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SPLIT QUATERNIONS AND THE LIE GROUP S_3^4

Abstract. In this work we showed that the pseudosphere S_3^4 is a Lie group. We obtained a Lie algebra and a Lie product rule of this group. Moreover examining actions of this group we gave some theorems.

1. Introduction

The only spheres which have Lie group structure in Euclidean space are the circle S^1 and the sphere S^3. S^1 is considered as the set of unit complex numbers, S^3 is considered as the set of unit quaternions and their group structures are constructed in [1].

Pseudo spheres which have Lie group structure in semi-Riemannian spaces are more than Euclidean spheres.

Lie group structures of the hyperbol $\overline{S_1^1}$, Lorentzian sphere the S_1^4 and the 2 winged hyperboloid H_0^2 are examined in [3], [4]. Also in [3] the group structure of the Lorentzian sphere S_1^4 constructed by means of quaternion product.

In this work we considered the pseudosphere S_3^4 in the semi-Euclidean space

$$R_3^5 = \left(R^5, (+,+,-,-,-)\right).$$

We defined a group operation on the pseudosphere S_3^4 by means of split quaternion product. We showed that S_3^4 is a Lie group together with this operation. We obtained the Lie algebra of this group. We found the rule of Lie product of left invariant vector fields. We gave some theorems. Furthermore we defined a $C^\infty-$ action which is transitive and effective from the Lie group S_3^4 onto the Lorentz manifold $S_3^4(r)$ with an arbitrary radius. We showed that some transformations which are obtained by means of this action are isometries. Finally, we showed how the orbits of points on $S_3^4(d)$ change when the action is restricted to geodesics on the Lie group S_3^4.

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In this work for $v = (v_1, ..., v_5)$, $w = (w_1, ..., w_5) \in R_3^5$, we take

$$\langle v, w \rangle = v_1 w_1 + v_2 w_2 - v_3 w_3 - v_4 w_4 - v_5 w_5.$$

If $\langle v, v \rangle > 0$, v is a space-like vector. If $\langle v, v \rangle < 0$, v is a time-like vector. If $\langle v, v \rangle = 0$, v is a null vector.

2. Split quaternions

DEFINITION 2.1. Let us consider

$$H' = \{a = a_0 1 + a_1 i + a_2 j + a_3 k : a_0, a_1, a_2, a_3 \in R\},$$

where the products of $\{1, i, j, k\}$ are given as in the following table

$\otimes$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	1	$-i$
k	k	j	i	1

Elements of H' are said to be split quaternions.

H' is an algebra. A norm of $a = a_0 1 + a_1 i + a_2 j + a_3 k$ is defined as

$$N(a) = a \otimes \overline{a} = a_0^2 + a_1^2 - a_2^2 - a_3^2,$$

where

$$\overline{a} = a_0 1 - a_1 i - a_2 j - a_3 k$$

is the conjugate of a. Thus we can consider the pseudosphere S_2^3 as the set of split quaternions with norm 1. That is

$$S_2^3 = \{a = a_0 1 + a_1 i + a_2 j + a_3 k : N(a) = 1\}.$$

S_2^3 is a Lie group with the split quaternion product operation. The Lie algebra of this group is the set of pure quaternions. This is isomorphic to the Lorentz space R_1^3. (See [5, 6]).

3. Lie group structure of pseudosphere S_3^4

DEFINITION 3.1. Note that

$$S_3^4 = \{X = (x_1, x_2, x_3, x_4, x_5) \in R_3^5 : x_1^2 + x_2^2 - x_3^2 - x_4^2 - x_5^2 = 1, \quad \text{i.e. } \langle X, X \rangle = 1\}.$$

Let us define the following operation on S_3^4,

$$\odot : S_3^4 \times S_3^4 \to S_3^4,$$

given by

$$X \odot Y = \left(\frac{\sqrt{1+x_5^2}\sqrt{1+y_5^2}+x_5y_5}{\sqrt{1+x_5^2}\sqrt{1+y_5^2}} X^* \otimes Y^*, x_5\sqrt{1+y_5^2}+y_5\sqrt{1+x_5^2} \right), \tag{3.1}$$

where $X^* = (x_1, x_2, x_3, x_4)$, $Y^* = (y_1, y_2, y_3, y_4)$ and $\otimes$ is the split quaternion product.

