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SOME INEQUALITIES OF ACZEL TYPE FOR GRAMIANS IN n-INNER PRODUCT SPACES

Abstract. Some inequalities of Aczel type for Gramians in $n-$inner product spaces which generalize pecaric result for $n-$inner product spaces are given. Applications connected to Schwartz's inequality for $n-$inner product spaces are also noted.

1. Introduction

The concept of n-inner product and n-inner product space is a generalization of the concept of an inner product space ($n = 1$) and of 2-inner product space ($n = 2$). The theory of n-inner product spaces have been intensively studied by many authors. A systematic presentation of the recent results related to the theory of n-inner product spaces as well as an extensive list of the related references can be found in [3].

The main aim of this paper is to point out some new inequalities of Aczel type for Gramians in $n-$inner product spaces, which also generalizes and extends the results of Pečaric and complement, in a sense of, Chapter XX of the book [13], but for $n-$inner product spaces. Some applications to real or complex numbers which are closely connected with those embodied in chapter IV of [13], for $n-$inner product space, are also given.

2. Preliminaries

(1) Here we give basic definition and elementary properties of $n-$inner products. Let n be a natural number and X a linear space of dimension ≥ 1 over the field $\Bbbk = \mathbb{R}$, of real numbers or the field $\Bbbk = \mathbb{C}$, of complex numbers. Suppose that $(.,. \mid .,\dots,.)$ be a $\Bbbk$-valued function on $X^{n+1} = X \times X \times \cdots \times X$ ($n+1$ times) satisfying the following conditions:

(i) $(x, x \mid a_2, \dots, a_n) \geq 0$ and $(x, x \mid a_2, \dots, a_n) = 0$ if and only if the vectors x $, a_2, \dots, a_n$ are linearly dependent,

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(ii) $(x, x \mid a_2, \ldots, a_n) = (a_2, a_2 \mid x, a_3, \ldots, a_n)$,

(iii) $(x, y \mid a_2, \ldots, a_n) = (x, y \mid a_{i_2}, \ldots, a_{i_n})$ for every permutation $(i_2, \ldots, i_n)$ of $(2, \ldots, n)$,

(iv) $(x, y \mid a_2, \ldots, a_n) = \overline{(y, x \mid a_2, \ldots, a_n)}$,

(v) $(\alpha x, y \mid a_2, \ldots, a_n) = \alpha(x, y \mid a_2, \ldots, a_n)$ for every scalar $\alpha \in K$,

(vi) $(x + x', y \mid a_2, \ldots, a_n) = (x, y \mid a_2, \ldots, a_n) + (x', y \mid a_2, \ldots, a_n)$.

A function $(.,. \mid ., \ldots, .)$ is called an n-inner product on X and $(X, (.,. \mid ., \ldots, .))$ is called an n-inner product space (or an n-pre-Hilbert space). Some basic properties of n-inner product can be obtained as follows:

$(1-1)$ If $\Bbbk = \mathbb{R}$, then (iv) reduces to

$$(x, y \mid a_2, \ldots, a_n) = (y, x \mid a_2, \ldots, a_n).$$

$(1-2)$ From (ii) and (iii), it follows that

$$(x, x \mid a_2, \ldots, a_n) = (a_i, a_i \mid a_2, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)$$

holds for any $i \in \{2, \ldots, n\}$.

$(1-3)$ From (iv) and (v), we have

$$(0, y \mid a_2, \ldots, a_n) = 0, \ (x, 0 \mid a_2, \ldots, a_n) = 0$$

and also

$$(x, \alpha y \mid a_2, \ldots, a_n) = \overline{\alpha}(x, y \mid a_2, \ldots, a_n). \tag{2.1}$$

$(1-4)$ Using (ii)–(vi), we get

$$\begin{aligned}(a_2, a_2 \mid x \pm y, a_3, \ldots, a_n) &= (x \pm y, x \pm y \mid a_2, \ldots, a_n) \\ &= (x, x \mid a_2, \ldots, a_n) + (y, y \mid a_2, \ldots, a_n) \pm 2\,\mathrm{Re}(x, y \mid a_2, \ldots, a_n).\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Re}(x, y \mid a_2, \ldots, a_n) \\ &= \frac{1}{4}[(a_2, a_2 \mid x + y, a_3, \ldots, a_n) - (a_2, a_2 \mid x - y, a_3, \ldots, a_n)].\end{aligned} \tag{2.2}$$

In the real case $\Bbbk = \mathbb{R}$, (2.2) reduces to

$$(x, y \mid a_2, \ldots, a_n) = \frac{1}{4}[(a_2, a_2 \mid x+y, \ldots, a_n) - (a_2, a_2 \mid x-y, \ldots, a_n)]. \tag{2.3}$$

Using this formula it is easy to see that, for any $\alpha \in \mathbb{R}$, we have

$$(x, y \mid \alpha a_2, \ldots, a_n) = \alpha^2(x, y \mid a_2, \ldots, a_n). \tag{2.4}$$

In the complex case $\Bbbk = \mathbb{C}$, using (2.1) and (2.2), we get

$$\begin{aligned}\mathrm{Im}(x, y \mid a_2, \ldots, a_n) &= \mathrm{Re}[-i(x, y \mid a_2, \ldots, a_n)] \\ &= \frac{1}{4}[(a_2, a_2 \mid x + iy, \ldots, a_n) - (a_2, a_2 \mid x - iy, \ldots, a_n)]\end{aligned}$$

which, on combination with (2.2), yields

$$(x, y \mid a_2, \dots, a_n) = \frac{1}{4}[(a_2, a_2 \mid x + y, \dots, a_n) - (a_2, a_2 \mid x - y, \dots, a_n)] + \frac{i}{4}[(a_2, a_2 \mid x + iy, \dots, a_n) - (a_2, a_2 \mid x - iy, \dots, a_n)].$$

Using the above formula and (2.1), we get, for any $\alpha \in \mathbb{C}$

$$(x, y \mid \alpha a_2, \dots, a_n) = |\alpha|^2 (x, y \mid a_2, \dots, a_n). \tag{2.5}$$

However, for $\alpha \in \mathbb{R}$, (2.5) reduces to (2.4). Also, from (2.5) and (iii), we get

$$(x, y \mid a_2, \dots, a_{i-1}, \alpha a_i, a_{i+1}, \dots, a_n) = |\alpha|^2 (x, y \mid a_2, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n)$$

and

$$(x, y \mid a_2, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_n) = 0,$$

for some $i \in \{2, \dots, n\}$.

(2) In any given n-inner product space $(X, (.,. \mid ., \dots, .))$, we can define a function $\|., \dots, .\|$on X^n as

$$\|x_1, a_2, \dots, a_n\| = \sqrt{(x_1, x_1 \mid a_2, \dots, a_n)}$$

for any $x_1, a_2, \dots, a_n \in X$. It is easy to see that this function satisfies the following conditions:

(i) $\|x_1, a_2, \dots, a_n\| = 0$ if and only if $x_1, a_2, \dots, a_n$ are linearly dependent,

(ii) $\|x_1, a_2, \dots, a_n\| = \|x_1, a_{i_2}, ., \dots, a_{i_n}\|$ for every permutation of $(i_2, \dots, i_n)$ of $(2, \dots, n)$,

(iii) $\|\alpha x_1, a_2, \dots, a_n\| = |\alpha| \, \|x_1, a_2, \dots, a_n\|$ for every real α,

(iv) $\|x_1 + x_1', a_2, \dots, a_n\| \le \|x_1, a_2, \dots, a_n\| + \|x_1', a_2, \dots, a_n\|$.

