i) Compatible mappings are R-weakly commuting mappings of type (A_f) or type (A_g) but converse is not true.

(ii) R-weakly commuting mappings are not necessarily R-weakly commuting of type (A_f) or R-weakly commuting of type (A_g).

Definition 2.7 ([7]). The mappings $f : X \to X$ and $F : X \to B(X)$ are said to be weakly commuting of type (KB) at x if there exists some positive real number R such that

$$\delta(ffx, Ffx) \leq R\delta(fx, Fx).$$

Here f and F are weakly commuting of type (KB) on X if above inequality holds for all $x \in X$. If f is single-valued self mapping of X the definition of weak commutativity of type (KB) reduces to Definition 2.6.

Example 2.1. Let $X = [1, 15]$ and d be the usual metric on X. Define $f : X \to X$ and $F : X \to B(X)$ by

$$fx = \begin{cases} x & \text{if } 1 \leq x \leq 10, \\ \frac{x+2}{4} & \text{if } 10 < x \leq 15. \end{cases}$$

$$Fx = \begin{cases} [1, x] & \text{if } 1 \leq x \leq 3, \\ [3, x] & \text{if } 3 < x \leq 10, \\ [3, \frac{x-1}{3}] & \text{if } 10 < x \leq 15. \end{cases}$$

Let $x_n = 10 + \frac{1}{n}$, $n = 1, 2, \ldots$. Then

$$\lim_{n\to\infty} fx_n = 3 \quad \text{and} \quad \lim_{n\to\infty} Fx_n = \{3\}.$$

Also $fFx_n \in B(X)$ and $\delta(Ffx_n, fFx_n) = \delta([3, 3 + \frac{1}{4n}], [3, 3 + \frac{1}{3n}]) \to 0$ as $n \to \infty$. Therefore the pair $\{F, f\}$ is δ-compatible.

On the other hand if we take $x = 2$ then $ffx = 2$, $Ffx = [1, 2]$ and clearly f and F are weakly commuting of type (KB) at $x = 2$.

Example 2.2. Let $X = [1, \infty)$ and d be the usual metric on X. Define $f : X \to X$ and $F : X \to B(X)$ by $f(x) = 2x$ and $Fx = [1, 2x - 1]$ for all $x \in X$. Then $ffx = 4x$, $Ffx = [1, 4x - 1]$ and for $R > 3$ we can see that $\delta(ffx, Ffx) < R\delta(fx, Fx)$ for all $x \in X$. Thus f and F are weakly commuting of type (KB) on X but there exists no sequence $\{x_n\}$ in X such that condition of compatibility is satisfied.

3. Main results

THEOREM 3.1. *Let $f, g : X \to X$ and $F, G : X \to B(X)$ be mappings such that*

(1) $F(X) \subseteq g(X),\ G(X) \subseteq f(X)$,

(2) $$\delta^2(Fx, Gy) \le c_1 \max\{d^2(fx, gy), \delta^2(fx, Fx), \delta^2(gy, Gy)\} + c_2 \max\{\delta(fx, Fx).D(fx, Gy), D(gy, Fx).\delta(gy, Gy)\} + c_3 D(fx, Gy).D(gy, Fx)$$

for all $x, y \in X$ where $c_1 + 2c_2 < 1$, $c_2 + c_3 < 1$, c_1, c_2, $c_3 \ge 0$,

(3) one of $f(X)$ or $g(X)$ is complete,

(4) the pairs $\{F, f\}$ and $\{G, g\}$ are weakly commuting of type (KB) at coincidence points in X.

Then there exists a unique fixed point z in X such that $\{z\} = \{fz\} = \{gz\} = Fz = Gz$.

Proof. Let $x_0 \in X$ be an arbitrary point in X. By (1) we choose a point x_1 in X such that $gx_1 \in Fx_0 = Z_1$ and for this point x_1 there exists a point x_2 in X such that $fx_2 \in Gx_1 = Z_2$ and so on continuing in this manner we can define a sequence $\{x_n\}$ as follows:

$$gx_{2n+1} \in Fx_{2n} = Z_{2n}, \quad fx_{2n+2} \in Gx_{2n+1} = Z_{2n+1} \quad \text{for } n = 0, 1, 2, \ldots$$