THEOREM 3.1. *$(S_3^4, \odot)$ is a Lie group.*

Proof. S_3^4 is a differentiable manifold. $e = (1,0,0,0,0) \in S_3^4$ is the unit element and for $X = (x_1, x_2, x_3, x_4, x_5)$, $X^{-1} = (x_1, -x_2, -x_3, -x_4, -x_5) \in S_3^4$. Hence the operation $\odot$ is differentiable. Thus S_3^4 is a Lie group. ■

COROLLARY 3.2. *The Lie group S_2^3 is a Lie subgroup of S_3^4.*

Proof. Substituting $X = (x_1, x_2, x_3, x_{4,}0)$ and $X = (y_1, y_2, y_3, y_{4,}0)$ into equation (3.1) gives $X \odot Y = X^* \otimes Y^*$. ■

4. Lie algebra of Lie group S_3^4

We can write the Lie group S_3^4 as parametrized by

$$S_3^4 = \left\{ \alpha(t, u_1, u_2, u_3) = \begin{pmatrix} \cosh t \cosh u_1 \cosh u_2 \cos u_3, \\ \cosh t \cosh u_1 \cosh u_2 \sin u_3, \\ \cosh t \cosh u_1 \sinh u_2, \\ \cosh t \sinh u_1, \\ \sinh t \end{pmatrix} \right\}. \tag{4.1}$$

Since unit element of Lie group S_3^4 is $e = (1,0,0,0,0)$, it is valid

$$T_e S_3^4 = Sp\{v_1, v_2, v_3, v_4\},$$

where

$$\begin{aligned} v_1 &= (0,0,0,0,1) \\ v_2 &= (0,0,0,1,0) \\ v_3 &= (0,0,1,0,0) \\ v_4 &= (0,1,0,0,0) \end{aligned}$$

The Lie algebra $T_e S_3^4$ is 4-dimensional. The space of left invariant vector fields of Lie group S_3^4, denoted by $\chi_l(S_3^4)$, then is isomorphic to $T_e S_3^4$:

$$\chi_l(S_3^4) \cong T_e S_3^4.$$

THEOREM 4.1. *Let $\chi_i = \chi_l(S_3^4)$ be defined by $X_i|_e = v_i$, $i = 1,2,3,4$. Then*

$$\chi_l(S_3^4) = Sp\{X_1, X_2, X_3, X_4\}.$$

The bracket product of X_i's are

$$[X_1, X_j] = 0, (j = 1, 2, 3, 4)$$
$$[X_2, X_3] = 2X_4,$$
$$[X_3, X_4] = -2X_2,$$
$$[X_4, X_2] = -2X_3.$$

Proof. The proof is straight forward. ∎

Since

$$[X_i, X_j] \mid_e = [v_i, v_j],$$

Lie product on the Lie algebra $T_e\left(S_3^4\right)$ of S_3^4 is given by

$$[v_1, v_j] = 0,$$
$$[v_2, v_3] = 2v_4,$$
$$[v_3, v_4] = -2v_2,$$
$$[v_4, v_2] = -2v_3.$$

Moreover, the Killing bilinear form of $T_e\left(S_3^4\right)$ is

$$K(X, Y) = \text{trace}\,(adX\,adY) = 8\left(\alpha_2\beta_2 + \alpha_3\beta_3 - \alpha_4\beta_4\right),$$

where

$$X = \sum_{i=1}^{4} \alpha_i v_i,$$
$$Y = \sum_{i=1}^{4} \beta_i v_i.$$

COROLLARY 4.2. *Since*

$$K(X, Y) = -8\langle X, Y\rangle$$

for $X, Y \in T_e\left(S_2^3\right)$, the transformation Ad is an isometry for the Lie group S_2^3.

Proof. Indeed

$$K(X, Y) = K(Ad_g X, Ad_g Y), \langle X, Y\rangle = \langle Ad_g X, Ad_g Y\rangle. \ ∎$$

5. C^∞ action of the group S_3^4 on the manifold $S_3^4(r)$

DEFINITION 5.1. Let us define the pseudo sphere $S_3^4(r)$ as

$$S_3^4(r) = \{X = (x_1, x_2, x_3, x_4, x_5) \in R_3^4 \mid x_1^2 + x_2^2 - x_3^2 - x_4^2 - x_5^2 = r^2, r > 0, \text{ i.e. } r \in R^+\}.$$

Let us consider the mapping

$$\theta : S_3^4 \times S_3^4(r) \rightarrow S_3^4(r),$$

for any

$$X = (x_1, x_2, x_3, x_4, x_5) \in S_3^4$$

and

$$Y = (y_1, y_2, y_3, y_4, y_5) \in S_3^4(r),$$

given by

$$\theta(X,Y) = \left(\frac{\sqrt{1+x_5^2}\sqrt{r^2+y_5^2}+x_5y_5}{\sqrt{1+x_5^2}\sqrt{r^2+y_5^2}} X^* \otimes Y^*, x_5\sqrt{r^2+y_5^2}+y_5\sqrt{1+x_5^2} \right),$$

where $X^* = (x_1, x_2, x_3, x_4)$, $Y^* = (y_1, y_2, y_3, y_4)$.

