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JORDAN STRUCTURE ON PRIME RINGS WITH CENTRALIZERS

Abstract. Our object in this paper is to study the generalization of Borut Zalar result in [1] on Jordan centralizer of semiprime rings by prove the following result: Let R be a prime of characteristic different from 2, and U be a Jordan ideal of R. If T is an additive mapping from R to itself satisfying the following condition

$$T(ur + ru) = uT(r) + T(r)u,$$

then $T(ur) = uT(r)$, for all $r \in R$, $u \in U$.

1. Introduction

Let R be an associative ring with the center Z. R is called prime ring if $aRb = 0$ implies $a = 0$ or $b = 0$ and semiprime if $aRb = 0$ implies $a = 0$. An additive subgroup U of R is said to be a Jordan ideal of R if $ur + ru \in U$, for all $u \in U$, $r \in R$. An additive mapping $T : R \to R$ which satisfies $T(xy) = T(x)y$ ($T(xy) = xT(y)$), for all $x, y \in R$, then T is left (right) centralizer. A centralizer is both left and right centralizer. Many studies were done on Jordan structure of an associative ring and also on Jordan structure of an associative ring with derivation see [2], [3], [5], [6]. But now we want to study the Jordan structure of an associative ring with centralizer. If we introduce a new product in R given by $x \circ y = xy + yx$, then Jordan derivation is an additive mapping D which satisfies

$$D(x \circ y) = D(x) \circ y = x \circ D(y), \text{ for all } x, y \in R$$

and Jordan homomorphism is an additive mapping A which satisfies

$$A(x \circ y) = A(x) \circ A(y), \text{ for all } x, y \in R.$$

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Therefore we can define Jordan centralizer to be an additive mapping T which satisfies

$$T(x \circ y) = T(x) \circ y = T(x) \circ y = x \circ T(y), \text{ for all } x, y \in R.$$

An easy computation shows that every centralizer is Jordan centralizer. In [1] Borut Zalar proved that every Jordan centralizer of semiprime ring of characteristic different from 2 is a centralizer. Our object in this paper is to study the generalization of this result on Jordan ideal of prime ring by proving the following result; Let R be a prime ring of characteristic different from 2, and U be a Jordan ideal of R. Let T be an additive mapping from R to itself satisfying the following condition

$$T(ur + ru) = uT(r) + T(r)u,$$

then $T(ur) = uT(r)$, for all $r \in R$, $u \in U$.

Now we must give some useful theorems need it to prove the main theorem in this paper.

THEOREM A [5]. *Let R be a semiprime ring of 2-tortion free, then any non-zero Jordan ideal of R contains a non-zero ideal of R.*

THEOREM B [2]. *Let R be a semiprime ring of 2-tortion free. Suppose that $a \in R$, such that a commutes with every $[a, x]$, $x \in R$, then $a \in Z$.*

THEOREM C [7]. *Let R be a semiprime ring of characteristic different from 2, and U be a Jordan ideal of R, suppose that $t \in R$, such that t commute with u^2 for all $u \in U$, then t commute with every element of U.*

THEOREM D [7]. *Let R be a semiprime ring of characteristic different from 2, and U be a Jordan ideal of R. Suppose that $t \in R$, such that commutes with every element of $[U, U]$, then t commutes with every element of U.*

2. Results

Let R be an associative prime ring of characteristic different from 2, U be a Jordan ideal of R, and T be an additive mapping from R into itself satisfying the condition

$$T(ur + ru) = uT(r) + T(r)u, \text{ for all } r \in R,\ u \in U. \qquad (*)$$

In particular, if $r = u$ in equation $(*)$, then we get

$$2T(u^2) = T(u)u + uT(u), \text{ for all } u \in U.$$

Put

$$(u)^r = T(ur) - uT(r), \text{ for all } r \in R \text{ and } u \in U,$$

and

$$(r)^u = T(ru) - T(r)u, \text{ for all } r \in R \text{ and } u \in U.$$

So we can show by using equation $(*)$ that: $(u)^r = -(r)^u$.