(3) A natural extension of the Cauchy-Schwartz-Buniakowsky inequality, in $n-$inner product space [3],

$$|(x, y \mid a_2, \dots, a_n)|^2 \le (x, x \mid a_2, \dots, a_n)(y, y \mid a_2, \dots, a_n), \tag{2.6}$$

is the Gram's inequality [1],

$$\Gamma(x_1, x_2, \dots, x_n \mid a_2, \dots, a_n) \ge 0, \tag{2.7}$$

which holds for any choice of vectors $x_1, x_2, \dots, x_n \in X$ and is strict unless

$x_1, x_2, \dots, x_n, a_2, \dots, a_n$ are linearly dependent, where

$$\Gamma(x_1, x_2, \dots, x_n \mid a_2, \dots, a_n) = \begin{vmatrix} (x_1, x_1 \mid a_2, \dots, a_n) & (x_1, x_2 \mid a_2, \dots, a_n) & \dots & (x_1, x_n \mid a_2, \dots, a_n) \\ (x_2, x_1 \mid a_2, \dots, a_n) & (x_2, x_2 \mid a_2, \dots, a_n) & \dots & (x_2, x_n \mid a_2, \dots, a_n) \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ (x_n, x_1 \mid a_2, \dots, a_n) & (x_n, x_2 \mid a_2, \dots, a_n) & \dots & (x_n, x_n \mid a_2, \dots, a_n) \end{vmatrix}.$$

The Gram's determinant, $\Gamma(x_1, x_2, \dots, x_n \mid a_2, ., \dots, a_n)$, of the vectors $x_1, x_2, \dots, x_n$ with respect to the vectors $a_2, \dots, a_n$ is given by

$$\Gamma(x_1, x_2, \dots, x_n \mid a_2, ., \dots, a_n) = \det G(x_1, x_2, \dots, x_n \mid a_2, ., \dots, a_n),$$

where

$$G(x_1, x_2, \dots, x_n \mid a_2, \dots, a_n) = \begin{bmatrix} (x_1, x_1 \mid a_2, \dots, a_n) & (x_1, x_2 \mid a_2, \dots, a_n) & \dots & (x_1, x_n \mid a_2, \dots, a_n) \\ (x_2, x_1 \mid a_2, \dots, a_n) & (x_2, x_2 \mid a_2, \dots, a_n) & \dots & (x_2, x_n \mid a_2, \dots, a_n) \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ (x_n, x_1 \mid a_2, \dots, a_n) & (x_n, x_2 \mid a_2, \dots, a_n) & \dots & (x_n, x_n \mid a_2, \dots, a_n) \end{bmatrix}.$$

(4) In 1956 J. Aczel established an interesting inequality (page 117 of [13]).

Let $a = (a_1, a_2, \dots, a_n)$ and $b = (b_1, b_2, \dots, b_n)$ be two sequences of real numbers such that either

$$(a_1^2 - a_2^2 - \dots - a_n^2) > 0 \text{ or } (b_1^2 - b_2^2 - \dots - b_n^2) > 0,$$

then

$$(a_1^2 - a_2^2 - \dots - a_n^2)(b_1^2 - b_2^2 \dots - b_n^2) \leq (a_1 b_1 - a_2 b_2 - \dots - a_n b_n)^2, \tag{2.8}$$

with equality if and only if the sequences a and b are proportional.

In [11], S. Kurepa pointed out the following inequality of Aczel type which holds in Hilbert spaces (page 602 of [13]).

Let X be a real Hilbert space and c, a unit vector in X. Suppose that $a, b \in X$ are given vectors such that

$$(u^2 - \|a_0\|^2) \times (v^2 - \|b_0\|^2) \geq 0, \tag{2.9}$$

where $u = (a, c)$, $v = (b, c)$, $a_0 = a - uc$, and $b_0 = b - vc$, then

$$(u^2 - \|a_0\|^2)(v^2 - \|b_0\|^2) \le (uv - (a_0, b_0))^2. \tag{2.10}$$

If a and b are independent and strict inequality holds in (2.9), then strict inequality also holds in (2.10).

In [14] and also (page 603 of [13]) J. E. Pecarič proved an interesting converse of a known inequality of Kurepa (page 599 of [13]) which asserts that,

$$\left|\det \begin{bmatrix} (x_1, y_1) & \dots & (x_1, y_m) \\ . & \dots & . \\ . & \dots & . \\ . & \dots & . \\ (x_m, y_1) & \dots & (x_m, y_m) \end{bmatrix}\right|^2 \le \Gamma(x_1, x_2, \dots, x_m)\Gamma(y_1, y_2, \dots, y_m), \tag{2.11}$$

where $x_i, y_i \in X, (i = \overline{1, m})$ and X is an inner product space over the real or complex number field $\mathbb{k}$ and Γ is the Gramian of the vectors involved above. Pecarič's result is:

$$\begin{aligned} &\left(u^2 - \Gamma(x_1, x_2, \dots, x_m)\right)\left(v^2 - \Gamma(y_1, y_2, \dots, y_m)\right) \\ &\le \left(uv - \det \begin{bmatrix} (x_1, y_1) & \dots & (x_1, y_m) \\ . & \dots & . \\ . & \dots & . \\ . & \dots & . \\ (x_m, y_1) & \dots & (x_m, y_m) \end{bmatrix}\right)^2 \end{aligned} \tag{2.12}$$

provided that

$$u^2 - \Gamma(x_1, x_2, \dots, x_m) > 0 \text{ or } v^2 - \Gamma(y_1, y_2, \dots, y_m) > 0,$$

where $x_{i,}\ y_i\ (i = \overline{1, m})$ are vectors in a real inner product space X. Note that this result is a generalization for Gramians of the Aczel inequality (2.8).

3. Main results

THEOREM 1. *Let X be a real $n-$inner product space and $c, a_2, \dots, a_n \in X$ with $\|c, a_2, \dots, a_n\| = 1$. Suppose that $a, b \in X$ are given vectors and $u, v \in \mathbb{R}$ such that,*

$$(u^2 - \|a_0, a_2, \dots, a_n\|^2)(v^2 - \|b_0, a_2, \dots, a_n\|^2) \ge 0, \tag{3.1}$$

where

$$u = (a, c \mid a_2, \dots, a_n),\ \ v = (b, c \mid a_2, \dots, a_n),$$
$$a_0 = a - uc \text{ and } b_0 = b - vc. \tag{3.2}$$

Then

$$(u^2 - \|a_0, a_2, \dots, a_n\|^2)(v^2 - \|b_0, a_2, \dots, a_n\|^2)$$
$$\le (uv - (a_0, b_0 \mid a_2, \dots, a_n))^2. \tag{3.3}$$

Proof. Consider

$$(a_0, b_0 \mid a_2, \dots, a_n) = (a - uc, b - vc \mid a_2, \dots, a_n)$$
$$= (a, b \mid a_2, \dots, a_n) - (a, c \mid a_2, \dots, a_n)(b, c \mid a_2, \dots, a_n) \tag{3.4}$$

and

$$\|a_0, a_2, \dots, a_n\|^2 = (a - uc, a - uc \mid a_2, \dots, a_n)$$
$$= \|a, a_2, \dots, a_n\|^2 - (a, c \mid a_2, \dots, a_n)^2, \tag{3.5}$$

$$\|b_0, a_2, \dots, a_n\|^2 = (b - vc, b - vc \mid a_2, \dots, a_n)$$
$$= \|b, a_2, \dots, a_n\|^2 - (b, c \mid a_2, \dots, a_n)^2. \tag{3.6}$$

Since

$$(u^2 - \|a_0, a_2, \dots, a_n\|^2)(v^2 - \|b_0, a_2, \dots, a_n\|^2) \ge 0 \tag{3.7}$$

implies

$$((a, c \mid a_2, \dots, a_n)^2 - \|a, a_2, \dots, a_n\|^2 + (a, c \mid a_2, \dots, a_n)^2)$$
$$\times((b, c \mid a_2, \dots, a_n)^2 - \|b, a_2, \dots, a_n\|^2 + (b, c \mid a_2, \dots, a_n)^2) \ge 0,$$

using $(m^2 - n^2)(p^2 - q^2) \le (mp - nq)^2$, for m, n, p, q to be real, we get

$$2(a, c \mid a_2, \dots, a_n)^2 - \|a, a_2, \dots, a_n\|^2 \ge 0,$$
$$2(b, c \mid a_2, \dots, a_n)^2 - \|b, a_2, \dots, a_n\|^2 \ge 0. \tag{3.8}$$

Now consider

$$(u^2 - \|a_0, a_2, \dots, a_n\|^2)(v^2 - \|b_0, a_2, \dots, a_n\|^2)$$
$$= \left[(a, c \mid a_2, \dots, a_n)^2 - \|a, a_2, \dots, a_n\|^2 + (a, c \mid a_2, \dots, a_n)^2\right]$$
$$\times \left[(b, c \mid a_2, \dots, a_n)^2 - \|b, a_2, \dots, a_n\|^2 + (b, c \mid a_2, \dots, a_n)^2\right]$$
$$\le (2(a, c \mid a_2, \dots, a_n)(b, c \mid a_2, \dots, a_n) - \|a, a_2, \dots, a_n\| \, \|b, a_2, \dots, a_n\|)^2 .$$

