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APPLICATION OF THE FOURIER SERIES TO CONVEX CURVES WITH AXES OF SYMMETRY

Abstract. In this paper we consider a class $\mathcal{O}$ of all ovals. A sufficient and necessary condition for existence of axes of symmetry in the class $\mathcal{O}$ is given. Moreover, a form of Fourier series expansion of a support function of a curve $K \in \mathcal{O}$ is also given.

1. Preliminaries

Our geometric results complete and deepen the properties obtained in [2] and [3] and [4].

DEFINITION 1.1. A plane, closed, simple, positively oriented curve of positive curvature is said to be an oval (cf. [6]).

This article is concerned with ovals with axes of symmetry. The class of all ovals we denote by $\mathcal{O}$. Obviously each oval K: $z = z(t)$, $t \in R$ of the class $\mathcal{O}$ belongs to the class C^2 and R means real numbers.

We write down some sentences about a special parametrization by an oriented support function. This parametrization is a natural generalization of the ordinary one in [1], [3], [5] or [6]. Let us consider an oval K: $z = z(s)$ parametrized by arc lenght. Let a point O be the origin of our coordinate system and suppose that the curve K is considered in this system. Let us fix a point $z_o = z(s_o)$ and consider the tangent line at z_o. We can assume that z_o is chosen in such a way, that the tangent line is perpedicular to the x-axis. For an arbitrary point $z(s)$ we define a vector $e^{it} = \cos t + i \sin t$, where t is an oriented angle between the positive direction of the x-axis and the vector e^{it}. Now we consider an oriented distans $p(t)$ from the origin O of the coordinate system to the tangent line to K. Fix a point $z(s)$. Than we take e^{it} as normal vector ($n(s) = \frac{z''(s)}{\|z''(s)\|}$) to K at this point (the mark $'$ denotes in this article the differentiation with respect to the parameter t). If the vector e^{it} points to this half-plane which contains O then we put $p(t)$ equals the negative of the ordinary distance between O and the tangent line at $z(s)$. If not, we define $p(t)$ as the ordinary distance between O and the tangent line at $z(s)$. Since the oval K is convex and regular, the function $p(t)$, $t \in R$ has got the following properties:

1. it is a periodic function (the period $T = 2\pi$);
2. it is at least at the class C^1;
3. it is a positive one if only O$\in intV$, where $\partial V = K$.

Using $p(t)$ we obtain the special parametrization of K, given by

$$z(t) = p(t)e^{it} + p'(t)ie^{it}, \qquad \text{for } t \in R. \tag{1.1}$$

DEFINITION 1.2. The function p constructed above is called an oriented Minkowski support function.

Furthermore, we let S^1 denotes the boundary of the closed unit ball in the Euclidean 2-dimensional space centered at O, that is, the unit sphere in E^2. The spherical Lebesque measure on S^1 is denoted by σ. We also let $L(S^1)$ denote the class of integrable function on S^1, and $L_2(S^1)$ the class of square integrable functions on S^1. Thus, $L_2(S^1)$ consists of all real valued Lebesque integrable functions F on S^1 with the property that $\int_{S^1} F(u)^2 d\sigma(u) < \infty$.

DEFINITION 1.3. If $F \in L_2(S^1)$ and $H_0, H_1, \ldots$ is a given orthogonal sequence, then the numbers

$$\alpha_i = \frac{\langle F, H_i\rangle}{\|H_i\|^2} \tag{1.2}$$

are called *the Fourier coefficients of* F (with respect to the given orthogonal sequence), and the series

$$\sum_{i=1}^{\infty} \alpha_i H_i \tag{1.3}$$

is called *the Fourier series of* F (with respect to the sequence $H_0, H_1, \ldots$). We denote this fact by $F \sim \sum_{i=1}^{\infty} \alpha_i H_i$.

2. Conditions for existence of axes of symmetry

DEFINITION 2.1. Let $K \in \mathcal{O}$. With each support function p of K we associate a function $f : R \times R \to R$ given by the formula

$$f(t,a) = p(t) - p(\pi + 2a - t) - [p(a) - p(\pi + a)] \cos(t - a)\,. \tag{2.1}$$

THEOREM 2.2. *The function f does not depend on a choice of a support functtion.*

Proof. Let p, q be support functions of K. Then we have $q(t) = p(t) + A\cos t + B \sin t$ for some $A, B \in R$. We note that

$$q(t) - q(\pi + 2a - t) - [q(a) - q(\pi + a)] \cos(t - a)$$
$$= p(t) + A\cos t + B\sin t - p(\pi + 2a - t) - A\cos(\pi + 2a - t) - B\sin(\pi + 2a - t).$$