Let $V_n = \delta(Z_n, Z_{n+1})$ for $n = 0, 1, 2, \ldots$ By (2) we have

$$\begin{aligned} V_{2n}^2 &= \delta^2(Z_{2n}, Z_{2n+1}) = \delta^2(Fx_{2n}, Gx_{2n+1}) \\ &\le c_1 \max\{d^2(fx_{2n}, gx_{2n+1}), \delta^2(fx_{2n}, Fx_{2n}), \delta^2(gx_{2n+1}, Gx_{2n+1})\} \\ &\quad + c_2 \max\{\delta(fx_{2n}, Fx_{2n}).D(fx_{2n}, Gx_{2n+1}), \\ &\quad D(gx_{2n+1}, Fx_{2n}).\delta(gx_{2n+1}, Gx_{2n+1})\} + c_3 D(fx_{2n}, Gx_{2n+1}).D(gx_{2n+1}, Fx_{2n}) \\ &\le c_1 \max\{V_{2n-1}^2, V_{2n}^2\} + c_2 V_{2n-1}(V_{2n-1} + V_{2n}). \end{aligned}$$

If $V_{2n} > V_{2n-1}$ then we have

$$V_{2n}^2 \le (c_1 + 2c_2) V_{2n}^2 < V_{2n}^2,$$

since $c_1 + 2c_2 < 1$, which is a contradiction. Thus $V_{2n} < hV_{2n-1}$, where $h = \sqrt{c_1 + 2c_2} < 1$. Similarly we have $V_{2n+1} < hV_{2n}$ and so

$$V_{2n} = \delta(Z_{2n}, Z_{2n+1}) = \delta(Fx_{2n}, Gx_{2n+1}) \le \cdots \le h^{2n}\delta(Fx_0, Gx_1)$$

for $n = 1, 2, \ldots$ Let z_n be an arbitrary point in Z_n for $n = 0, 1, 2, \ldots$. Thus we have

$$d(z_n, z_{n+1}) \le \delta(Z_n, Z_{n+1}) \le \cdots \le h^n \delta(Fx_0, Gx_1).$$

Since $h < 1$, therefore the sequence $\{z_n\}$ is a Cauchy sequence in X and hence any subsequence thereof is a Cauchy sequence in X. Suppose that $g(X)$ is complete. Since

$$gx_{2n+1} \in Fx_{2n} = Z_{2n} \quad \text{for } n = 0, 1, 2, \ldots$$

then

$$d(gx_{2m+1}, gx_{2n+1}) \leq \delta(Z_{2m}, Z_{2n}) < \epsilon$$

for $m, n \geq n_0$, $n_0 = 1, 2, 3, \ldots$ Therefore $\{gx_{2n+1}\}$ is Cauchy and hence $gx_{2n+1} \to z = gv \in g(X)$ for $v \in X$. But $fx_{2n} \in Gx_{2n-1} = Z_{2n-1}$ so we have

$$d(fx_{2n}, gx_{2n+1}) \leq \delta(Z_{2n-1}, Z_{2n}) = V_{2n-1} \to 0.$$

Consequently, $fx_{2n} \to z$. Moreover we have for $n = 1, 2, 3, \ldots$

$$\delta(Fx_{2n}, z) \leq \delta(Fx_{2n}, fx_{2n}) + \delta(fx_{2n}, z),$$

Therefore $\delta(Fx_{2n}, z) \to 0$. Similarly $\delta(Gx_{2n-1}, z) \to 0$.

By (2) for $n = 1, 2, 3, \ldots$ we have

$$\begin{aligned}
&\delta^2(Fx_{2n}, Gv) \\
&\quad \leq c_1 \max\{d^2(fx_{2n}, gv), \delta^2(fx_{2n}, Fx_{2n}), \delta^2(gv, Gv)\} \\
&\qquad + c_2 \max\{\delta(fx_{2n}, Fx_{2n}).D(fx_{2n}, Gv), D(gv, Fx_{2n}).\delta(gv, Gv)\} \\
&\qquad + c_3 D(fx_{2n}, Gv).D(gv, Fx_{2n}) \\
&\quad \leq c_1 \max\{d^2(fx_{2n}, gv), \delta^2(fx_{2n}, Fx_{2n}), \delta^2(gv, Gv)\} \\
&\qquad + c_2 \max\{\delta(fx_{2n}, Fx_{2n}).\delta(fx_{2n}, Gv), \delta(gv, Fx_{2n}).\delta(gv, Gv)\} \\
&\qquad + c_3 \delta(fx_{2n}, Gv).\delta(gv, Fx_{2n}),
\end{aligned}$$

and since $\delta(fx_{2n}, Gv) \to \delta(z, Gv)$ when $fx_{2n} \to z$ we get as $n \to \infty$

$$\delta^2(z, Gv) \leq c_1 \delta^2(z, Gv),$$

since $c_1 < 1$, we see that $Gv = \{z\} = \{gv\}$.