By simple calculation, using (3.8), and

$$
\begin{aligned}
&\left(m^2-n^2\right)\left(p^2-q^2\right)\\
&\leq (mp-nq)^2, \text{ for } m,\ n,\ p,\ q \text{ to be real, we get}\\
&(u^2-\|a_0,a_2,\ldots,a_n\|^2)(v^2-\|b_0,a_2,\ldots,a_n\|^2)\\
&\leq ((a,c\mid a_2,\ldots,a_n)(b,c\mid a_2,\ldots,a_n)-(a_0,b_0\mid a_2,\ldots,a_n))^2.
\end{aligned}
$$

Using (3.2), we get

$$
\begin{aligned}
&(u^2-\|a_0,a_2,\ldots,a_n\|^2)(v^2-\|b_0,a_2,\ldots,a_n\|^2)\\
&\leq (uv-(a_0,b_0\mid a_2,\ldots,a_n))^2.
\end{aligned}
$$

If a and b are independent then strict inequality holds in (3.1), then strict inequality also holds in (3.2). ∎

In [1], the authors proved an interesting result which asserts that:

$$
\begin{aligned}
&\left|\det\begin{bmatrix}(x_1,y_1\mid a_2,\ldots,a_n) & \ldots & (x_1,y_m\mid a_2,\ldots,a_n)\\ . & \ldots & .\\ . & \ldots & .\\ . & \ldots & .\\ (x_m,y_1\mid a_2,\ldots,a_n) & \ldots & (x_m,y_m\mid a_2,\ldots,a_n)\end{bmatrix}\right|^2\\
&\leq \Gamma(x_1,x_2,\ldots,x_m\mid a_2,\ldots,a_n)\Gamma(y_1,y_2,\ldots,y_m\mid a_2,\ldots,a_n),
\end{aligned}
\tag{3.9}
$$

where $x_i,y_i\in X,(i=\overline{1,m})$ and X is an n-inner product space over the real or complex number field $\mathbb{k}$ and Γ is the Gramian of the vectors involved above. However the converse of (3.9) is proved in following theorem.

THEOREM 2. *Let $\{x_1,\ldots,x_n\}$ and $\{y_1,\ldots,y_n\}$ be two set of linearly independent vectors in $n-$inner product space X. Set*

$$Y=L(x_1,\ldots,x_n,y_1,\ldots,y_n)$$

take $a_2,\ldots,a_n\notin Y$ and $u,v\in\mathbb{R}$. Let $\widetilde{\Gamma}$ be a matrix defined by

$$
\widetilde{\Gamma}=\begin{bmatrix}(x_1,y_1\mid a_2,\ldots,a_n) & \ldots & (x_1,y_m\mid a_2,\ldots,a_n)\\ . & \ldots & .\\ . & \ldots & .\\ . & \ldots & .\\ (x_m,y_1\mid a_2,\ldots,a_n) & \ldots & (x_m,y_m\mid a_2,\ldots,a_n)\end{bmatrix}.
$$

Then

$$(3.10)\quad (u^2 - \widetilde{\Gamma}(x_1, x_2, \dots, x_m \mid a_2, \dots, a_n))(v^2 - \widetilde{\Gamma}(y_1, y_2, \dots, y_m \mid a_2, \dots, a_n)) \leq \left(uv - \det \begin{bmatrix} (x_1, y_1 \mid a_2, \dots, a_n) & \dots & (x_1, y_m \mid a_2, \dots, a_n) \\ . & \dots & . \\ . & \dots & . \\ . & \dots & . \\ (x_m, y_1 \mid a_2, \dots, a_n) & \dots & (x_m, y_m \mid a_2, \dots, a_n) \end{bmatrix} \right)^2$$

provided that

$$u^2 - \widetilde{\Gamma}(x_1, x_2, \dots, x_m \mid a_2, \dots, a_n) > 0$$

or

$$v^2 - \widetilde{\Gamma}(y_1, y_2, \dots, y_m \mid a_2, \dots, a_n) > 0.$$

Proof. Since

$$u^2 - \widetilde{\Gamma}(x_1, x_2, \dots, x_m \mid a_2, \dots, a_n) > 0,$$

and

$$v^2 - \widetilde{\Gamma}(y_1, y_2, \dots, y_m \mid a_2, \dots, a_n) > 0.$$

We know that

$$(m^2 - n^2)(p^2 - q^2) \leq (mp - nq)^2,$$

then

$$\begin{aligned} &\left(u^2 - \widetilde{\Gamma}(x_1, x_2, \dots, x_m \mid a_2, \dots, a_n)\right) \left(v^2 - \widetilde{\Gamma}(y_1, y_2, \dots, y_m \mid a_2, \dots, a_n)\right) \\ &\leq (uv - \widetilde{\Gamma}(x_1, x_2, \dots, x_m \mid a_2, \dots, a_n)^{\frac{1}{2}} \widetilde{\Gamma}(y_1, y_2, \dots, y_m \mid a_2, \dots, a_n)^{\frac{1}{2}})^2 \\ &\leq \left(uv - \det \begin{bmatrix} (x_1, y_1 \mid a_2, \dots, a_n) & \dots & (x_1, y_m \mid a_2, \dots, a_n) \\ . & \dots & . \\ . & \dots & . \\ . & \dots & . \\ (x_m, y_1 \mid a_2, \dots, a_n) & \dots & (x_m, y_m \mid a_2, \dots, a_n) \end{bmatrix} \right)^2 . \quad \blacksquare \end{aligned}$$

4. Some inequalities of Aczel type for Gramians

Let us denote by $\widetilde{\Gamma}(x_1, y_1, \dots, x_m, y_m \mid a_2, \dots, a_n)$, the determinant given by,

$$\widetilde{\Gamma}(x_1, y_1, \dots, x_m, y_m \mid a_2, \dots, a_n) = \det \begin{bmatrix} (x_1, y_1 \mid a_2, \dots, a_n) & (x_1, y_2 \mid a_2, \dots, a_n) & \dots & (x_1, y_m \mid a_2, \dots, a_n) \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ (x_m, y_1 \mid a_2, \dots, a_n) & (x_m, y_2 \mid a_2, \dots, a_n) & \dots & (x_m, y_m \mid a_2, \dots, a_n) \end{bmatrix},$$

where $x_1,\ y_1, \dots, x_m, y_m,\ a_2, \dots, a_n$ are vectors in $n-$ inner product space $(X; (.,. \mid ., \dots, .))$. In addition, we observe that if $y_1 = x_1, \dots,\ y_m = x_m$, then

$$\widetilde{\Gamma}(x_1, y_1, \dots, x_m, y_m \mid a_2, \dots, a_n) = \widetilde{\Gamma}(x_1, x_2, \dots, x_m \mid a_2, \dots, a_n).$$

By the use of this notation, the inequality (3.9) may be written as:

$$\begin{aligned} \text{(4.1)} \qquad & \widetilde{\Gamma}(x_1, x_2, \dots, x_m \mid a_2, \dots, a_n)\widetilde{\Gamma}(y_1, y_2, \dots, y_m \mid a_2, \dots, a_n) \\ & \geq \left|\widetilde{\Gamma}(x_1, y_1, \dots, x_m, y_m \mid a_2, \dots, a_n)\right|^2 \end{aligned}$$

for all $x_i, y_i \in X$, $(i = \overline{1,m})$. The first result which gives a converse of (4.1) is embodied in the following theorem:

THEOREM 3. *Let $(X; (.,. \mid ., \dots, .))$ be an n-inner product space over the real or complex number field $\mathbb{k}$ and a, b, c be three real numbers satisfying the following condition:*

$$a, c \geq 0 \quad and \quad b^2 \geq ac.$$

Then for all $x_i, y_i \in X, (i = \overline{1,m})$, such that either

$$a \geq \widetilde{\Gamma}(x_1, x_2, \dots, x_m \mid a_2, \dots, a_n),$$

or

$$c \geq \widetilde{\Gamma}(y_1, y_2, \dots, y_m \mid a_2, \dots, a_n),$$

then we have the inequality

$$\text{(4.2)} \quad [a - \widetilde{\Gamma}(x_1, x_2, \dots, x_m \mid a_2, \dots, a_n)][c - \widetilde{\Gamma}(y_1, y_2, \dots, y_m \mid a_2, \dots, a_n)]$$

$$\begin{aligned}
&\leq \min[(b \pm \operatorname{Re}\widetilde{\Gamma}(x_1, y_1, \ldots, x_m, y_m \mid a_2, \ldots, a_n))^2,\\
&\quad (b \pm |\operatorname{Re}\widetilde{\Gamma}(x_1, y_1, \ldots, x_m, y_m \mid a_2, \ldots, a_n)|)^2,\\
&\quad (b \pm \operatorname{Im}\widetilde{\Gamma}(x_1, y_1, \ldots, x_m, y_m \mid a_2, \ldots, a_n))^2,\\
&\quad (b \pm |\operatorname{Im}\widetilde{\Gamma}(x_1, y_1, \ldots, x_m, y_m \mid a_2, \ldots, a_n)|)^2,\\
&\quad (b \pm |\widetilde{\Gamma}(x_1, y_1, \ldots, x_m, y_m \mid a_2, \ldots, a_n)|^2].
\end{aligned}$$

Proof. Suppose that $a > \widetilde{\Gamma}(x_1, x_2, \ldots, x_m \mid a_2, \ldots, a_n)$ and consider the polynomial

$$P(t) = at^2 - 2bt + c, \ t \in \mathbb{R}.$$

Since $a > 0$ and $b^2 \geq ac$, it follows that there exists a $t_0 \in \mathbb{R}$ such that $P(t_0) = 0$.