Let f be a continuous function on the closed interval $[a, b]$ and have a derivative at every x in the open interval (a, b). Then there is at least one number c in the open interval (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. Then

$$\begin{aligned}
-[p(a) + A\cos a + B\sin a - p(\pi + a) - A\cos(\pi + a) - B\sin(\pi + a)] \cos(t - a) \\
= p(t) - p(\pi + 2a - t) + A\cos t + B\sin t \\
+ A\cos(2a - t) + B\sin(2a - t) - [p(a) - p(\pi + a)] \cos(t - a) \\
- [A\cos a + B\sin a + A\cos a + B\sin a] \cos(t - a) \\
= p(t) - p(\pi + 2a - t) - [p(a) - p(\pi + a)] \cos(t - a) \\
+ A[\cos t + \cos(2a - t) - 2\cos a \cos(t - a)] \\
+ B[\sin t + \sin(2a - t) - 2\sin t \cos(t - a)] \\
= p(t) - p(\pi + 2a - t) - [p(a) - p(\pi + a)] \cos(t - a). \blacksquare
\end{aligned}$$

Now, we present a necessery and sufficient condition for existence of axes of symmetry of the curve K.

THEOREM 2.3. *A curve K has an axis of symmetry in direction ie^{ia} if and only if*

$$f(t, a) = 0 \qquad \textit{for arbitrary } t \in R. \tag{2.2}$$

Proof. *Necessity.* We assume that K has an axis of symmetry in the direction ie^{ia}. We note that the symmetric point to $z(t)$ with respect to the axis of symmetry in the directtion ie^{ia} is a point $z(\pi + 2a - t)$. We will use the notation $\{u + iv, x + iy\} = uy - vx$ for arbitrary complex numbers $u + iv, x + iy$. Thus, we have

$$\{z(\pi + 2a - t) - z(t), e^{ia}\} = 0 \qquad \text{for arbitrary } t \in R. \tag{2.3}$$

Hence, we have

$$\begin{aligned}
0 &\equiv \{z(\pi + 2a - t), e^{ia}\} - \{z(t), e^{ia}\} \\
&= \{p(\pi + 2a - t) e^{i(\pi+2a-t)} + p'(\pi + 2a - t) ie^{i(\pi+2a-t)}, e^{ia}\} \\
&\quad - \{p(t)e^{it} + p'(t)ie^{it}, e^{ia}\} \\
&= p(\pi + 2a - t) \{e^{i(\pi+2a-t)}, e^{ia}\} + p'(\pi + 2a - t) \{ie^{i(\pi+2a-t)}, e^{ia}\} \\
&\quad - p(t)\{e^{it}, e^{ia}\} - p'(t)\{ie^{it}, e^{ia}\}
\end{aligned}$$

$$= -p(\pi+2a-t)\sin(\pi+a-t) - p'(\pi+2a-t)\cos(\pi+a-t)$$
$$- p(t)\sin(a-t) + p'(t)\cos(a-t)$$
$$= p(\pi+2a-t)\sin(a-t) + p'(\pi+2a-t)\cos(a-t)$$
$$- p(t)\sin(a-t) + p'(t)\cos(a-t)$$
$$= -[p(t)-p(\pi+2a-t)]\sin(a-t) + [p(t)-p(\pi+2a-t)]'\cos(a-t),$$

so

$$-[p(t)-p(\pi+2a-t)]\sin(a-t)\,[p(t)-p(\pi+2a-t)]'\cos(a-t) = 0 \tag{2.4}$$

for arbitrary $t \in R$.

We note that (2.4) can be treated as an ordinary differential equation. Solving this equation we obtain (2.2).

Sufficiency. We assume that a support function p of K satisfies the identity (2.2), i.e.

$$p(t) - p(\pi+2a-t) - [p(a)-p(\pi+a)]\cos(t-a) \equiv 0.$$

We prove that

$$\left\{\frac{z(\pi+2a-t)+z(t)}{2} - \frac{z(\pi+2a-s)+z(s)}{2}, ie^{ia}\right\} \equiv 0$$

