DEMONSTRATIO MATHEMATICA
Vol. XL No 3 2007

Gabriela Petruşel

GENERALIZED MULTIVALUED CONTRACTIONS WHICH ARE QUASI-BOUNDED

Abstract. The purpose of this note is to study the following problem: which multivalued generalized contractions are quasi-bounded with the quasi-norm strictly less than 1 ? As consequences, some surjectivity results are given.

1. Preliminaries

If X is a normed space, then an operator $f : X \to X$ is said to be quasi-bounded if there are $a, b > 0$ such that $\|f(x)\| \leq a \cdot \|x\| + b$, for all $x \in X$. Also, by definition the quasi-norm of f is

$$|f| := \inf\{a \in \mathbb{R} : \text{there is } b>0 \text{ such that } \|f(x)\| \leq a\|x\| + b, \text{ for all } x \in X\}.$$

A quasi-bounded operator having the quasi-norm strictly less than 1 is called a norm-contraction.

In Aldea F. [1] and Anisiu M.C. [2] are considered several classes of singlevalued generalized contractions which are quasi-bounded with the quasi-norm strictly less than 1. It is proved that Ciric, Bianchini and Hardy-Rogers type operators, as well as, generalized φ-contractions are quasi-bounded, and in certain conditions, the quasi-norm is strictly less than 1.

The purpose of this paper is to establish similar results for the case of multivalued operators.

Throughout this paper, the standard notations and terminologies in nonlinear analysis are used. For the convenience of the reader we recall some of them.

Let (X, d) be a metric space. If Y is a subset of X then $diam(Y) := \sup\{d(a,b) | \ a, b \in Y\}$ denote the diameter of the set Y.

Key words and phrases: multivalued operator, fixed point, quasi-bounded operator, surjectivity theorem.
2000 *Mathematics Subject Classification*: 47H10, 54H25.

Also we will use the following symbols:
$\mathcal{P}(X) = \{Y|\ Y \text{ is a subset of } X\}$,
$P(X) = \{Y \in \mathcal{P}(X)|\ Y \text{ is nonempty}\}$,
$P_b(X) := \{Y \in P(X)|\ Y \text{ is bounded }\}$,
$P_{cl}(X) := \{Y \in P(X)|\ Y \text{ is closed}\}$,
$P_{b,cl}(X) := \{Y \in P(X)|\ Y \text{ is bounded and closed}\}$,
$P_{ac}(X) := \{Y \in P(X)|\ Y \text{ is acyclic }\}$.
If X is a normed space then denote $P_{cv}(X) := \{Y \in P(X)|\ Y \text{ is convex}\}$.

If $F : X \to P(X)$ is a multivalued operator then for $Y \in P(X)$, $F(Y) := \bigcup_{x\in Y} F(x)$ will denote the image of the set Y, $I(F) := \{Y \in P(X)|F(Y) \subset Y\}$ is the set of all invariant subsets of F, while the graph of the multivalued operator F is denoted by $Graf(F) := \{(x,y) \in X \times X \mid y \in F(x)\}$. Throughout the paper $FixF := \{x \in X|\ x \in F(x)\}$ denotes the fixed point set of the multivalued operator F.

The following (generalized) functionals are used in the main section of the paper.

The gap functional

(1) $$D : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\},$$

$$D(A,B) = \begin{cases} \inf\{d(a,b)|\ a \in A,\ b \in B\}, & A \neq \emptyset \neq B \\ 0, & A = \emptyset = B \\ +\infty, & \text{otherwise.} \end{cases}$$

The excess generalized functional

(3) $$\rho : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\},$$

$$\rho(A,B) = \begin{cases} \sup\{D(a,B)|\ a \in A\}, & A \neq \emptyset \neq B \\ 0, & A = \emptyset \\ +\infty, & B = \emptyset \neq A. \end{cases}$$

Pompeiu-Hausdorff generalized functional

(4) $$H : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\},$$

$$H(A,B) = \begin{cases} \max\{\rho(A,B), \rho(B,A)\}, & A \neq \emptyset \neq B \\ 0, & A = \emptyset = B \\ +\infty, & \text{othewise.} \end{cases}$$

It is well-known that $(P_{b,cl}(X), H)$ is a complete metric space provided (X,d) is a complete metric space.