Now put

$$\begin{aligned}
Q_1(t) = P(t) - [&\widetilde{\Gamma}(x_1, x_2, \ldots, x_m \mid a_2, \ldots, a_n)t^2\\
&\mp 2\operatorname{Re}\widetilde{\Gamma}(x_1, y_1, \ldots, x_m, y_m \mid a_2, \ldots, a_n)t\\
&+ \widetilde{\Gamma}(y_1, y_2, \ldots, y_m \mid a_2, \ldots, a_n)],
\end{aligned}$$

for $t \in \mathbb{R}$, and

$$\begin{aligned}
\overline{Q_1(t)} = P(t) - [&\widetilde{\Gamma}(x_1, x_2, \ldots, x_m \mid a_2, \ldots, a_n)t^2\\
&\mp 2|\operatorname{Re}\widetilde{\Gamma}(x_1, y_1, \ldots, x_m, y_m \mid a_2, \ldots, a_n)|t\\
&+ \widetilde{\Gamma}(y_1, y_2, \ldots, y_m \mid a_2, \ldots, a_n)]
\end{aligned}$$

for $t \in \mathbb{R}$.

A simple calculation gives

$$\begin{aligned}
Q_1(t) = &(a - \widetilde{\Gamma}(x_1, x_2, \ldots, x_m \mid a_2, \ldots, a_n))t^2\\
&-2(b \pm \operatorname{Re}\widetilde{\Gamma}(x_1, y_1, \ldots, x_m, y_m \mid a_2, \ldots, a_n))t\\
&+(c - \widetilde{\Gamma}(y_1, y_2, \ldots, y_m \mid a_2, \ldots, a_n)), \text{ for } t \in \mathbb{R}
\end{aligned}$$

and

$$\begin{aligned}
\overline{Q_1(t)} = &(a - \widetilde{\Gamma}(x_1, x_2, \ldots, x_m \mid a_2, \ldots, a_n))t^2\\
&-2(b \pm |\operatorname{Re}\widetilde{\Gamma}(x_1, y_1, \ldots, x_m, y_m \mid a_2, \ldots, a_n)|)t\\
&+(c - \widetilde{\Gamma}(y_1, y_2, \ldots, y_m \mid a_2, \ldots, a_n)), \text{ for } t \in \mathbb{R}.
\end{aligned}$$

Since

$$\begin{aligned}
Q_1(t_0) &= -[\widetilde{\Gamma}(x_1, x_2, \ldots, x_m \mid a_2, \ldots, a_n)t_0^2\\
&\qquad \mp 2\operatorname{Re}\widetilde{\Gamma}(x_1, y_1, \ldots, x_m, y_m \mid a_2, \ldots, a_n)t_0\\
&\qquad + \widetilde{\Gamma}(y_1, y_2, \ldots, y_m \mid a_2, \ldots, a_n)]\\
&\leq 0
\end{aligned}$$

as, by (3.9), one has

$$\begin{aligned}&|\operatorname{Re}\widetilde{\Gamma}(x_1,y_1,\ldots,x_m,y_m \mid a_2,\ldots,a_n)|^2\\ &\quad\le \widetilde{\Gamma}(x_1,x_2,\ldots,x_m \mid a_2,\ldots,a_n)\widetilde{\Gamma}(y_1,y_2,\ldots,y_m \mid a_2,\ldots,a_n),\end{aligned}$$

which gives

$$\begin{aligned}&[\widetilde{\Gamma}(x_1,x_2,\ldots,x_m \mid a_2,\ldots,a_n)t^2 \mp 2\operatorname{Re}\widetilde{\Gamma}(x_1,y_1,\ldots,x_m,y_m \mid a_2,\ldots,a_n)t\\ &\qquad + \widetilde{\Gamma}(y_1,y_2,\ldots,y_m \mid a_2,\ldots,a_n) \ge 0, \text{ for all } t \in \mathbb{R}.\end{aligned}$$

Hence we conclude that Q_1 has at least one solution in $\mathbb{R}$, i.e.,

$$\begin{aligned}0 \le \frac{1}{4}\Delta_1 &= (b \pm \operatorname{Re}\widetilde{\Gamma}(x_1,y_1,\ldots,x_m,y_m \mid a_2,\ldots,a_n))^2\\ &\quad -(a-\widetilde{\Gamma}(x_1,x_2,\ldots,x_m \mid a_2,\ldots,a_n))(c-\widetilde{\Gamma}(y_1,y_2,\ldots,y_m \mid a_2,\ldots,a_n)).\end{aligned}$$

Similarly, $\overline{Q_1}$ has at least one solution in $\mathbb{R}$ which is equivalent to

$$\begin{aligned}0 \le \frac{1}{4}\overline{\Delta}_1 &= (b \pm |\operatorname{Re}\widetilde{\Gamma}(x_1,y_1,\ldots,x_m,y_m \mid a_2,\ldots,a_n)|)^2\\ &\quad -(a-\widetilde{\Gamma}(x_1,x_2,\ldots,x_m \mid a_2,\ldots,a_n))(c-\widetilde{\Gamma}(y_1,y_2,\ldots,y_m \mid a_2,\ldots,a_n)),\end{aligned}$$

and the first part of (4.2) is proved.

The last part can be proved similarly by considering the polynomial:

$$\begin{aligned}Q_2(t) &= P(t) - [\widetilde{\Gamma}(x_1,x_2,\ldots,x_m \mid a_2,\ldots,a_n)t^2\\ &\quad \mp 2\operatorname{Im}\widetilde{\Gamma}(x_1,y_1,\ldots,x_m,y_m \mid a_2,\ldots,a_n)t\\ &\quad + \widetilde{\Gamma}(y_1,y_2,\ldots,y_m \mid a_2,\ldots,a_n)],\\ \overline{Q_2(t)} &= P(t) - [\widetilde{\Gamma}(x_1,x_2,\ldots,x_m \mid a_2,\ldots,a_n)t^2\\ &\quad \mp 2\operatorname{Im}\left|\widetilde{\Gamma}(x_1,y_1,\ldots,x_m,y_m \mid a_2,\ldots,a_n)\right| t\\ &\quad + \widetilde{\Gamma}(y_1,y_2,\ldots,y_m \mid a_2,\ldots,a_n)]\end{aligned}$$

and

$$\begin{aligned}Q_3(t) &= P(t) - [\widetilde{\Gamma}(x_1,x_2,\ldots,x_m \mid a_2,\ldots,a_n)t^2\\ &\quad \mp 2\left|\widetilde{\Gamma}(x_1,y_1,\ldots,x_m,y_m \mid a_2,\ldots,a_n)\right| t\\ &\quad + \widetilde{\Gamma}(y_1,y_2,\ldots,y_m \mid a_2,\ldots,a_n)]\end{aligned}$$

respectively. This completes the proof. ∎

Remark 1. Let $(X;\ (.,. \mid .,\ldots,.))$ be an $n-$inner product space and $M_1, M_2 \in \mathbb{R}$. Then for all $x_i, y_i \in X$ $(i=\overline{1,m})$ with

$$\widetilde{\Gamma}(x_1,x_2,\ldots,x_m \mid a_2,\ldots,a_n) \le |M_1| \ ,$$

or

$$\widetilde{\Gamma}(y_1,y_2,\ldots,y_m \mid a_2,\ldots,a_n) \le |M_2| \,.$$

One has the inequality:

$$
\begin{aligned}
&(M_1^2 - \widetilde{\Gamma}(x_1, x_2, \dots, x_m \mid a_2, \dots, a_n))(M_2^2 - \widetilde{\Gamma}(y_1, y_2, \dots, y_m \mid a_2, \dots, a_n)) \\
&\le \min[(M_1M_2 - \operatorname{Re}\widetilde{\Gamma}(x_1, y_1, \dots, x_m, y_m \mid a_2, \dots, a_n))^2, \\
&\qquad (M_1M_2 - |\operatorname{Re}\widetilde{\Gamma}(x_1, y_1, \dots, x_m, y_m \mid a_2, \dots, a_n)|)^2, \\
&\qquad (M_1M_2 - \operatorname{Im}\widetilde{\Gamma}(x_1, y_1, \dots, x_m, y_m \mid a_2, \dots, a_n))^2, \\
&\qquad (M_1M_2 - |\operatorname{Im}\widetilde{\Gamma}(x_1, y_1, \dots, x_m, y_m \mid a_2, \dots, a_n)|)^2, \\
&\qquad (M_1M_2 - \widetilde{\Gamma}(x_1, y_1, \dots, x_m, y_m \mid a_2, \dots, a_n))^2],
\end{aligned}
$$

which improves result in (3.10).

