DEMONSTRATIO MATHEMATICA
Vol. XL No 3 2007

Mohammad Nawaz

QUASI-SYMMETRIC ⅂B -CATEGORIES

Abstract. This paper deals with ⅂B-categories where ⅂B is a quantaloid obtained from a right Gelfand quantale Q. Quantale is non-commutative extension of concept of locale. A notion of sheaf for Q is introduced; it turns out that these sheaves are precisely quasi-symmetric skeletal Cauchy-complete ⅂B-categories. In particular if Q is a locale, this construction reduces to that given in *"Sheaves and Cauchy-complete Categories"* by R.F.C. Walters.

1. Introduction

Quantum theory asserts that most pairs of observations are incompatible, and can not be made simultaneously (Principle of Non-commutativity of Observations) [2]. Referring to this principle, in a well circulated note summarizing observations made during Category Meeting at Oberwolfach in September 1983 (a more developed form of this note was presented in Topology Meeting in Taormina, Sicily in April, 1984 and was published in 1986 [11]), C. J. Mulvey asks that with what rules of deduction are physical observations naturally manipulated. He proposes a logical operation & which is not to be assumed commutative. Citing few more instances, he conjectures that the logic which arises in these situations is the one characterized by a complete lattice together with an associative product which is distributive on both sides over arbitrary joins. He coins the term quantale for such lattices. He gave a talk on Quantales in 27th meeting of Peripatetic Seminars on Sheaves and Logic at Institut Poincare, Paris in first week of March 1984. However, C. J. Mulvey himself [12] and some authors [16, 17] refer to [11] as the article which introduced the concept of quantale. C. J. Mulvey does not require a quantale to possess unit for the (possibly non-commutative) multiplication. Hence his quantales are semigroups in *SUP*, the closed category of sup-lattices. He calls a quantale unital if it is equipped with unit for

2000 *Mathematics Subject Classification*: 06F07, 18B35, 18D05, 18D20.
Key words and phrases: quantales, quantaloids, bicategory, sheaf, Cauchy-complete, bimodule.

multiplication [10, 13]. Unital quantales are monoids in *SUP* and what [17, 20] refer to as quantales are unital quantales in the sense of Mulvey. An example of unital quantale from [7] is the set of all sup-lattice endomorphisms of a sup-lattice S, joins of this quantale are calculated pointwise, its multiplication is composition and its unit is the identity map on S. The lattice of two sided ideals *Tidl(R)* of a (possibly noncommutative) ring R (with unit) is also a unital quantale with top element as unit. *Lidl(R)*, the left ideals of R form a quantale with top as unit from left hand side. Similarly *Ridl(R)*, the right ideals of R form a quantale with top as unit from right. For details see [16].

Let A be a unital C^* algebra, then the closed linear subspaces of A form a unital quantale where join of a family of closed linear subspaces is given by taking the closure of algebraic sum, and the product & of any two closed linear subspaces of A is given by taking the closure of their algebraic product. Unit for this quantale is given by the closed linear subspace generated by the unit of A while the top element of this quantale is the C^* algebra A itself. Thus the top element is different from the unit. Let us write this quantale as *Max A*. Those elements of *Max A* on which the top element (of *Max A*) acts as a unit on the right, are exactly the closed right ideals of A and again form a quantale with respect to operations of *Max A*. However this quantale may no longer be unital. For any closed right ideal I of A, we have $I\&I = I$. Thus the closed right ideals of a C^* algebra A form an idempotent quantale in which the top element acts as unit from right. Following [10] we shall call such quantale as right Gelfand quantale (some authors [9] call such lattices as right-sided idempotent quantales). Similarly the lattice of closed left ideals of a C^* algebra is a quantale with respect to multiplication of left ideals; again the top element of the lattice is unit from left hand side.

For an ordinary category C, the hom functors $C(A,-)$ and $C(-,B)$ take values in *SET*, the category of abstract sets and functions. Category theorists have studied categories for which the hom functors *hom(A, -)* and *hom(-, B)* take values in a more abstract category V and have built a theory (called V-enriched category theory) which is parallel to ordinary category theory in many ways provided V is symmetric, monoidal and closed. Such categories are called V-categories [4, 8]. Some examples from [8] are: the *SET*-category which is just an ordinary small category; the *2*-category (where *2* is object of truth values) which is an ordinary poset; the R-category is an ordinary (generalized) metric space where R is the category whose objects are the extended non-negative reals $R^+ \cup \{\infty\}$ and the morphisms are the greater-than-or-equal-to relations, and an ordinary ring which is an *Ab*-category with one object.