If X, Y are metric spaces and $T : X \to P(Y)$, then the multivalued operator T is said to be upper semi-continuous on X (briefly u. s. c.) if for any open set $U \subset X$ the set $T^+(U) := \{x \in X|T(x) \subset U\}$ is open in X.

For more details and basic results concerning the above notions see for example [3], [4], [9], etc.

Let X be a Banach space and $T : X \to P_b(X)$. By definition, (see [13], [10]) the operator T is called quasi-bounded if there exist $m, M \in \mathbb{R}_+^*$ such that

$$(1.1) \qquad \|y\| \leq m \cdot \|x\| + M, \text{ for each } (x, y) \in Graf(T).$$

The number

$$|T| := \inf\{m > 0| \text{ there exists } M > 0 \text{ such that the relation (1.1) holds}\},$$

is called the quasi-norm of T. If $|T| < 1$ then T is said to be a multivalued norm-contraction. Also, $\|T(x)\| := H(T(x), \{0\}), x \in X$.

Let X be a Frechet space, i. e. a locally convex space which is metrizable and complete. A mapping $\alpha : P_b(X) \to \mathbb{R}_+$ is called an abstract measure of noncompactness on X if the following conditions hold:

1) (Regularity) $\alpha(A) = 0$ implies $\overline{A}$ is compact.

2) (Convex hull property) $\alpha(\overline{conv}A) = \alpha(A)$, for each $A \in P_b(X)$.

3) (Non-singularity) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$, for each $A, B \in P_b(X)$.

4) (Cantor type property) If $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence of closed subset of X with $\lim_{n\to+\infty} \alpha(A_n) = 0$, then $\bigcap_{n=1}^{+\infty} A_n$ is nonempty and compact.

As consequence, we also have that $\alpha(A) \leq \alpha(B)$ if $A \subset B$. Kuratowski and Hausdorff measures of noncompactness are examples of abstract measures of noncompactness.

In this setting, a multivalued operator $T : X \to P(X)$ is said to be densifying with respect to α, if $\alpha(T(A)) < \alpha(A)$, for each $A \in P_b(X) \cap I(T)$, with $\alpha(A) > 0$. It is known that compact multivalued operators are densifying with respect to any measure of noncompactness.

By homology we mean throughout the paper, Čech homology with rational coefficients and we call a compact metric space X acyclic if it has the same homology as a one point space. In particular, any convex or star-shaped subset of a normed space is acyclic.

The paper is organized as follows. In Section 2, we will present some classes of multivalued generalized contractions which are quasi-bounded. In Section 3 some surjectivity theorems are proved. The theorems of the present paper are generalizations of some results of Iannacci [10], Martelli [12] and Martelli-Vignoli [13] and they are extensions to the multivalued case of some theorems in Aldea [1], Anisiu [2], Rus [18] and Rus-Petruşel A.-Petruşel G. [21].

2. Generalized multivalued contractions which are norm-contractions

The aim of this section is to give some examples of multivalued norm-contractions.

Let $(X, \|\cdot\|)$ be a normed space. Then $T : X \to P(X)$ is said to be a Reich type multivalued operator if there exist $a, b, c \in \mathbb{R}_+$ with $a+b+c<1$ such that

$$H(T(x),T(y)) \le a\|x-y\| + bD(x,T(x)) + cD(y,T(y)), \text{ for all } x,y \in X.$$

THEOREM 2.1. *Let $(X, \|\cdot\|)$ be a normed space and $T : X \to P_b(X)$ be a Reich type multivalued operator. Then T is a multivalued norm-contraction.*

Proof. Let us take $y = 0$ in Reich type condition. It follows that

$$\begin{aligned} H(T(x),T(0)) &\le a\|x\| + bD(x,T(x)) + cD(0,T(0)) \\ &\le a\|x\| + bD(x,T(x)) + c\|T(0)\|, \quad \text{for each } x \in X. \end{aligned}$$

Then

$$\begin{aligned} \|T(x)\| &= H(T(x),\{0\}) \le H(T(x),T(0)) + H(T(0),\{0\}) \\ &\le a\|x\| + bD(x,T(x)) + c\|T(0)\| + \|T(0)\|, \quad \text{for each } x \in X. \end{aligned}$$