Now, using the above theorem, we can give the following inverse of inequality (3.10), in $n-$inner product spaces.

COROLLARY 1. *Suppose that a, b, c, x_i, y_i $(i = \overline{1, m})$ are as in Theorem 3. Then we have the inequalities:*

$$
\begin{aligned}
0 &\le \widetilde{\Gamma}(x_1, x_2, \dots, x_m \mid a_2, \dots, a_n)\widetilde{\Gamma}(y_1, y_2, \dots, y_m \mid a_2, \dots, a_n) \\
&\quad -[\operatorname{Re}\widetilde{\Gamma}(x_1, y_1, \dots, x_m, y_m \mid a_2, \dots, a_n)]^2 \\
&\le b^2 - ac + a\widetilde{\Gamma}(y_1, y_2, \dots, y_m \mid a_2, \dots, a_n) \\
&\quad +c\widetilde{\Gamma}(x_1, x_2, \dots, x_m \mid a_2, \dots, a_n) \\
&\quad +2\min[\pm b \operatorname{Re}\widetilde{\Gamma}(x_1, y_1, \dots, x_m, y_m \mid a_2, \dots, a_n); \\
&\quad \pm b|\operatorname{Re}\widetilde{\Gamma}(x_1, y_1, \dots, x_m, y_m \mid a_2, \dots, a_n)|],
\end{aligned}
$$

$$
\begin{aligned}
0 &\le \widetilde{\Gamma}(x_1, x_2, \dots, x_m \mid a_2, \dots, a_n)\widetilde{\Gamma}(y_1, y_2, \dots, y_m \mid a_2, \dots, a_n) \\
&\quad -[\operatorname{Im}\widetilde{\Gamma}(x_1, y_1, \dots, x_m, y_m \mid a_2, \dots, a_n)]^2 \\
&\le b^2 - ac + a\widetilde{\Gamma}(y_1, y_2, \dots, y_m \mid a_2, \dots, a_n) \\
&\quad +c\widetilde{\Gamma}(x_1, x_2, \dots, x_m \mid a_2, \dots, a_n) \\
&\quad +2\min[\pm b \operatorname{Im}\widetilde{\Gamma}(x_1, y_1, \dots, x_m, y_m \mid a_2, \dots, a_n); \\
&\quad \pm b|\operatorname{Im}\widetilde{\Gamma}(x_1, y_1, \dots, x_m, y_m \mid a_2, \dots, a_n)|]
\end{aligned}
$$

and

$$
\begin{aligned}
0 &\le \widetilde{\Gamma}(x_1, x_2, \dots, x_m \mid a_2, \dots, a_n)\widetilde{\Gamma}(y_1, y_2, \dots, y_m \mid a_2, \dots, a_n) \\
&\quad -|\widetilde{\Gamma}(x_1, y_1, \dots, x_m, y_m \mid a_2, \dots, a_n)|^2 \\
&\le b^2 - ac + a\widetilde{\Gamma}(y_1, y_2, \dots, y_m \mid a_2, \dots, a_n) \\
&\quad +c\widetilde{\Gamma}(x_1, x_2, \dots, x_m \mid a_2, \dots, a_n) \\
&\quad \pm 2b|\widetilde{\Gamma}(x_1, y_1, \dots, x_m, y_m \mid a_2, \dots, a_n)|.
\end{aligned}
$$

The proof follows by a simple calculation from (4.2). We omit the details.

The following result also holds.

COROLLARY 2. *Let $(X; (.,. \mid .,\dots,.))$ be an $n-$inner product space given as above and $M \in \mathbb{R}$, for all $x_i, y_i \in X$ $(i = \overline{1,m})$ with*

$$[\widetilde{\Gamma}(x_1, x_2, \dots, x_m \mid a_2, \dots, a_n)]^{\frac{1}{2}} \le M$$
$$\text{or} \quad [\widetilde{\Gamma}(y_1, y_2, \dots, y_m \mid a_2, \dots, a_n)]^{\frac{1}{2}} \le M,$$

then we have the inequality

$$\begin{aligned} 0 &\le \widetilde{\Gamma}(x_1, x_2, \dots, x_m \mid a_2, \dots, a_n)\widetilde{\Gamma}(y_1, y_2, \dots, y_m \mid a_2, \dots, a_n) \\ &\quad -[\operatorname{Re}\widetilde{\Gamma}(x_1, y_1, \dots, x_m, y_m \mid a_2, \dots, a_n)]^2 \\ &\le M^2[\widetilde{\Gamma}(x_1, x_2, \dots, x_m \mid a_2, \dots, a_n) \\ &\quad -2\operatorname{Re}\widetilde{\Gamma}(x_1, y_1, \dots, x_m, y_m \mid a_2, \dots, a_n) \\ &\quad +\widetilde{\Gamma}(y_1, y_2, \dots, y_m \mid a_2, \dots, a_n)]. \end{aligned}$$

It is important to note that a similar theorem can also be stated.

THEOREM 4. *Let $(X; (.,. \mid .,\dots,.))$ be an $n-$ inner product space over the real or complex number field and α, β, γ be the three real numbers with the property that*

$$\alpha, \gamma \ge 0 \quad and \quad \beta^2 \ge \alpha\gamma,$$

then for all $x_i, y_i \in X, (i = \overline{1,m})$, such that either

$$\alpha \ge [\widetilde{\Gamma}(x_1, x_2, \dots, x_m, a_2, \dots, a_n)]^{\frac{1}{2}}$$
$$\text{or} \quad \gamma \ge [\widetilde{\Gamma}(y_1, y_2, \dots, y_m, a_2, \dots, a_n)]^{\frac{1}{2}}.$$

We have the inequality

$$\begin{aligned} &[\alpha - [\widetilde{\Gamma}(x_1, x_2, \dots, x_m, a_2, \dots, a_n)]^{\frac{1}{2}}][\gamma - [\widetilde{\Gamma}(y_1, y_2, \dots, y_m, a_2, \dots, a_n)]^{\frac{1}{2}}] \\ &\quad \le \min[(\beta \pm |\operatorname{Re}\widetilde{\Gamma}(x_1, y_1, \dots, x_m, y_m \mid a_2, \dots, a_n)|^{\frac{1}{2}})^2, \\ &\qquad (\beta \pm |\operatorname{Re}\widetilde{\Gamma}(x_1, y_1, \dots, x_m, y_m \mid a_2, \dots, a_n)|^{\frac{1}{2}})^2, \\ &\qquad (\beta \pm |\widetilde{\Gamma}(x_1, y_1, \dots, x_m, y_m \mid a_2, \dots, a_n)|^{\frac{1}{2}})^2]. \end{aligned}$$

Proof. Suppose that $\alpha > [\widetilde{\Gamma}(x_1, x_2, \dots, x_m, a_2, \dots, a_n)]^{\frac{1}{2}}$ and consider the polynomial

$$P(t) = \alpha t^2 - 2\beta t + \gamma, \qquad t \in \mathbb{R}.$$

Since $\alpha > 0$ and $\beta^2 \ge \alpha\gamma$, it follows that there exists a $t_0 \in \mathbb{R}$ such that $P(t_0) = 0$.