The following lemmas help us to prove the main theorem in this paper.

LEMMA 1. *For all $r \in R$ and $u \in U, T(uru) = uT(r)u$.*

Proof. In $(*)$, replace r by $u.2r + 2ru$, then we get

$$T(u(u.2r + 2ru) + (u.2r + 2ru)u) = uT(u.2r + 2ru) + T(u.2r + 2ru)u,$$
$$T(2u^2r + 4uru + 2ru^2) = 2u^2T(r) + 4uT(r)u + 2T(r)u^2,$$
$$T(2u^2.r + r.2u^2) + 4T(uru) = 2u^2T(r) + 4uT(r)u + 2T(r)u^2.$$

Since $2u^2 \in U$ we have

$$2u^2T(r) + T(r)2u^2 + 4T(uru) = 2u^2T(r) + 4uT(r)u + 2T(r)u^2.$$

Then

$$4T(uru) = 4uT(r)u.$$

Since the characteristic of R is different from 2, we get

$$T(uru) = uT(r)u, \text{ for all } r \in R,\ u \in U. \ \blacksquare$$

If we replace u by $u + v$ in last equation, we get the following

COROLLARY 1. *For all $u, v, r \in R$*

$$T(urv + vru) = uT(r)v + vT(r)u.$$

LEMMA 2 ([5]). *For any $t \in R$, if $tv^2 + v^2t = 0$ for all $v \in U$, then $t = 0$.*

LEMMA 3. *For all $v \in U$ and $r \in R$, $[v^2, r](v^2)^r = 0$ and $(v^2)^r[v^2, r] = 0$.*

Proof. Since $T(ur + ru) = uT(r) + T(r)u$, for all $r \in R$, $u \in U$, we get $2T(u^2) = uT(u) + T(u)u$, for all $u \in U$. Then by using Theorem 1 in[4] T is centralizer on U.

$$\text{i.e. } (T(uv) - T(u)v) = 0, \text{ for all } u \in U.$$
$$\text{i.e. } (uv - vu)(T(uv) - T(u)v) = 0.$$

Replace u by $2vr + 2rv$, for all $r \in R$, we get

$$((2vr + 2rv)v - v(2vr + 2rv))(T((2vr + 2rv)v) - T(2vr + 2rv)v) = 0,$$
$$(2vrv + 2rv^2 - 2v^2r - 2vrv)(T(2vr + 2rv) - 2vT(r)v - 2T(r)v^2 = 0).$$

By using Lemma 1, we have

$$(2rv^2 - 2v^2r)(T(2rv^2) - 2T(r)v^2) = 0.$$

By using the relation $(u)^r = -(r)^u$ for all $u \in U$ and $r \in R$, we get

$$(2v^2r - 2rv^2)(T(2v^2r) - 2v^2T(r)) = 0.$$

Since R has characteristic different from 2, we get

$$(v^2r - rv^2)(T(v^2r) - v^2T(r)) = 0,$$

i.e. $[v^2, r](v^2)^r = 0$, for all $v \in U$ and $r \in R$. Similarly, we can prove $(v^2)^r\ [v^2, r] = 0$. ∎

After replacing r by $r + s$ for all $s \in R$ in Lemma 3 we get the following:

COROLLARY 2.

1. $[u^2, r](u^2)^s + [u^2, s](u^2)^r = 0$.
2. $(u^2)^s[u^2, r] + (u^2)^r[u^2, s] = 0$, *for all* $u \in U$ *and* $r, s \in R$.

LEMMA 4. *For all* $u \in U$ *and* $r \in R$, $(u^2)^r = 0$.