for arbitrary $t, s \in R$. We have

$$\frac{1}{2}\{z(\pi+2a-t)+z(t)-(z(\pi+2a-s)+z(s)), ie^{ia}\}$$
$$= \frac{1}{2}[\{z(\pi+2a-t), ie^{ia}\} + \{z(t), ie^{ia}\} - \{z(\pi+2a-s), ie^{ia}\} - \{z(s), ie^{ia}\}]$$
$$= \frac{1}{2}[p(\pi+2a-t)\cos(a-\pi-2a+t) + p'(\pi+2a-t)\sin(a-\pi-2a+t)$$
$$+ p(t)\cos(a-t) + p'(t)\sin(a-t) - p(\pi+2a-s)\cos(a-\pi-2a+s)$$
$$- p'(\pi+2a-s)\sin(a-\pi-2a+s) - p(s)\cos(a-s) - p'(s)\sin(a-s)]$$
$$= \frac{1}{2}[(p(t)-p(\pi+2a-t))\cos(t-a) + (p(t)-p(\pi+2a-t))'(-\sin(t-a))$$
$$- (p(s)-p(\pi+2a-s))\cos(s-a)$$
$$- (p(s)-p(\pi+2a-s))'(-\sin(s-a))]$$
$$= \frac{1}{2}[(p(a)-p(\pi+a))\cos^2(t-a) + (p(a)-p(\pi+a))\sin^2(t-a)$$
$$- (p(a)-p(\pi+a))\cos^2(s-a) - (p(a)-p(\pi+a))\sin^2(s-a)] = 0.$$

Thus, there exists an axis of symmetry of K in the direction ie^{ia}. ■

3. Fourier series expansion of a support function of a convex body with axes of symmetry

It is clear from context which kind of Fourier series is meant. A routine calculations shows that the sequence $1, \cos t, \sin t, \cos 2t, \sin 2t, \dots$ is an orthogonal sequence as discussed in Preliminaries, where the points of S^1 are in the usual way identified with the angle t which, for purpose of integration, is assumed to range between 0 and 2π. Let $a \in R$ be fixed. We consider a function

$$f_a(t) = p(t) - p(\pi + 2a - t) - [p(a) - p(\pi + a)]\cos(t - a), \quad t \in R. \tag{3.1}$$

We will find a Fourier series expansion of f_a on the base of the properties of a support function p. Let the support function p of K have a Fourier series expansion, as follows

$$p(t) \sim \frac{1}{2}a_0 + \sum_{m\geq 1}(a_m \cos mt + b_m \sin mt). \tag{3.2}$$

Then we have

$$\begin{aligned}
&p(\pi + 2a - t)\\
&\quad\sim \frac{1}{2}a_0 + \sum_{m\geq 1}(\mathrm{a}_m \cos m(\pi + 2a - t) + \mathrm{b}_m \sin m(\pi + 2a - t))\\
&\quad= \frac{1}{2}a_0 + \sum_{m\geq 1}(\mathrm{a}_m \cos(m\pi + m(2a - t)) + \mathrm{b}_m \sin(m\pi + m(2a - t)))\\
&\quad= \frac{1}{2}a_0 + \sum_{m\geq 1}(-1)^m[a_m \cos(2ma - mt) + b_m \sin(2ma - mt)]\\
&\quad= \frac{1}{2}a_0 + \sum_{m\geq 1}(-1)^m[a_m(\cos 2ma \cos mt) + \sin 2ma \sin mt)\\
&\qquad+ b_m(\sin 2ma \cos mt - \cos 2ma \sin mt)]\\
&\quad= \frac{1}{2}a_0 + \sum_{m\geq 1}(-1)^m[(a_m \cos 2ma + b_m \sin 2ma)\cos mt\\
&\qquad+ (a_m \sin 2ma - b_m \cos 2ma)\sin mt]
\end{aligned}$$

and

$$\begin{aligned}
p(t) - p(\pi + 2a - t) \sim \sum_{m\geq 1}\{&[a_m - (-1)^m(a_m \cos 2ma + b_m \sin 2ma)]\cos mt\\
&+ [b_m - (-1)^m(a_m \sin 2ma - b_m \cos 2ma)]\sin mt\}.
\end{aligned}$$

Moreover, we have

$$p(a) \sim \frac{1}{2}a_0 + \sum_{m\geq 1}(a_m \cos ma + b_m \sin ma),$$

$$p(\pi + a) \sim \frac{1}{2}a_0 + \sum_{m\geq 1}(a_m \cos(m\pi + ma) + b_m \sin(m\pi + ma))$$

$$= \frac{1}{2}a_0 + \sum_{m\geq 1}(-1)^m(a_m \cos ma + b_m \sin ma)$$

and

$$p(a) - p(\pi + a) \sim \sum_{m\geq 1}(1 - (-1)^m)(a_m \cos ma + b_m \sin ma)$$

$$= 2\sum_{k\geq 1}(a_{2k-1} \cos(2k-1)a + b_{2k-1} \sin(2k-1)a) = c.$$

Thus, for a fixed a, we have

$$\begin{aligned}(3.3)\quad f_a(t) \sim\ & (a_1 + a_1 \cos 2a + b_1 \sin 2a - c\cos a)\cos t\\ &+ (b_1 + a_1 \sin 2a - b_1 \cos 2a - c\sin a)\sin t\\ &+ \sum_{m\geq 2}\{[a_m - (-1)^m(a_m \cos 2ma + b_m \sin 2ma)]\cos mt\\ &+ [b_m - (-1)^m(a_m \sin 2ma - b_m \cos 2ma)]\sin mt\}.\end{aligned}$$