A very special category of this type is quantaloid whose hom-sets are sup-lattices in which composition distributes on both sides over arbitrary suprema of morphisms [16], thus a quantaloid is a SUP-category. For any object a of a quantaloid, the homset $hom(a,a)$ is a unital quantale [15], hence a quantaloid with one object is a unital quantale as well. Simplest example of quantaloid is the category SUP. The category of relations in a Grothendieck topos is also a quantaloid [16]. Given a ring R (possibly non-commutative) with unit, a quantaloid with two objects 0 and 1 may be defined as having following hom-sets; $hom(0,0) =$ *additive* subgroups of R, $hom(0,1) = Lidl(R)$, $hom(1,0) = Ridl(R)$ and $hom(1,1) = Tidl(R)$, the identity arrows on *1* and *0* are R and the center $Z(R)$ of R respectively; details may be seen in [16, 17]. Another example of quantaloid is *Relations(H)* [21] for which objects are the elements of H and the hom-lattices are given by $hom(u,v) = \{w \in H/w \leq u \wedge v\}$ where H is a locale.

A V-functor f from V-category X to V-category Y assigns to every object a of X an object $f(a)$ of Y and to every morphism of $X(a,\ b)$, a morphism of $Y(f(a),\ f(b))$ subject to commutativity of certain well understood diagrams. A bimodule from a V-category X to a V-category Y is a V-valued relation from X to Y (for this reason, bimodules are also called profunctors, as, roughly speaking, a bimodule is to a functor what a relation is to a mapping). Lawere [8] mentions that the bimodules are sometimes called profunctors if the base category $V = SET$. A bimodule is given by a V-functor ϕ from $Y^{op} \times X$ to V together with morphisms

$$Y(y',y) \otimes \phi(y,x) \longrightarrow \phi(y',x),$$

$$\phi(y,x) \otimes X(x,x') \longrightarrow \phi(y,x').$$

Here $\otimes$ is tensor product in V which is just the cartesian product in case V is cartesian closed. A V-functor $X \longrightarrow Y$ can be seen as a bimodule by postcomposing it with the obvious Yoneda embedding.

Can we replace the symmetric monoidal closed category V by a general bicategory [1] B and develop a theory of B-categories? Answer to this question is in affirmative and quite good work has been done in this direction [16-20]. A lattice (in fact any poset) can be considered as a category, hence a quantaloid which is a SUP-enriched category becomes a bicategory. In this paper we obtain a bicategory (a quantaloid) ⌉B from a right Gelfand quantale Q, then define a ⌉B -category and introduce notion of sheaf on Q. We then generalize the construction given in [21] and prove that these sheaves are quasi-symmetric skeletal Cauchy-complete ⌉B -categories.

We begin the paper with definition of right Gelfand quantale and main results about such quantales, details of these results may be seen in [10, 14].

2. Right Gelfand quantales

A *right Gelfand quantale* [10] is a complete lattice Q equipped with a binary operation (possibly non-commutative) $\& : Q \times Q \longrightarrow Q$ satisfying:

i. $p \,\& \bigvee_I q_i = \bigvee_I (p \,\& q_i)$, $(\bigvee_I q_i) \,\&\, p = \bigvee_I (q_i \,\&\, p)$

ii. $(p \,\&\, q) \,\&\, r = p \,\&\, (q \,\&\, r)$

iii. $p \,\&\, p = p$

iv. $p \,\&\, \mathit{1} = p$

where $\mathit{1}$ is the top element of Q.

For any $p \in Q$, the left and right distributive laws give two order preserving maps $p \,\&- : Q \longrightarrow Q$ and $-\&p : Q \longrightarrow Q$ and two types of implications [9, 14] $p \rightarrow \cdot -$ and $p \cdot \rightarrow -$ as their right adjoints respectively. Results about right Gelfand quantales are summed up in the following proposition, details may be found in [14].