Now put

$$\begin{aligned} Q_1(t) &= P(t) - [(\widetilde{\Gamma}(x_1, x_2, \dots, x_m \mid a_2, \dots, a_n))^{\frac{1}{2}} t^2 \\ &\quad \mp 2\phi_i t + |\widetilde{\Gamma}(y_1, y_2, \dots, y_m \mid a_2, \dots, a_n)|^{\frac{1}{2}}] \quad \text{for} \quad t \in \mathbb{R},\ i = 1, 2, 3. \end{aligned}$$

where

$$\phi_1 = |\operatorname{Re}\widetilde{\Gamma}(x_1, y_1, \ldots, x_m, y_m \mid a_2, \ldots, a_n)|,$$
$$\phi_2 = |\operatorname{Im}\widetilde{\Gamma}(x_1, y_1, \ldots, x_m, y_m \mid a_2, \ldots, a_n)|,$$
$$\phi_3 = |\widetilde{\Gamma}(x_1, y_1, \ldots, x_m, y_m \mid a_2, \ldots, a_n)|.$$

The proof is same as in Theorem 3. ∎

Now, using the above theorem, we can give the following converse of inequality in (3.9) for n-inner product spaces.

COROLLARY 3. *Suppose that $\alpha, \beta, \gamma, x_i, y_i \in X$ $(i = \overline{1, m})$ are as in theorem 4, then we have the inequalities:*

$$\begin{aligned}
0 \leq\ & \widetilde{\Gamma}(x_1, x_2, \ldots, x_m \mid a_2, \ldots, a_n)^{\frac{1}{2}} \widetilde{\Gamma}(y_1, y_2, \ldots, y_m \mid a_2, \ldots, a_n)^{\frac{1}{2}} \\
& -|\operatorname{Re}\widetilde{\Gamma}(x_1, y_1, \ldots, x_m, y_m \mid a_2, \ldots, a_n)| \\
& \leq \beta^2 - \alpha\gamma + \alpha\widetilde{\Gamma}(x_1, x_2, \ldots, x_m \mid a_2, \ldots, a_n)^{\frac{1}{2}} \\
& +\gamma\widetilde{\Gamma}(y_1, y_2, \ldots, y_m \mid a_2, \ldots, a_n)^{\frac{1}{2}} \\
& \pm 2\beta[\operatorname{Re}\widetilde{\Gamma}(x_1, y_1, \ldots, x_m, y_m \mid a_2, \ldots, a_n)\ ],
\end{aligned}$$

$$\begin{aligned}
0 \leq\ & \widetilde{\Gamma}(x_1, x_2, \ldots, x_m \mid a_2, \ldots, a_n)^{\frac{1}{2}} \widetilde{\Gamma}(y_1, y_2, \ldots, y_m \mid a_2, \ldots, a_n)^{\frac{1}{2}} \\
& -|\operatorname{Im}\widetilde{\Gamma}(x_1, y_1, \ldots, x_m, y_m \mid a_2, \ldots, a_n)| \\
& \leq \beta^2 - \alpha\gamma + \alpha\widetilde{\Gamma}(x_1, x_2, \ldots, x_m \mid a_2, \ldots, a_n)^{\frac{1}{2}} \\
& +\gamma\widetilde{\Gamma}(y_1, y_2, \ldots, y_m \mid a_2, \ldots, a_n)^{\frac{1}{2}} \\
& \pm 2\beta|\operatorname{Im}\widetilde{\Gamma}(x_1, y_1, \ldots, x_m, y_m \mid a_2, \ldots, a_n)|
\end{aligned}$$

and

$$\begin{aligned}
0 \leq\ & \widetilde{\Gamma}(x_1, x_2, \ldots, x_m \mid a_2, \ldots, a_n)^{\frac{1}{2}} \widetilde{\Gamma}(y_1, y_2, \ldots, y_m \mid a_2, \ldots, a_n)^{\frac{1}{2}} \\
& -|\ \widetilde{\Gamma}(x_1, y_1, \ldots, x_m, y_m \mid a_2, \ldots, a_n)| \\
& \leq \beta^2 - \alpha\gamma + \alpha\widetilde{\Gamma}(x_1, x_2, \ldots, x_m \mid a_2, \ldots, a_n)^{\frac{1}{2}} \\
& +\gamma\widetilde{\Gamma}(y_1, y_2, \ldots, y_m \mid a_2, \ldots, a_n)^{\frac{1}{2}} \\
& \pm 2\beta|\widetilde{\Gamma}(x_1, y_1, \ldots, x_m, y_m \mid a_2, \ldots, a_n)|.
\end{aligned}$$

COROLLARY 4. *Let $(X; (.,. \mid .,\ldots,.))$ be an $n-$ inner product space and $M > 0$. Then for all $x_i, y_i \in X$ $(i = \overline{1, m})$ with*

$$[\widetilde{\Gamma}(x_1, x_2, \ldots, x_m \mid a_2, \ldots, a_n)]^{\frac{1}{2}} \leq M$$

or

$$[\widetilde{\Gamma}(y_1, y_2, \ldots, y_m \mid a_2, \ldots, a_n)]^{\frac{1}{2}} \leq M,$$

then we have the inequality,

$$\begin{aligned} 0 &\le \widetilde{\Gamma}(x_1, x_2, \dots, x_m \mid a_2, \dots, a_n)^{\frac{1}{2}} \widetilde{\Gamma}(y_1, y_2, \dots, y_m \mid a_2, \dots, a_n)^{\frac{1}{2}} \\ &\quad -|\widetilde{\Gamma}(x_1, y_1, \dots, x_m, y_m \mid a_2, \dots, a_n)| \\ &\le M \begin{pmatrix} (\widetilde{\Gamma}(x_1, x_2, \dots, x_m \mid a_2, \dots, a_n))^{\frac{1}{2}} \\ -2|\widetilde{\Gamma}(x_1, y_1, \dots, x_m, y_m \mid a_2, \dots, a_n)|^{\frac{1}{2}} \\ +(\widetilde{\Gamma}(y_1, y_2, \dots, y_m \mid a_2, \dots, a_n))^{\frac{1}{2}} \end{pmatrix}. \end{aligned}$$

The following inequality [1], *analogue of Hadamard's inequality, in $n-$inner product space, $(X, (.,. \mid ., \dots, .))$ for the Gram's determinant is*

$$\widetilde{\Gamma}(x_1, x_2, \dots, x_m \mid a_2, \dots, a_n) \le \prod_{i=1}^{m} \|x_i, a_2, \dots, a_n\|^2 \tag{4.3}$$

for all $x_i \in X$ and $a_i \in X$ $(i = \overline{1, m})$ such that

$$L(x_1, x_2, \dots, x_m) \cap L(a_2, \dots, a_n) = \{0\}.$$

Equality holds in (4.3) *if*

(i) *there exists at least one $j \in \{1, \dots, m\}$ such that $x_j = 0$;*

(ii) *the vectors $x_1, \dots, x_m$ are linearly independent and*

$$(x_i, x_j \mid a_2, \dots, a_n) = 0, \quad 1 < i < j < m.$$

In the next theorem we point out a converse inequality for the inequality (4.3).

THEOREM 5. *Let $(X; (.,. \mid ., \dots, .))$ be an $n-$inner product space over the real or complex number field $\mathbb{k}$ and a, b, c three real numbers satisfying condition:*

$$a, c \ge 0 \text{ and } b^2 \ge ac$$

then for all $x_i \in X, (i = \overline{1, m})$ $(m \ge 2)$, such that either

$$a \ge \prod_{i=1}^{k} \|x_i, a_2, \dots, a_n\|^4 \text{ or } c \ge \prod_{i=k+1}^{m} \|x_i, a_2, \dots, a_n\|^4,$$

where $1 \le k \le m$, we have the inequality

$$\begin{aligned} &\left(a - \prod_{i=1}^{k} \|x_i, a_2, \dots, a_n\|^4\right)\left(c - \prod_{i=k+1}^{m} \|x_i, a_2, \dots, a_n\|^4\right) \\ &\qquad \le (b \pm \widetilde{\Gamma}(x_1, x_2, \dots, x_m \mid a_2, \dots, a_n))^2. \end{aligned} \tag{4.4}$$

Proof. Fix $k \in \{1, 2, \dots, m\}$ and suppose that $a \ge \prod_{i=1}^{k} \|x_i, a_2, \dots, a_n\|^4$. Consider the polynomial

$$P(t) = at^2 - 2bt + c, \qquad t \in \mathbb{R}.$$

Since $a > 0$ and $b^2 \geq ac$, it follows that there exist a $t_0 \in \mathbb{R}$ such that $P(t_0) = 0$. Now put

$$Q_1(t) = P(t) - \Big[\prod_{i=1}^{k} \|x_i, a_2, \dots, a_n\|^4 t^2 \mp 2b\widetilde{\Gamma}(x_1, \dots, x_m \mid a_2, \dots, a_n)t + \prod_{i=k+1}^{m} \|x_i, a_2, \dots, a_n\|^4\Big]$$

for $\alpha \in \mathbb{R}$.