Proof. By (2) of Corollary 2, we have

$$(u^2)^s[u^2, r] + (u^2)^r[u^2, s] = 0, \quad \text{for all } u \in U \text{ and } r, s \in R. \tag{1}$$

So

$$[u^2, z](u^2)^r[u^2, s] + [u^2, z](u^2)^s[u^2, r] = 0.$$

By equation (1), we have

$$[u^2, r](u^2)^s[u^2, z] + [u^2, z](u^2)^s[u^2, r] = 0. \tag{2}$$

Replace z by zt in equation (2) and by Jacobi's identities, we get

$$[u^2, r](u^2)^s(z[u^2, t] + [u^2, z]t) + (z[u^2, t] + [u^2, z]t)(u^2)^s[u^2, r] = 0.$$

By equation (2), we get

$$[u^2, r](u^2)^s z[u^2, t] - z[u^2, r](u^2)^s[u^2, t] -$$
$$-[u^2, z](u^2)^s[u^2, r]t + [u^2, z](u^2)^s[u^2, r] = 0,$$

i.e.

$$[[u^2, r](u^2)^s, z][u^2, t] + [u^2, z](t(u^2)^s[u^2, r] - (u^2)^s[u^2, r]t) = 0,$$
$$[[u^2, r](u^2)^s, z][u^2, t] + [u^2, z](-t(u^2)^r[u^2, s] - (u^2)^r[u^2, s]t) = 0,$$
$$[[u^2, r](u^2)^s, z][u^2, t] + [u^2, z][(u^2)^r[u^2, s], t] = 0. \tag{3}$$

Replace z by $[u^2, z]$ in equation (3), we get

$$[u^2, z][u^2, z][(u^2)^r[u^2, s], t] + z[u^2, [u^2, z]][(u^2)^r[u^2, s], t] +$$
$$+z[[u^2, r](u^2)^s, [u^2, z]][u^2, t] + [[u^2, r](u^2)^s, z][u^2, z][u^2, t] = 0.$$

In view of equation (3), we get

$$[u^2, z][u^2, z][(u^2)^r[u^2, s], t] + [[u^2, r](u^2)^s, z][u^2, z][u^2, t] = 0.$$

Again in view of equation (3), we get

$$-[u^2, z][[u^2, r](u^2)^s, z][u^2, t] + [[u^2, r](u^2)^s, z][u^2, z][u^2, t] = 0,$$

i.e.

(4) $$[[[u^2,r](u^2)^s,z],[u^2,z]][u^2,t]=0, \text{ for all } r,s,t,z\in R, u\in U.$$

Replace t by ct in equation (4) and by Jacobi's identities, we get

$$[[[u^2,r](u^2)^s,z],[u^2,z]][u^2,ct]=0,$$

(5) $$[[[u^2,r](u^2)^s,z],[u^2,z]](c[u^2,t]+[u^2,c]t)=0.$$

In view of equation (4), the second term is zero and equation (5) becomes

$$[[[u^2,r](u^2)^s,z],[u^2,z]]R[u^2,t]=0.$$

Since R is prime ring, so

$$\text{either } [u^2,t]=0, \text{ for all } t\in R \text{ or } [[[u^2,r](u^2)^s,z],\ [u^2,z]]=0.$$

If $[u^2,t]=0$, then $u^2\in Z$. So, $(u^2)^r=0$, for all $u\in U$ and $r\in R$. If

$$[[[u^2,r](u^2)^s,z],[u^2,z]]=0, \text{ for all } r,s,z\in R,\ u\in U,$$

i.e.

(6) $$[[u^2,r](u^2)^s,z][u^2,z]=[u^2,z][[u^2,r](u^2)^s,z].$$

Put $t=z$ in equation (3), we get

$$[u^2,z][(u^2)^r[u^2,s],z]+[[u^2,r](u^2)^s,z][u^2,z]=0.$$

In view of equation (6), the last equation becomes

$$[u^2,z][(u^2)^r[u^2,s],z]+[u^2,z][[u^2,r](u^2)^s,z]=0.$$

By Jacobi's identities, we get

$$[u^2,z][(u^2)^r[u^2,s]+[u^2,r](u^2)^s,z]=0.$$

By equation (1), we get

$$[u^2,z][(u^2)^r[u^2,s]-[u^2,s](u^2)^r,z]=0,$$

i.e.