Also

$$\begin{aligned} D(x,T(x)) &\le \|x-y\| + D(y,T(x)) \le \|x-y\| + \|y\| + \sup_{z\in T(x)} \|z\| \\ &= \|x-y\| + \|y\| + \|T(x)\|, \quad \text{for each } x,y \in X. \end{aligned}$$

For $y := 0$ we obtain $D(x,T(x)) \le \|x\| + \|T(x)\|$, for each $x \in X$. Then

$$\|T(x)\| \le a\|x\| + b[\|x\| + \|T(x)\|] + (c+1)\|T(0)\|,$$

for $x \in X$. So we immediately get

$$(2.1) \qquad \|T(x)\| \le \frac{a+b}{1-b}\|x\| + \frac{c+1}{1-b}\|T(0)\|, \text{ for } x \in X.$$

If in the Reich type condition we change the position of x with y and we follow a similar approach as above, then we obtain:

$$(2.2) \qquad \|T(x)\| \le \frac{a+c}{1-c}\|x\| + \frac{b+1}{1-c}\|T(0)\|, \text{ for } x \in X.$$

The condition $a+b+c<1$ implies that either $a+2b<1$ or $a+2c<1$. Then, from (2.1) and (2.2), we have $|T| \le \frac{a+b}{1-b} < 1$ or $|T| \le \frac{a+c}{1-c} < 1$, proving the conclusion. ∎

Next, we will consider the case of multivalued Ciric operators.

Let $(X, \|\cdot\|)$ be a normed space. Then $T : X \to P(X)$ is said to be a Ciric type multivalued operator if there exist $a \in \mathbb{R}_+$ with $a < 1$ such that

$$H(T(x),T(y)) \leq$$
$$a \cdot \max\{\|x-y\|, D(x,T(x)), D(y,T(y)), \frac{1}{2}[D(x,T(y)) + D(y,T(x))]\},$$

for all $x, y \in X$.

THEOREM 2.2. *Let $(X, \|\cdot\|)$ be a normed space and $T : X \to P_b(X)$ be a Ciric type multivalued operator. Then:*

i) *T is quasi-bounded.*

ii) *If $a < \frac{1}{2}$ then T is a multivalued norm-contraction.*

Proof. Let us take $y = 0$ in Ciric type condition. It follows that

$$H(T(x),T(0)) \leq$$
$$a \cdot \max\{\|x\|, D(x,T(x)), D(0,T(0)), \frac{1}{2}[D(x,T(0)) + D(0,T(x))]\},$$

for each $x \in X$. Hence

$$\begin{aligned}\|T(x)\| &= H(T(x),\{0\}) \\ &\leq H(T(x),T(0)) + H(T(0),\{0\}) \\ &\leq a \cdot \max\{\|x\|, D(x,T(x)), D(0,T(0)), \\ &\qquad \frac{1}{2}[D(x,T(0)) + D(0,T(x))]\} + \|T(0)\| \\ &\leq a \cdot \max\{\|x\|, D(x,T(x)), \|T(0)\|, \\ &\qquad \frac{1}{2}[D(x,T(0)) + \|T(x)\|]\} + \|T(0)\|,\end{aligned}$$

for each $x \in X$.

In a similar way as in the proof of Theorem 2.1. we have $D(x,T(x)) \leq \|x\| + \|T(x)\|$, for each $x \in X$. Also, $D(x,T(0)) \leq \|x\| + \|T(0)\|$.

Hence

$$\|T(x)\| \leq$$
$$a \cdot \max\{\|x\|, \|x\| + \|T(x)\|, \|T(0)\|, \frac{1}{2}[\|x\| + \|T(0)\| + \|T(x)\|]\} + \|T(0)\| =$$
$$a \cdot \max\{\|x\| + \|T(x)\|, \|T(0)\|\} + \|T(0)\|,$$

for each $x \in X$.

We distinguish two cases:

a) If $\max\{\|x\| + \|T(x)\|, \|T(0)\|\} = \|x\| + \|T(x)\|$, then $\|T(x)\| \leq a \cdot [\|x\| + \|T(x)\|] + \|T(0)\|$. Hence $\|T(x)\| \leq \frac{a}{1-a}\|x\| + \frac{1}{1-a}\|T(0)\|$.

b) If $\max\{\|x\| + \|T(x)\|, \|T(0)\|\} = \|T(0)\|$, then $\|T(x)\| \leq (a+1) \cdot \|T(0)\|$.