PROPOSITION. *For a right Gelfand quantale Q and $p, p', q, q', r, v \in Q$,*

1. $q \le q' \Rightarrow p \,\&\, q \le p \,\&\, q'$
2. $q \le q' \Rightarrow q \,\&\, p \le q' \,\&\, p$
3. $p \le p'$, $q \le q' \Rightarrow p \,\&\, q \le p' \,\&\, q'$
4. $p \,\&\, q \le p$
5. $p \le q \Rightarrow p = p \,\&\, q$
6. $p \wedge q \le p \,\&\, q$
7. $p \,\&\, 0 = 0 \,\&\, p = 0$
8. $p \le 1 \,\&\, p$
9. $r \le p \,\&q \Rightarrow r = r \,\&\, q$
10. $r \le p \rightarrow \cdot\, q \Leftrightarrow p \,\&\, r \le q$
11. $r \le p \cdot\!\rightarrow q \Leftrightarrow r \,\&\, p \le q$
12. $\mathit{1} \cdot\!\rightarrow p = p$
13. $\mathit{1} \rightarrow \cdot\, p \le p \le \hat{p}$ *(where $\hat{p} = \mathit{1} \,\&\, p$)*
14. $p \,\&\, q \,\&\, r = p \,\&\, r \,\&\, q$
15. *Call p two-sided if $p = \hat{p} = \mathit{1} \,\&\, p$. A right Gelfand quantale Q is a locale if and only if every element of Q is two-sided.*
16. $p \,\&\, q = p \wedge \hat{q}$
17. *If v is two-sided then $p \,\&\, v = p \,\&\, (p \rightarrow \cdot\, v)$.*
18. $p \,\&\, v \,\&\, (v \rightarrow \cdot\, p) = v \,\&\, p \,\&\, (v \rightarrow \cdot\, p)$ *and* $v \le p \rightarrow \cdot\, v$.
19. $p \rightarrow \cdot\, q$ *is two-sided.*
20. $\downarrow p = \{r \in Q \mid r \le p\}$ *is a right Gelfand quantale with the induced operations.*
21. $(p \,\&\, q) \rightarrow \cdot\, r = q \rightarrow \cdot\, (p \rightarrow \cdot\, r)$ *holds but* $(p \,\&\, q) \rightarrow \cdot\, r = p \rightarrow \cdot\, (q \rightarrow \cdot\, r)$ *may fail.*
22. $p \,\&\, (p \rightarrow \cdot\, q) \le p \,\&\, q$ *holds but* $p \,\&\, q \le p \,\&\, (p \rightarrow \cdot\, q)$ *may fail.*
23. $(p \rightarrow \cdot\, q) \,\&\, (q \rightarrow \cdot\, r) \le p \rightarrow \cdot\, r$

24. $\bigwedge_I (p_i \to \cdot\, q) = (\bigvee_I p_i) \to \cdot\, q$ *where* $\{p_i \mid i \in I\} \subseteq Q$
25. $(p \,\&\, q) \to \cdot\, r = q \to \cdot\, (p \to \cdot\, r)$
26. $p \,\&\, (p \to \cdot q) \leq p \,\&\, q$, ($\geq$ *holds if and only if* $p \,\&\, q \leq q \,\&\, p$)
27. $p \leq q \Leftrightarrow p \to \cdot\, q = 1$
28. $p \to \cdot\, q \leq p \to \cdot\, \hat{q}$
29. $p \to \cdot\, \hat{q} = \hat{p} \to \cdot\, \hat{q}$
30. $(p \to \cdot\, q) \,\&\, (q \to \cdot\, p) = (p \to \cdot\, \hat{q}) \,\&\, (q \to \cdot\, \hat{p}) \Leftrightarrow p \,\&\, q = q \,\&\, p$
31. $r \leq p \Rightarrow p \to \cdot\, q \leq r \to \cdot\, q$
32. $(p \to \cdot\, q) \,\&\, (r \to \cdot\, u) \leq (p \,\&\, r) \to \cdot\, (q \,\&\, u)$
33. $r \,\&\, (p \to \cdot\, q) \leq p \to \cdot\, (q \,\&\, r)$
34. $r \,\&\, (p \to \cdot\, q) \leq (p \,\&\, r) \to \cdot\, q$ ■

Further references on quantales are [3, 5].

3. Quantaloids from right Gelfand quantales

From a right Gelfand quantale Q we obtain a bicategory (1) $\urcorner\mathrm{B}$ as follows. Ob $\urcorner\mathrm{B}$: Elements of Q

$$\urcorner\mathrm{B}(p,q) : \{r \in Q \mid r \leq p \,\&\, q,\ r = p\&r\}$$

so that

Arrows: Elements $r \in Q$ such that $r \leq p \,\&\, q,\ r = p\&r$

2-Cells: order in Q.

The identity arrow of $\urcorner\mathrm{B}$ (p,p) is p itself, we denote it by I_p.

Composition: For $p,\ q, r \in Q$

$$\begin{array}{rcl} \urcorner\mathrm{B}\ (p,q) \times \urcorner\mathrm{B}(q,r) & \xrightarrow{C(p,q,r)} & \urcorner\mathrm{B}\ (p\ ,\ r) \\ (u,v) & \mapsto & u \,\&\, v \\ (u \leq u',\ v \leq v') & \mapsto & u \,\&\, v \leq u' \,\&\, v'. \end{array}$$

Left identity for this composition is consequence of the condition

$$r \leq p \,\&\, q,\ r = p \,\&\, r$$

imposed on the elements r of $\urcorner\mathrm{B}$ $(p,\ q)$ and the right identity follows from the fact that $r \leq p \,\&\, q \Rightarrow r = r \,\&\, q$.