A simple calculation gives

$$Q_1(t) = \Big[\Big(a - \prod_{i=1}^{k} \|x_i, a_2, \dots, a_n\|^4\Big)t^2 - 2(b \pm \widetilde{\Gamma}(x_1, \dots, x_m \mid a_2, \dots, a_n))t + \Big(c - \prod_{i=k+1}^{m} \|x_i, a_2, \dots, a_n\|^4\Big)\Big] \text{ for } t \in \mathbb{R}.$$

By inequality in (4.3), one has:

$$\begin{aligned}\widetilde{\Gamma}^2(x_1, \dots, x_m \mid a_2, \dots, a_n) &\leq \prod_{i=1}^{m} \|x_i, a_2, \dots, a_n\|^2 \\ &= \Big(\prod_{i=1}^{k} \|x_i, a_2, \dots, a_n\|^4\Big)\Big(\prod_{i=k+1}^{m} \|x_i, a_2, \dots, a_n\|^4\Big),\end{aligned}$$

which gives

$$\Big[\Big(\prod_{i=1}^{k} \|x_i, a_2, \dots, a_n\|^4\Big)t^2 - 2\widetilde{\Gamma}(x_1, \dots, x_m \mid a_2, \dots, a_n)t + \prod_{i=k+1}^{m} \|x_i, a_2, \dots, a_n\|^4\Big] \geq 0,$$

for all t$\in \mathbb{R}$.

Since

$$Q_1(t_0) = -\Big[\prod_{i=1}^{k} \|x_i, a_2, \dots, a_n\|^4)t_0^2 - 2\widetilde{\Gamma}(x_1, \dots, x_m \mid a_2, \dots, a_n))t_0 + \prod_{i=k+1}^{m} \|x_i, a_2, \dots, a_n\|^4\Big] \leq 0,$$

we conclude that Q_1 has at least one solution in $\mathbb{R}$, i.e.,

$$0 \le \frac{1}{4}\Delta_1 = (b \pm \widetilde{\Gamma}(x_1, \dots, x_m \mid a_2, \dots, a_n))^2 - \Big(a - \prod_{i=1}^{k} \|x_i, a_2, \dots, a_n\|^4\Big)\Big(c - \prod_{i=k+1}^{m} \|x_i, a_2, \dots, a_n\|^4\Big),$$

and the theorem is proved. ∎

The following converse of inequality (4.3) holds.

COROLLARY 5. *Suppose that a, b, c and $x_i \in X$ $(i = \overline{1,m})$ are as above. Then one has the following inequality:*

$$\begin{aligned} 0 &\le \prod_{i=1}^{k} \|x_i, a_2, \dots, a_n\|^4 - \widetilde{\Gamma}^2(x_1, \dots, x_m \mid a_2, \dots, a_n) \\ &\le b^2 - ac + a \prod_{i=k+1}^{m} \|x_i, a_2, \dots, a_n\|^4 + c \prod_{i=1}^{k} \|x_i, a_2, \dots, a_n\|^4 \\ &\quad \pm 2\widetilde{\Gamma}(x_1, \dots, x_m \mid a_2, \dots, a_n) b. \end{aligned}$$

COROLLARY 6. *Suppose that $M > 0$ and $x_i \in X, (i = \overline{1,m})$, with the property that*

$$\prod_{i=1}^{k} \|x_i, a_2, \dots, a_n\|^2 \le M \ \text{ or } \prod_{i=k+1}^{m} \|x_i, a_2, \dots, a_n\|^2 \le M.$$

Then one has the inequality:

$$\begin{aligned} 0 &\le \prod_{i=1}^{m} \|x_i, a_2, \dots, a_n\|^4 - \widetilde{\Gamma}^2(x_1, \dots, x_m \mid a_2, \dots, a_n) \\ &\le M^2 [\prod_{i=1}^{k} \|x_i, a_2, \dots, a_n\|^4 + \prod_{i=k+1}^{m} \|x_i, a_2, \dots, a_n\|^4 \\ &\quad - 2\widetilde{\Gamma}(x_1, \dots, x_m \mid a_2, \dots, a_n)]. \end{aligned}$$

By a similar argument as in the proof of the last theorem, we also have:

THEOREM 6. *Let $(X; (.,. \mid .,\dots,.))$ be an $n-$ inner product space over the real or complex number field $\Bbbk$ and α, β, γ be the real numbers with $\alpha, \gamma > 0$ and $\beta^2 \ge \alpha\gamma$. Then for all $x_i \in X$ $(i = \overline{1,m})$ such that*

$$\prod_{i=1}^{k} \|x_i, a_2, \dots, a_n\|^2 \le \alpha \quad \text{or} \quad \prod_{i=k+1}^{m} \|x_i, a_2, \dots, a_n\|^2 \le \gamma \ \ (1 \le k \le m).$$

We have the inequality,

$$\left(\alpha - \prod_{i=1}^{k} \|x_i, a_2, \ldots, a_n\|^2\right)\left(\gamma - \prod_{i=k+1}^{m} \|x_i, a_2, \ldots, a_n\|^2\right)$$
$$\leq (\beta \pm |\widetilde{\Gamma}(x_1, \ldots, x_m \mid a_2, \ldots, a_n)|^{\frac{1}{2}})^2.$$

Now, by the use of the above theorem, we can also state the following converse of inequality (4.3)*:*

COROLLARY 7. *Suppose that α, β, γ and $x_i \in X$ $(i = \overline{1,m})$ are as above, then*

$$0 \leq \prod_{i=1}^{m} \|x_i, a_2, \ldots, a_n\|^2 - \widetilde{\Gamma}(x_1, \ldots, x_m \mid a_2, \ldots, a_n)$$
$$\leq \beta^2 - \alpha\gamma + \alpha \prod_{i=k+1}^{m} \|x_i, a_2, \ldots, a_n\|^2 + \gamma \prod_{i=1}^{k} \|x_i, a_2, \ldots, a_n\|^2$$
$$\pm 2\beta[\widetilde{\Gamma}(x_1, \ldots, x_m \mid a_2, \ldots, a_n)]^{\frac{1}{2}}.$$

COROLLARY 8. *Let $M > 0$ and $x_i \in X$, $(i = \overline{1,m})$, be such that*

$$\prod_{i=1}^{k} \|x_i, a_2, \ldots, a_n\|^2 \leq M \text{ or } \prod_{i=k+1}^{m} \|x_i, a_2, \ldots, a_n\|^2 \leq M,$$

then one has the inequality:

$$0 \leq \prod_{i=1}^{m} \|x_i, a_2, \ldots, a_n\|^2 - \widetilde{\Gamma}(x_1, \ldots, x_m \mid a_2, \ldots, a_n)$$
$$\leq M \begin{pmatrix} \prod_{i=1}^{k} \|x_i, a_2, \ldots, a_n\|^2 + \prod_{i=k+1}^{m} \|x_i, a_2, \ldots, a_n\|^2 \\ -2[\widetilde{\Gamma}(x_1, \ldots, x_m \mid a_2, \ldots, a_n)]^{\frac{1}{2}} \end{pmatrix}.$$

In addition, we note that the following inequality improving inequality (4.3) *also holds:*

$$\text{(4.5)} \quad \widetilde{\Gamma}(x_1, \ldots, x_m \mid a_2, \ldots, a_n)$$
$$\leq \widetilde{\Gamma}(x_1, \ldots, x_k \mid a_2, \ldots, a_n)\widetilde{\Gamma}(x_{k+1}, \ldots, x_m \mid a_2, \ldots, a_n).$$

(*see* [1]), *where $x_i \in X$, $(i = \overline{1,m})$ and $1 \leq k \leq m$.*

By the use of this inequality and a similar argument as above, we can obtain the following converses of (4.5).