$$[u^2,z][[(u^2)^r,[u^2,s]],z]=0, \text{ for all } r,s,z\in R,\ u\in U.$$

Linearized on z, we get

(7) $$[u^2,z][[(u^2)^r,[u^2,s]],t]+\ [u^2,t][[(u^2)^r,\ [u^2,s]],z]=0,$$

for all $r,s,t,z\in R$, $u\in U$. Replace t by u^2t in equation (7), we get

$$[u^2,z](u^2[[(u^2)^r,[u^2,s]],t]+[[(u^2)^r,[u^2,s]],u^2]t)$$
$$+([u^2,u^2]t+u^2[u^2,t])[[(u^2)^r,[u^2,s]],\ z]=0.$$

Now, since $[u^2,u^2]=0$, then the third term is zero, and in view of equation (7) second term is zero so the last equation becomes

$$u^2[u^2,t][[(u^2)^r,[u^2,s]],z]+[u^2,z]u^2[[(u^2)^r,[u^2,s]],t]=0,$$

and in view of equation (7), we get

$$-u^2[u^2,z][[(u^2)^r,[u^2,s]],t]+[u^2,z]u^2[[(u^2)^r,[u^2,s]],t]=0,$$

i.e.

$$[[u^2,z],u^2][[(u^2)^r,[u^2,s]],t]=0, \text{ for all } r,s,t,z\in R, u\in U.$$

Then

$$[[u^2,z],u^2]R[[(u^2)^r,[u^2,s]],t]=0.$$

Since R is prime ring, then

$$\text{either } [[u^2,z],u^2]=0, \text{ for all } u\in U,\ r\in R,$$

$$\text{or } [[(u^2)^r,[u^2,s]],t]=0, \text{ for all } r,s,t\in R, u\in U.$$

If

$$[[u^2,z],u^2]=0, i.e. [u^2,\ [u^2,z]]=0,$$

by Theorem B, we get $u^2\in Z$, then $(u^2)^r=0$ for all $u\in U$, $r\in R$.

If

$$[[(u^2)^r,[u^2,s]],t]=0, \text{ for all } r,s,t\in R, u\in U,$$

then $[(u^2)^r,[u^2,s]]\in Z$, i.e. $(u^2)^r[u^2,s]-[u^2,s](u^2)^r\in Z$. Put $\alpha=(u^2)^r[u^2,s]$ and $\beta=[u^2,s](u^2)^r$. Now, trivially we have $\alpha^2=0$ and $\beta^2=0$, so $(\alpha-\beta)^3=\beta\alpha\beta-\alpha\beta\alpha$. Now, since $[(u^2)^r,[u^2,s]]\in Z$, then

$$[u^2,s][(u^2)^r,[u^2,s]]=[(u^2)^r,[u^2,s]][u^2,s].$$

By expanding and using Corollary 2 and Lemma 3 itself, we get

(8) $$-[u^2,s][u^2,s](u^2)^r=(u^2)^r[u^2,s][u^2,s].$$

And also, from $[(u^2)^r,[u^2,s]]\in Z$, we get

$$(u^2)^r[(u^2)^r,[u^2,s]]=[(u^2)^r,[u^2,s]](u^2)^r.$$

Also, by expanding and using Lemma 3 and Corollary 2, we get

(9) $$(u^2)^r(u^2)^r[u^2,s]=-[u^2,s](u^2)^r(u^2)^r.$$

Now,

$$\alpha\beta=((u^2)^r[u^2,s])([u^2,s](u^2)^r).$$

By equation (8), we get

$$\alpha\beta=-[u^2,s][u^2,s](u^2)^r(u^2)^r,$$

by equation (9), we get

$$\alpha\beta=([u^2,s](u^2)^r)((u^2)^r[u^2,s])=\beta\alpha.$$

So, $(\alpha-\beta)^3=0$. Now, since R is prime ring and $\alpha-\beta\in Z$, then $\alpha-\beta=0$, i.e.