Finally from a) and b) we obtain that $\|T(x)\| \leq \frac{a}{1-a}\|x\| + \frac{1}{1-a}\|T(0)\|$.

When $a < \frac{1}{2}$ we obtain that $|T| \leq \frac{a}{1-a} < 1$. ∎

Finally, let us discuss the case of a multivalued generalized φ-contraction. In this respect, we need first some definitions.

A mapping $\varphi : \mathbb{R}_+^5 \to \mathbb{R}_+$ is said to be a generalized comparison function if φ is increasing (i.e. $r, s \in \mathbb{R}_+^5, r \leq s$ implies $\varphi(r) \leq \varphi(s)$) and the mapping $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) := \varphi(t,t,t,t,t)$ satisfies the condition $\psi^n(t) \to 0$, as $n \to +\infty$, for each $t \in \mathbb{R}_+$.

If (X, d) is a metric space, then by definition, $T : X \to P_{b,cl}(X)$ is said to be a multivalued generalized φ-contraction if $\varphi : \mathbb{R}_+^5 \to \mathbb{R}_+$ is a generalized comparison function and for each $x, y \in X$ we have:

$$H(T(x),T(y)) \leq \varphi(d(x,y), D(x,T(x)), D(y,T(y)), D(x,T(y)), D(y,T(x))).$$

If we choose $\varphi(t_1,t_2,t_3,t_4,t_5) := at_1+bt_2+ct_3$ (where $a+b+c < 1$), then, the above definition, implies the notion of multivalued Reich type operator. Moreover, if we consider $\varphi(t_1,t_2,t_3,t_4,t_5) := a \max\{t_1,t_2,t_3,\frac{1}{2}[t_4+t_5]\}$ (with $a < 1$), then we get the definition of a multivalued Ciric type contraction.

THEOREM 2.3. *Let $(X, \|\cdot\|)$ be a normed space and $T : X \to P_b(X)$ be a multivalued generalized φ-contraction. Suppose that there exists $\lambda \in]0,1[$ such that $\varphi(t,t,t,t,t) < \lambda t$, for each $t \in \mathbb{R}_+$. Then:*

i) *T is quasi-bounded*

ii) *If $\lambda < \frac{1}{2}$ then T is a multivalued norm-contraction.*

Proof. From the multivalued generalized φ-contraction assumption, written for $y := 0$, we have:

$$\begin{aligned} H(T(x),T(0)) &\leq \varphi(\|x\|, D(x,T(x)), D(0,T(0)), D(x,T(0)), D(0,T(x))) \\ &\leq \varphi(\|x\|, D(x,T(x)), \|T(0)\|, D(x,T(0)), \|T(x)\|). \end{aligned}$$

As before $D(x,T(x)) \leq \|x\| + \|T(x)\|$ and $D(x,T(0)) \leq \|x\| + \|T(0)\|$, for each $x \in X$. Hence

$$H(T(x),T(0)) \leq \varphi(\|x\|, \|x\| + \|T(x)\|, \|T(0)\|, \|x\| + \|T(0)\|, \|T(x)\|).$$

Then, for each $x \in X$, we have

$$\begin{aligned} \|T(x)\| &= H(T(x),\{0\}) \\ &\leq H(T(x),T(0)) + H(T(0),\{0\}) \\ &\leq \varphi(\|x\|, \|x\| + \|T(x)\|, \|T(0)\|, \|x\| + \|T(0)\|, \|T(x)\|) + \|T(0)\|. \end{aligned}$$

We distinguish the following two cases:

i) If $x \in X$ has the property $\|T(0)\| \leq \|T(x)\|$ then we have

$$\begin{aligned} \|T(x)\| \leq \varphi(&\|x\| + \|T(x)\|, \|x\| + \|T(x)\|, \|x\| + \|T(x)\|, \\ &\|x\| + \|T(x)\|, \|x\| + \|T(x)\|) + \|T(0)\| \\ \leq{}& \lambda \cdot (\|x\| + \|T(x)\|) + \|T(0)\|. \end{aligned}$$