THEOREM 5.1. *The mapping θ, defined above, is a $C^\infty-$ action of the Lie group S_3^4 onto the manifold $S_3^4(r)$. This action is transitive and effective.*

Proof. (i) For $X, Y \in S_3^4$ and $P \in S_3^4(r)$, the equality

$$\theta(X, \theta(Y, P)) = \theta(X \odot Y, P)$$

holds.

(ii) For $e = (1, 0, ..., 0) \in S_3^4$ and each $P \in S_3^4(r)$, $\theta(e, P) = P$. There exists a unique $X \in S_3^4$ such that $\theta(X, P) = Q$ for all $P, Q \in S_3^4(r)$. Thus θ is transitive. ■

COROLLARY 5.2. *Denoting by $(S_3^4)_{\{P\}}$ the orbit of the point $P \in S_3^4(r)$ we have $(S_3^4)_{\{P\}} = S_3^4(r)$.*

COROLLARY 5.3. *For every $g \in S_2^3$, the mapping $\theta_g : S_3^4(r) \rightarrow S_3^4(r)$, given by $\theta_g(X) = \theta(g, X)$, is an isometry.*

Proof. Since for all $X, Y \in S_3^4(r)$

$$d(X,Y) = d(\theta_g(X), \theta_g(Y)), \ where \ d(X,Y) = \sqrt{|\langle X-Y, X-Y\rangle|},$$

θ_g is an isometry. ■

LEMMA 5.4. *If S^1 is a space-like geodesic on S_3^4 then $((S_3^4)_{S^1})_{\{p\}}$ is a space-like geodesic on $S_3^4(r)$.*

Proof. Let

$$P \in \left(S_3^4\right)_r, p_1^2 + p_2^2 - p_3^2 - p_4^2 - p_5^2 = r^2.$$

If we take $t = u_1 = u_2 = 0$ in equation (4.1) we obtain

$$\alpha(u_3) = S^1 = (\cos u_3, \sin u_3, 0, 0, 0).$$

We can write

$$\left((S_3^4)_{S^1}\right)_{\{p\}} = \left\{\theta\left(S^1, P\right) : S^1 = (\cos u_3, \sin u_3, 0, 0, 0), 0 \leq u_3 \leq 2\pi\right\}$$

and from this, we obtain

$$\theta\left(S^1,P\right)=\beta\left(u_3\right),\qquad \left\langle \beta^{'}\left(u_3\right),\beta^{'}\left(u_3\right)\right\rangle = r^2+p_5^2>0.$$

Hence $\beta^{'}\left(u_3\right)$ is a space-like vector and $\beta\left(u_3\right)$ is a space-like curve. If $u_1=u_2=u_3=0$ in parametrical expression (4.1), then we have

$$\alpha\left(t\right)=\left(cht,0,0,0,sht\right),t\in R.$$

This is the parametrical expression of $\overline{S_1^1}$, which is a time-like geodesic on S_3^4. ∎

We give the following lemma.

LEMMA 5.5. *If $\overline{S_1^1}$ is a time-like geodesic on S_3^4 then $((S_3^4)_{\overline{S_1^1}})_{\{p\}}$ is a time-like geodesic on $S_3^4(r)$.*

Proof. Let

$$P\in S_3^4(r),p_1^2+p_2^2-p_3^2-p_4^2-p_5^2=r^2.$$

Note that

$$((S_3^4)_{\overline{S_1^1}})_{\{P\}}=\{\theta(\overline{S_1^1},P):\overline{S_1^1}=(cht,0,0,0,sht),t\in R\}=\beta(t).$$

Note also that $\langle\beta^{'}(t),\beta^{'}(t)\rangle\ \langle\ 0$. Hence $\beta^{'}(t)$ is a time-like vector and thus $\beta(t)$ is a time-like curve. A null geodesic of S_3^4 given by parametrical expression (4.1) which starts from the point

$$e=(1,0,0,0,0)\in S_3^4$$

is the line

$$n=\{\alpha(m)=(1,\sqrt{3}m,m,m,m):m\in R\}$$

that lies on S_3^4. ∎

In this case, we give the following.

LEMMA 5.6. *Let $((S_3^4)_n)_{\{P\}}$ be the orbit of P with respect to the action $\theta\mid_n$. If n is a null geodesic on S_3^4, then in general, $((S_3^4)_n)_{\{P\}}$ is not a null geodesic on $S_3^4(r)$. But if $P\in S_2^3(r)$, then it is a null geodesic.*

Proof. Let $\beta(m)=((S_3^4)_e)_{\{P\}}$. Then, in general, the equality $\langle\beta^{'}(m),\beta^{'}(m)\rangle=0$ is not satisfied. If $P\in S_2^3(r)$, then $\langle\beta^{'}(m),\beta^{'}(m)\rangle=0$. Hence $\beta^{'}(m)$ is a null vector and $\beta(m)$ is a null geodesic. ∎

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