(10) $$[(u^2)^r,\ [u^2,s]]=0, \text{ for all } r,s\in R,\ u\in U.$$

If we replace s by st in the equation (10), we get

$$[(u^2)^r, s[u^2, t] + [u^2, s]t] = 0.$$

By Jacobi's identities, we get

$$[(u^2)^r, s[u^2, t]] + [(u^2)^r, [u^2, s]t] = 0.$$

Also Jacobi's identities yields

$$s[(u^2)^r, [u^2, t]] + [(u^2)^r, s][u^2, t] + [(u^2)^r, [u^2, s]]t + [u^2, s][(u^2)^r, t] = 0.$$

In view of equation (10), the first and third terms of the last equation are zero so, we get

$$[(u^2)^r, s][u^2, t] + [u^2, s][(u^2)^r, t] = 0,$$

for all $r, s, t \in R$, $u \in U$. Put $s = [u^2, s]$ in last equation, we get

$$[(u^2)^r, [u^2, s]][u^2, t] + [u^2, [u^2, s]][(u^2)^r, t] = 0.$$

Again by equation (10), we get

$$[u^2, [u^2, s]][(u^2)^r, t] = 0, \text{ for all } r, s, t \in R, \ u \in U.$$

So,

$$[u^2, [u^2, s]]R[(u^2)^r, t] = 0.$$

Since R is prime ring, so either $[u^2, [u^2, s]] = 0$ or $[(u^2)^r, t] = 0$.

If $[u^2, [u^2, s]] = 0$ for all $s \in R$, $u \in U$, then by Theorem B we obtain that $u^2 \in Z$ and hence $(u^2)^r = 0$ for all $r \in R$, $u \in U$. In case $[(u^2)^r, t] = 0$, for all $t \in R$. i.e. $(u^2)^r \in Z$. Then by Lemma 3

$$(u^2)^r[u^2, r] = 0, \text{ for all } r \in R, u \in U.$$

So, if for some u and r, $(u^2)^r \neq 0$, since R is prime ring, then $[u^2, r] = 0$, so $u^2 \in Z$. We get $(u^2)^r = 0$. Hence $(u^2)^r = 0$, for all $u \in U$, $r \in R$. ∎

Now, we can prove the main theorem in this paper which state:

THEOREM. *Let R be a prime ring of characteristic different from 2, U be a Jordan ideal of R and $T : R \to R$ be an additive mapping such that*

$$T(ur + ru) = uT(r) + T(r)u, \qquad (*)$$

for all $u \in U$, $r \in R$. Then

$$T(ur) = uT(r), \quad \textit{for all } u \in U, r \in R.$$

Proof. Replace r by ur in equation $(*)$, then

(1) $$T(\mathrm{u}\,ur + uru) = uT(ur) + T(ur)u$$

So,

(2) $$T(\mathrm{u}\,ur + uru) = T(u^2r + uru) = T(u^2r) + uT(r)u$$

But, by Lemma 4

$$(u^2)^r = 0 = T(u^2r) - u^2T(r), \qquad \text{for all } u \in U, r \in R,$$

i.e. $T(u^2r) = u^2T(r)$. So, equation (2) becomes

$$T(u^2r + uru) = u^2T(r) + uT(r)u. \tag{3}$$

By comparing equation (1) and (3), we get

$$u^2T(r) + uT(r)u = uT(ur) + T(ur)u.$$
$$u(uT(r) - T(ur)) = (T(ur) - uT(r))u.$$
$$u.(u)^r + (u)^r.u = 0, \text{ for all } u \in U, r \in R.$$

Linearizing the above equation on u, we get

$$u.(v)^r + v.(u)^r + (u)^r.v + (v)^r.u = 0.$$

Replace v by $2v^2$, and use Lemma 4, we get

$$2v^2.(u)^r + 2(u)^r.v^2 = 0,$$

i.e.

$$v^2.(u)^r + (u)^r.v^2 = 0, \text{ for all } u, v \in U, r \in R,$$

and so by Lemma 2, we get $(u)^r = 0$, for all $u \in U$, $r \in R$, i.e. $T(ur) = uT(r)$, for all $u \in U$, $r \in R$. ∎

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