Hence we have:

$$\|T(x)\| \leq \frac{\lambda}{1-\lambda}\|x\| + \frac{1}{1-\lambda}\|T(0)\|. \tag{2.3}$$

ii) If $x \in X$ has the property $\|T(x)\| \leq \|T(0)\|$ then we can write

$$\begin{aligned}\|T(x)\| \leq \varphi(&\|x\| + \|T(0)\|, \|x\| + \|T(0)\|, \|x\| + \|T(0)\|,\\ &\|x\| + \|T(0)\|, \|x\| + \|T(0)\|) + \|T(0)\|\\ \leq\ &\lambda \cdot (\|x\| + \|T(0)\|) + \|T(0)\|.\end{aligned}$$

Hence we have proved that:

$$\|T(x)\| \leq \lambda\|x\| + (1+\lambda)\|T(0)\|. \tag{2.4}$$

As a conclusion, from (2.3) and (2.4), for each $x \in X$ we have:

$$\begin{aligned}\|T(x)\| &\leq \max\left\{\frac{\lambda}{1-\lambda}, \lambda\right\}\|x\| + \max\left\{\frac{1}{1-\lambda}, \lambda+1\right\}\|T(0)\|\\ &= \frac{\lambda}{1-\lambda}\|x\| + \frac{1}{1-\lambda}\|T(0)\|.\end{aligned}$$

Of course, for $\lambda < \frac{1}{2}$ we get that $|T| \leq \frac{\lambda}{1-\lambda} < 1$. ∎

COROLLARY 2.4. *Let $(X, \|\cdot\|)$ be a normed space and $T : X \to P_b(X)$ be a multivalued operator. Suppose that there exist $a_1, a_2, a_3, a_4, a_5 \in \mathbb{R}_+$ with $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ such that*

$$\begin{aligned}H(T(x), T(y)) \leq\ & a_1\|x-y\| + a_2 D(x, T(x))\\ &+ a_3 D(y, T(y)) + a_4 D(x, T(y)) + a_5 D(y, T(x)),\end{aligned}$$

for each $x, y \in X$. Then T is a multivalued norm-contraction.

Proof. Define $\varphi(t_1, t_2, t_3, t_4, t_5) := \sum_{i=1}^{5} a_i t_i$. From the proof of Theorem 2.3 we have

$$\|T(x)\| \leq \varphi(\|x\|, \|x\| + \|T(x)\|, \|T(0)\|, \|x\| + \|T(0)\|, \|T(x)\|) + \|T(0)\|.$$

Denote $r_1 := \|x\|$, $r_2 := \|T(x)\|$ and $r_3 := \|T(0)\|$. Then

$$r_2 \leq a_1 r_1 + a_2(r_1 + r_2) + a_3 r_3 + a_4(r_1 + r_3) + a_5 r_2 + r_3$$

and finally

$$r_2 \leq \frac{a_1 + a_2 + a_4}{1 - a_2 - a_5} \cdot r_1 + \frac{a_3 + a_4 + 1}{1 - a_2 - a_5} \cdot r_3.$$

Hence $\|T(x)\| \leq m\|x\| + M$, for each $x \in X$, where $m := \frac{a_1+a_2+a_4}{1-a_2-a_5} < 1$. ∎

3. Surjectivity results

The following theorems are known.

THEOREM 3.1 (Iannaci [10]). *Let X be a Banach space and $F : X \to P_{cv}(X)$ be a multivalued norm-contraction, u.s.c. and compact (i.e. F sends bounded sets into relatively compact sets). Then $1_X - F$ is onto.*

THEOREM 3.2 (Martelli-Vignoli [13]). *Let E be a Banach space and $T : E \to P(E)$ be a multivalued operator satisfying the following two conditions:*

i) *T is a multivalued norm-contraction,*

ii) *for each $Z \in P_b(E)$ we have $T(Z) \in P_b(E)$.*

Then there exists $R > 0$ such that $\tilde{B}(0, R) \in I(T)$.