If $f_a(t) \equiv 0$ (i.e. K has an axis of symmetry), then we get

$$(3.4)\qquad \begin{cases} a_1 + a_1 \cos 2a + b_1 \sin 2a - c\cos a = 0,\\ b_1 + a_1 \sin 2a - b_1 \cos 2a - c\sin a = 0,\end{cases}$$

and

$$\begin{cases} [1 - (-1)^m \cos 2ma]\mathrm{a}_m - (-1)^m\mathrm{b}_m \sin 2ma = 0,\\ -(-1)^m\mathrm{a}_m \sin 2ma + (1 + (-1)^m \cos 2ma)\mathrm{b}_m = 0,\end{cases}$$

for $m \geq 2$.

We have a system of linear equations

$$(3.5)\qquad \begin{cases} ((-1)^m - \cos 2ma)\mathrm{a}_m - \mathrm{b}_m \sin 2ma = 0,\\ -\mathrm{a}_m \sin 2ma + ((-1)^m + \cos 2ma)\mathrm{b}_m = 0,\end{cases}$$

with the determinant of coefficients equals to 0. The solutions of (3.5) are of the form

$$(3.6)\qquad a_m \sin 2ma - ((-1)^m + \cos 2ma)b_m = 0.$$

If $m \equiv 0 (\mathrm{mod}\, 2)$ then we have

$$a_m \sin 2ma - (1 + \cos 2ma) b_m = 0,$$
$$a_m 2 \sin ma \cos ma - 2b_m \cos^2 ma = 0,$$
$$2 \cos ma (a_m \sin ma - b_m \cos ma) = 0.$$

Then

$$a_m \sin ma - b_m \cos ma = 0 \quad \text{for } \cos ma \neq 0. \tag{3.7}$$

For $m \geq 3$ and $m \equiv 1 (\mathrm{mod}\, 2)$ we have

$$a_m \sin 2ma + (1 - \cos 2ma) b_m = 0,$$
$$2a_m \sin ma \cos ma + 2b_m \sin^2 ma = 0,$$
$$2 \sin ma (a_m \cos ma + b_m \sin ma) = 0$$

which implies

$$a_m \cos ma + b_m \sin ma = 0 \quad \text{for } \sin ma \neq 0. \tag{3.8}$$

Of course, we see the other solutions of the system of linear equations (3.5), that is $\cos ma = 0$ for even m and $\sin ma = 0$ for odd m not less than 3. However, its must be omitted, because its make a sense of a as a function of independent integer variable m.

The above considerations lead us to the following theorem.

THEOREM 3.1. *Let p be a support function of $K \in \mathcal{O}$. If $f_a \equiv 0$, then the Fourier coefficients of p satisfy the conditions:*

a) *If $m \equiv 0(\mathrm{mod}\, 2)$, then $a_m \sin ma - b_m \cos ma = 0$ and $\cos ma \neq 0$;*

b) *If $m \geq 3$ and $m \equiv 1(\mathrm{mod}\, 2)$, then $a_m \cos ma + b_m \sin ma = 0$ and $\sin ma \neq 0$.*

Theorem 3 allow us to costruct examples of curves with axes of symmetry in the given directions. Namely

THEOREM 3.2. *If a curve $K \in \mathcal{O}$ has axes of symmetry in the directions ie^{ia} and $ie^{i\beta}$, the $\beta - \alpha$ is commensurable with π.*

Proof. Axess of symmetry of K have the directions ie^{ia} and $ie^{i\beta}$, then by Theorem 3, we have

- let $m \equiv 0(\mathrm{mod}\, 2)$. Then

$$\begin{cases} \mathrm{a}_m \sin ma - \mathrm{b}_m \cos ma = 0, \\ \mathrm{a}_m \sin m\beta - \mathrm{b}_m \cos m\beta = 0. \end{cases} \tag{3.9}$$

The determinant of this system is equal $\sin m(\beta - a)$ and it must be zero. Thus $\beta - a$ is commensurable with π.

- if $m \geq 3$ and $m \equiv 1(\mathrm{mod}\, 2)$, then

$$\begin{cases} \mathrm{a}_m \cos ma + \mathrm{b}_m \sin ma = 0, \\ \mathrm{a}_m \cos m\beta + \mathrm{b}_m \sin m\beta = 0. \end{cases} \tag{3.10}$$

The determinant of this system is equal $\sin m(\beta - a)$ and it must be zero too. Thus, once again $\beta - a$ is commensurable with π. ∎

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