But $G(X) \subset f(X)$, there exists $u \in X$ such that $\{fu\} = Gv = \{gv\} = \{z\}$. Now if $Fu \neq Gv$, $\delta(Fu, Gv) \neq 0$ so by (2), we have

$$\begin{aligned}
\delta^2(Fu, Gv) &\leq c_1 \max\{d^2(fu, gv), \delta^2(fu, Fu), \delta^2(gv, Gv)\} \\
&\quad + c_2 \max\{\delta(fu, Fu).D(fu, Gv), D(gv, Fu).\delta(gv, Gv)\} \\
&\quad + c_3 D(fu, Gv).D(gv, Fu) \\
&\leq c_1 \max\{d^2(fu, gv), \delta^2(fu, Fu), \delta^2(gv, Gv)\} \\
&\quad + c_2 \max\{\delta(fu, Fu).\delta(fu, Gv), \delta(gv, Fu).\delta(gv, Gv)\} \\
&\quad + c_3 \delta(fu, Gv).\delta(gv, Fu).
\end{aligned}$$

So we have $\delta^2(Fu, Gv) \le c_1\delta^2(Fu, Gv)$ and since $c_1 < 1$, we can see that

$$Fu = \{fu\} = \{gv\} = Gv = \{z\}.$$

Since $Fu = \{fu\}$ and the pair $\{F, f\}$ is weakly commuting of type (KB) at coincidence points in X we obtain $\delta(ffu, Ffu) \le R\delta(fu, Fu)$, which gives $\{fz\} = Fz$.

Again since $Gv = \{gv\}$ and the pair $\{G, g\}$ is weakly commuting of type (KB) at coincidence points in X we obtain $\delta(ggv, Ggv) \le R\delta(gv, Gv)$, which gives $\{gz\} = Gz$. By (2), we have

$$\begin{aligned}
\delta^2(Fz, z) &\le \delta^2(Fz, Gv) \\
&\le c_1 \max\{d^2(fz, gv), \delta^2(fz, Fz), \delta^2(gv, Gv)\} \\
&\quad + c_2 \max\{\delta(fz, Fz).D(fz, Gv), D(gv, Fz).\delta(gv, Gv)\} \\
&\quad + c_3 D(fz, Gv).D(gv, Fz) \\
&\le c_1 \max\{d^2(fz, gv), \delta^2(fz, Fz), \delta^2(gv, Gv)\} \\
&\quad + c_2 \max\{\delta(fz, Fz).\delta(fz, Gv), \delta(gv, Fz).\delta(gv, Gv)\} \\
&\quad + c_3\delta(fz, Gv).\delta(gv, Fz) \\
&\le (c_1 + c_3)\delta^2(Fz, z).
\end{aligned}$$

Since $c_1 + c_3 < 1$, it follows that $Fz = \{z\}$. Consequently, we have $\{z\} = Fz = \{fz\}$. Similarly $\{z\} = Gz = \{gz\}$. Therefore we have $\{z\} = \{fz\} = \{gz\} = Fz = Gz$.

Finally, we prove that z is unique. If not let w be another common fixed point such that $z \ne w$ and $\{w\} = \{fw\} = \{gw\} = Fw = Gw$. By (2), we have

$$\begin{aligned}
d^2(z, w) &\le \delta^2(Fz, Gw) \\
&\le c_1 \max\{d^2(fz, gw), \delta^2(fz, Fz), \delta^2(gw, Gw)\} \\
&\quad + c_2 \max\{\delta(fz, Fz).D(fz, Gw), D(gw, Fz).\delta(gw, Gw)\} \\
&\quad + c_3 D(fz, Gw).D(gw, Fz) \\
&\le (c_1 + c_3)d^2(z, w).
\end{aligned}$$

Since $c_1 + c_2 < 1$. Then $z = w$. This completes the proof.

REMARK 3.1. Theorem 3.1, improves and generalizes the result of Rashwan and Ahmad [14] in the sense that δ- compatibility is relaxed by weak commutativity of type (KB), continuity of any mapping is not required and completeness of the whole space X is replaced by completeness of $f(X)$ or $g(X)$.

If F and G are single-valued mappings in Theorem 3.1, then we get the following:

COROLLARY 3.2. *Let $f, g, F, G : X \to X$ be mappings satisfying the condition* (1), (3), (4) *and*

(5) $$d^2(Fx, Gy) \le c_1 \max\{d^2(fx, gy), d^2(fx, Fx), d^2(gy, Gy)\} + c_2 \max\{d(fx, Fx).d(fx, Gy), d(gy, Fx).d(gy, Gy)\} + c_3 d(fx, Gy).d(gy, Fx)$$

for all $x, y \in X$ where $c_1 + 2c_2 < 1$, $c_2 + c_3 < 1$, $c_1, c_2, c_3 \ge 0$.