(i) If $a \geq \widetilde{\Gamma}^2(x_1, \ldots, x_m \mid a_2, \ldots, a_n)$ or $c \geq \widetilde{\Gamma}^2(x_{k+1}, \ldots, x_m \mid a_2, \ldots, a_n)$ and $b^2 \geq ac > 0$, then

$$[a-\widetilde{\Gamma}^2(x_1,\dots,x_k \mid a_2,\dots,a_n)][c-\widetilde{\Gamma}^2(x_{k+1},\dots,x_m \mid a_2,\dots,a_n)]$$
$$\leq (b\pm\widetilde{\Gamma}(x_1,\dots,x_m \mid a_2,\dots,a_n))^2,$$

from which one can easily obtain

$$\begin{aligned}0 &\leq \widetilde{\Gamma}^2(x_1,\dots,x_k \mid a_2,\dots,a_n)\widetilde{\Gamma}^2(x_{k+1},\dots,x_m \mid a_2,\dots,a_n)\\ &\quad -\widetilde{\Gamma}^2(x_1,\dots,x_m \mid a_2,\dots,a_n)\\ &\leq b^2-ac+a\widetilde{\Gamma}^2(x_{k+1},\dots,x_m \mid a_2,\dots,a_n)\\ &\quad +c\widetilde{\Gamma}^2(x_1,\dots,x_k \mid a_2,\dots,a_n)\\ &\quad \pm 2\widetilde{\Gamma}(x_1,\dots,x_m \mid a_2,\dots,a_n)b.\end{aligned}$$

If $\widetilde{\Gamma}(x_1,\dots,x_k \mid a_2,\dots,a_n) \leq M$ or $\widetilde{\Gamma}(x_{k+1},\dots,x_m \mid a_2,\dots,a_n) \leq M$, then also

$$\begin{aligned}0 &\leq \widetilde{\Gamma}^2(x_1,\dots,x_k \mid a_2,\dots,a_n)\widetilde{\Gamma}^2(x_{k+1},\dots,x_m \mid a_2,\dots,a_n)\\ &\quad -\widetilde{\Gamma}^2(x_1,\dots,x_m \mid a_2,\dots,a_n)\\ &\leq M^2[\widetilde{\Gamma}^2(x_1,\dots,x_k \mid a_2,\dots,a_n)\\ &\quad +\widetilde{\Gamma}^2(x_{k+1},\dots,x_m \mid a_2,\dots,a_n)\\ &\quad -2\widetilde{\Gamma}(x_1,\dots,x_m \mid a_2,\dots,a_n)].\end{aligned}$$

(*ii*) If $\alpha,\gamma>0$ and $\beta^2\geq\alpha\gamma$ then for all $x_i\in H, (i=\overline{1,m})$ with $\alpha \geq \widetilde{\Gamma}(x_1,\dots,x_k \mid a_2,\dots,a_n)$ or $\gamma\geq\widetilde{\Gamma}(x_{k+1},\dots,x_m \mid a_2,\dots,a_n)$.

One has the inequality:

$$[\alpha-\widetilde{\Gamma}(x_1,\dots,x_k \mid a_2,\dots,a_n)][\gamma-\widetilde{\Gamma}(x_{k+1},\dots,x_m \mid a_2,\dots,a_n)]$$
$$\leq (\beta\pm|\widetilde{\Gamma}(x_1,\dots,x_m \mid a_2,\dots,a_n)|^{\frac{1}{2}})^2,$$

which gives the following converse of (4.5),

$$\begin{aligned}0 &\leq \widetilde{\Gamma}(x_1,\dots,x_k \mid a_2,\dots,a_n)\widetilde{\Gamma}(x_{k+1},\dots,x_m \mid a_2,\dots,a_n)\\ &\quad -\widetilde{\Gamma}(x_1,\dots,x_m \mid a_2,\dots,a_n)\\ &\leq \beta^2-\alpha\gamma+\alpha\widetilde{\Gamma}(x_{k+1},\dots,x_m \mid a_2,\dots,a_n)\\ &\quad +\gamma\widetilde{\Gamma}(x_1,\dots,x_k \mid a_2,\dots,a_n)\pm 2[\widetilde{\Gamma}(x_1,\dots,x_m \mid a_2,\dots,a_n)]^{\frac{1}{2}}\beta.\end{aligned}$$

If $\widetilde{\Gamma}(x_1,\dots,x_k \mid a_2,\dots,a_n) \leq M$ or $\widetilde{\Gamma}(x_{k+1},\dots,x_m \mid a_2,\dots,a_n) \leq M$, then also

$$\begin{aligned}0 &\leq \widetilde{\Gamma}(x_1,\dots,x_k \mid a_2,\dots,a_n)\widetilde{\Gamma}(x_{k+1},\dots,x_m \mid a_2,\dots,a_n)\\ &\quad -\widetilde{\Gamma}(x_1,\dots,x_m \mid a_2,\dots,a_n)\\ &\leq M[\widetilde{\Gamma}(x_1,\dots,x_k \mid a_2,\dots,a_n)\\ &\quad +\widetilde{\Gamma}(x_{k+1},\dots,x_m \mid a_2,\dots,a_n)-2[\widetilde{\Gamma}(x_1,\dots,x_m \mid a_2,\dots,a_n)]^{\frac{1}{2}}].\end{aligned}$$

5. Some applications

1. Suppose that $(X; (.,. \mid .,\dots,.))$ is an $n-$inner product space over the real or complex number field $\mathbb{k}$. If $x,\ y \in X$ and $M_1,\ M_2$ are real number such that

$$\|x, a_2, \dots, a_n\| \leq |M_1| \ \text{ or } \ \|y, a_2, \dots, a_n\| \leq |M_2|,$$

then the following generalization of Aczel's inequality in $n-$inner product spaces holds:

$$\begin{aligned}(M_1^2 - \|x, a_2, \dots, a_n\|^2)&(M_2^2 - \|y, a_2, \dots, a_n\|^2) \\ &\leq \min[(M_1M_2 - \operatorname{Re}(x, y \mid a_2, \dots, a_n))^2, \\ &\quad (M_1M_2 - |\operatorname{Re}(x, y \mid a_2, \dots, a_n)|)^2, \\ &\quad (M_1M_2 - \operatorname{Im}(x, y \mid a_2, \dots, a_n))^2, \\ &\quad (M_1M_2 - |\operatorname{Im}(x, y \mid a_2, \dots, a_n)|)^2, \\ &\quad (M_1M_2 - |(x, y \mid a_2, \dots, a_n)|)^2].\end{aligned}$$

This inequality is obvious from Theorem 3. We omit the details.

2. Suppose that $x, y \in X$ and $M_1, M_2 \in \mathbb{R}$ are as above. Then by the use of Theorem 4, we have the following interesting inequality of Aczel type in $n-$inner product spaces:

$$\begin{aligned}(M_1 - \|x, a_2, \dots, a_n\|)^{\frac{1}{2}}&(M_2 - \|y, a_2, \dots, a_n\|)^{\frac{1}{2}} \\ &\leq |M_1M_2|^{\frac{1}{2}} - |(x, y \mid a_2, \dots, a_n)|^{\frac{1}{2}},\end{aligned}$$

provided that

$$\|x, a_2, \dots, a_n\| \leq |M_1| \ \text{ and } \ \|y, a_2, \dots, a_n\| \leq |M_2|.$$

Note that if $a = (a_1, a_2, \dots, a_n), b = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n$ satisfy $a_1^2 - a_2^2 - \cdots - a_n^2 \geq 0$ and $b_1^2 - b_2^2 - \cdots - b_n^2 \geq 0$, then we have the inequality,

$$\begin{aligned}[|a_1| - (a_2^2 + \cdots + a_n^2)^{\frac{1}{2}}]&[|b_1| - (b_2^2 + \cdots + b_n^2)^{\frac{1}{2}}] \\ &\leq (|a_1b_1|^{\frac{1}{2}} - |a_2b_2 + \cdots + a_nb_n|^{\frac{1}{2}})^2.\end{aligned}$$

This is a new inequality of Aczel type for real numbers (see also [8]).

3. By the use of Corollary 2, we also have the following converse of Schwartz's inequality in $n-$inner product spaces:

$$\begin{aligned}0 &\leq \|x, a_2, \dots, a_n\|^2 \|y, a_2, \dots, a_n\|^2 - [\operatorname{Re}(x, y \mid a_2, \dots, a_n)]^2 \\ &\leq M^2 \min[\|x - y, a_2, \dots, a_n\|^2, \|x + y, a_2, \dots, a_n\|^2]\end{aligned}$$

provided that $x,\ y \in X$ with

$$\|x, a_2, \dots, a_n\| \leq M \ \text{ or } \ \|y, a_2, \dots, a_n\| \leq M.$$

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