THEOREM 3.3 (Fitzpatrick-Petryshyn [8]). *Let X be a Banach space, $Y \in P_{b,cl,cv}(X)$, and $T : Y \to P_{cl,ac}(Y)$ be u.s.c. and densifying (with respect to an abstract measure of noncompactness). Then $FixT \neq \emptyset$.*

Using the above mentioned results we have:

THEOREM 3.4. *Let X be a Banach space, $Z \in P_b(X)$ and $F : X \to P_{cv}(X)$. Suppose that:*

i) *F u.s.c. and compact on X*

ii) *There exist $a, b, c \in \mathbb{R}_+$, with $a+b+c < 1$, such that $H(F(x), F(y)) \leq a\|x - y\| + bD(x, F(x)) + cD(y, F(y))$, for all $x, y \in X \setminus Z$.*

Then $1_X - F$ is onto.

Proof. We will prove that T is a multivalued norm-contraction on X. For this purpose , let us consider the following cases:

a) $x \in Z$. Then $\|F(x)\| \leq \|F(Z)\|$.

b) $x \in X \setminus Z$. Let $x_0 \in Z$. Then

$$\begin{aligned}\|F(x)\| &= H(F(x), \{0\}) \leq H(F(x), F(x_0)) + H(F(x_0), \{0\}) \\ &\leq a\|x - x_0\| + bD(x, T(x)) + cD(x_0, F(x_0)) + \|F(x_0)\| \\ &\leq a\|x\| + a\|x_0\| + b(\|x\| + \|F(x)\|) + cD(x_0, F(x_0)) + \|F(x_0)\|.\end{aligned}$$

Hence we have:

$$\|F(x)\| \leq \frac{a+b}{1-b} \cdot \|x\| + \frac{a\|x_0\| + cD(x_0, F(x_0)) + \|F(x_0)\|}{1-b}.$$

From both cases, for each $x \in X$, we get

$$\|F(x)\| \leq \frac{a+b}{1-b} \cdot \|x\| + \max\left\{\frac{a\|x_0\| + cD(x_0, F(x_0)) + \|F(x_0)\|}{1-b}, \|F(Z)\|\right\}.$$

Hence F is a multivalued norm-contraction and Theorem 3.1. applies. ∎

Another surjectivity result is:

THEOREM 3.5. *Let X be a Banach space, $Z \in P_b(X)$ and $F : X \to P_{cl,ac}(X)$. Suppose that:*

i) *F is u.s.c. and densifying (with respect to an abstract measure of noncompactness) on X.*

ii) *There exist $a, b, c \in \mathbb{R}_+$, with $a+b+c<1$ such that $H(F(x), F(y)) \leq a\|x-y\| + bD(x, F(x)) + cD(y, F(y))$, for all $x, y \in X \setminus Z$.*
Then $1_X - F$ is onto.

Proof. Let $y \in E$. Denote $T(x) := y + F(x)$, for each $x \in E$. Then we have to find an element $x^* \in E$ such that $x^* \in T(x^*)$.

T is clearly u.s.c., densifying with compact and acyclic values. From ii), in a similar way to the proof of Theorem 3.4., we can get that F is norm-contraction. Then T is norm-contraction too. From Theorem 3.2. we deduce that there is $R > 0$ such that $\tilde{B}(0, R)$ is invariant with respect to T. From Theorem 3.3. there exists $x^* \in \tilde{B}(0, R)$ such that $x^* \in T(x^*)$. The proof is complete. ■

REMARK 3.6. Theorem 3.4. and Theorem 3.5. are true if we replace the condition ii) with one of the following assumption:

ii′) F is a multivalued Ciric operator on $X \setminus Z$ with a constant $a \in]0, \frac{1}{2}[$
or

ii″) F is a multivalued generalized φ-contraction on $X \setminus Z$ with respect to a generalized comparison function $\varphi : \mathbb{R}^5_+ \to \mathbb{R}_+$ satisfying the condition: there exists $\lambda \in]0, \frac{1}{2}[$ such that $\varphi(t, t, t, t, t) < \lambda t$, for each $t \in \mathbb{R}_+$.

REMARK 3.7. In particular, Theorem 3.4. and Theorem 3.5. imply that the fixed point set $FixF$ of the multivalued operator F is nonempty. From this point of view, all the surjectivity results of this section are extensions of some fixed point results given in [21].

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DEPARTMENT OF APPLIED MATHEMATICS
BABEŞ-BOLYAI UNIVERSITY CLUJ-NAPOCA
Kogălniceanu 1
400084, CLUJ-NAPOCA, ROMANIA
e-mail: gabip@math.ubbcluj.ro

Received May 11, 2006.