Then f, g, F and G have a unique common fixed point in X.

REMARK 3.2. Corollary 3.2, improves and generalizes the result of Tas, Telki and Fisher [17].

If we put $c_1 = c_2 = 0$ in Theorem 3.1, we obtain the following:

COROLLARY 3.3. *Let $f, g : X \to X$ and $F, G : X \to B(X)$ be mappings satisfying the conditions* (1), (3), (4) *and the following:*

(6) $$\delta^2(Fx, Gy) \le c_1 \max\{d^2(fx, gy), \delta^2(fx, Fx), \delta^2(gy, Gy)\}$$

for all $x, y \in X$ where $c_1 \ge 0$.

Then there exists a unique fixed point z in X such that $\{z\} = \{fz\} = \{gz\} = Fz = Gz$.

REMARK 3.3. Corollary 3.3 improves and generalizes the result of Fisher [2].

If we put $F = G$ and $f = g$ in Theorem 3.1 then we get the following:

COROLLARY 3.4. *Let $f : X \to X$ and $F : X \to B(X)$ be mappings such that*

(7) $F(X) \subseteq f(X)$,

(8) $$\delta^2(Fx, Fy) \le c_1 \max\{d^2(fx, fy), \delta^2(fx, Fx), \delta^2(fy, Fy)\} + c_2 \max\{\delta(fx, Fx).D(fx, Fy), D(fy, Fx).\delta(fy, Fy)\} + c_3 D(fx, Fy).D(fy, Fx)$$

for all $x, y \in X$ where $c_1 + 2c_2 < 1$, $c_2 + c_3 < 1$, $c_1, c_2, c_3 \ge 0$,

(9) *$f(X)$ is complete,*

(10) *the pair $\{F, f\}$ is weakly commuting of type (KB) at coincidence points in X.*

Then there exists a unique fixed point z in X such that $\{z\} = \{fz\} = Fz$.

For a set valued map $F : X \to B(X)$ (respectively a single-valued map $f : X \to X$), F^* (respectively f^*) will denote the set of fixed points of F (respectively f).

THEOREM 3.5. *Let $f, g : X \to X$ and $F, G : X \to B(X)$ be mappings. If condition* (2) *holds for all $x, y \in X$ then*

$$(f^* \cap g^*) \cap F^* = (f^* \cap g^*) \cap G^*.$$

Proof. Let $u \in (f^* \cap g^*) \cap F^*$ so

$$\begin{aligned}\delta^2(u, Gu) &= \delta^2(Fu, Gu)\\ &\leq c_1 \max\{d^2(fu, gu), \delta^2(fu, Fu), \delta^2(gu, Gu)\}\\ &\quad + c_2 \max\{\delta(fu, Fu).D(fu, Gu), D(gu, Fu).\delta(gu, Gu)\}\\ &\quad + c_3 D(fu, Gu).D(gu, Fu)\\ &= c_1 \delta^2(u, Gu).\end{aligned}$$

Since $c_1 < 1$, it follows that $\{u\} = Gu$. Thus

$$(f^* \cap g^*) \cap F^* \subseteq (f^* \cap g^*) \cap G^*.$$

Similarly one can show that

$$(f^* \cap g^*) \cap F^* \supseteq (f^* \cap g^*) \cap G^*.$$

Theorem 3.1 and Theorem 3.5 imply the following:

THEOREM 3.6. *Let $f, g : X \to X$ and $F_n : X \to B(X)$, $n \in N$ be mappings satisfying condition* (3) *and the following:*

(11) $$F_1(X) \subseteq g(X) and F_2(X) \subseteq f(X),$$

(12) $$\begin{aligned}&\delta^2(F_n x, F_{n+1} y)\\ &\leq c_1 \max\{d^2(fx, gy), \delta^2(fx, F_n x), \delta^2(gy, F_{n+1} y)\}\\ &\quad + c_2 \max\{\delta(fx, F_n x).D(fx, F_{n+1} y), D(gy, F_n x).\delta(gy, F_{n+1} y)\}\\ &\quad + c_3 D(fx, F_{n+1} y).D(gy, F_n x)\end{aligned}$$

for all $x, y \in X$ where $c_1 + 2c_2 < 1$, $c_2 + c_3 < 1$, $c_1, c_2, c_3 \geq 0$, $n \in N$.

(13) *the pairs $\{F_1, f\}$, $\{F_2, g\}$ are weakly commuting of type (KB) at coincidence points in X.*

Then f, g and $\{F_i\}_{i \in N}$ have a unique common fixed point in X.

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