We shall write $v \circ u$ for $u \,\&\, v$ whenever $u \,\&\, v$ stands for composition of arrows. It is evident that $\urcorner\mathrm{B}$ $(p,\ q)$ is a complete sup lattice. $\urcorner\mathrm{B}$ is therefore a SUP-enriched category that is a quantaloid. In [17] the author remarks that the quantaloid *Relations(H)* (as discussed above) equals its opposite. However the quantaloid $\urcorner\mathrm{B}$ does not necessarily equal its opposite as $p \in \urcorner\mathrm{B}$ $(p,\ q)$ and it is not necessary that p is also in $\urcorner\mathrm{B}$ $(q,\ p)$.

The quantaloid $\urcorner\mathrm{B}$ comes equipped with a contravariant functor,

$$(-)^* : \urcorner\mathrm{B} \longrightarrow \urcorner\mathrm{B}$$

which is an involution, is identity on objects and induces covariant functor

$$\urcorner\mathrm{B}\ (p,\ q) \longrightarrow \urcorner\mathrm{B}\ (q,\ p)$$
$$r \mapsto q \,\&\, r$$
$$r \leq r' \mapsto\ q \,\&\, r \leq\ q \,\&\, r'.$$

4. ⌝B-categories and sheaves

A ⌝B-*category* is a set X together with functions

$$e\ :\ X\ \longrightarrow Ob(\urcorner\mathrm{B})$$
$$d\ :\ X \times X \longrightarrow \mathrm{Mor}(\urcorner\mathrm{B})$$

satisfying

i. $d(x_1, x_2) : e(x_1) \longrightarrow e(x_2)$
ii. $I_{e(x)} \leq d(x, x)$
iii. $d(x_2, x_3) \circ d(x_1, x_2) \leq d(x_1, x_3)$.

A ⌝B-*functor* $F : X \longrightarrow Y$ is a function satisfying

i. $e(F(x)) = e(x)$
ii. $d(x_1, x_2) \leq d(F(x_1), F(x_2))$ for all $x_1, x_2 \in X$.

A ⌝B-category X is said to be *quasi-symmetric* if

$$(d(x', x))^* = d(x, x')$$

and *skeletal* if

$$e(x) = e(x') = w \quad \text{and} \quad I_w \leq d(x, x'),\ I_w \leq d(x', x) \Rightarrow x = x'.$$

A *bimodule* $\varphi : X \to Y$ where X and Y are ⌝B-categories is a function

$$\varphi : X \times Y \longrightarrow \mathrm{Mor}\,\urcorner\mathrm{B}$$

satisfying

i. $\varphi(x, y) : e(x) \longrightarrow e(y)$
ii. $\varphi(x, y) \circ\ d(x', x) \leq \varphi(x', y)$
iii. $d(y, y') \circ \varphi(x, y) \leq \varphi(x, y')$ for all $x, x' \in X$ and $y, y' \in Y$.

Given a bimodule $\varphi : X \longrightarrow Y$, the mapping

$$Y \times X \longrightarrow \mathrm{Mor}\,\urcorner\mathrm{B}\ ; (y, x) \mapsto (\varphi(x, y))^*$$

defines a bimodule $\varphi^* : Y \longrightarrow X$. We call φ *self-adjoint* (in notation $\varphi : X \twoheadrightarrow Y$) if

$$d(x, x') \leq \bigvee_y \varphi^*(y, x') \circ \varphi(x, y)$$

$$\varphi(x, y') \circ \varphi^*(y, x) \leq d(y, y') \quad \text{for all } x, x' \in X \text{ and } y, y' \in Y.$$

Any ⌝B-functor $F : X \longrightarrow Y$ gives rise to a self-adjoint bimodule $\bar{F}$ where

$$\bar{F}(x, y) = d(F(x), y).$$

Finally we define a $\urcorner$B-category X to be *Cauchy-complete* if every self-adjoint bimodule $Y \longrightarrow X$ arises from a $\urcorner$B-functor $F : Y \longrightarrow X$.

We shall write $D(x, x')$, $\Phi(x, y)$ for the elements representing $d(x, x')$, $\varphi(x, y)$ in Q. Following definitions agree with [6] when Q is a locale.

A *presheaf* on Q is a set A together with mappings

$$E : A \longrightarrow Q, \ \lfloor : Q \times A \longrightarrow A$$

satisfying

i. $(Ea) \lfloor a = a$

ii. $P \lfloor (q \lfloor a) = (p \,\&\, q) \lfloor a$

iii. $E(p \lfloor a) = p \,\&Ea.$

A *morphism of presheaves* is a mapping $f : A \longrightarrow B$ satisfying

i. $Ea = Ef(a)$

ii. $f(p \lfloor a) = p \lfloor f(a).$

E and $\lfloor$ induce a partial order $\prec$ on underlying set of a presheaf A given by

$$a \prec a' \Leftrightarrow Ea \leq Ea' \text{ and } a = Ea \lfloor a'.$$

Following lemma holds.

LEMMA. *$a \in A$ is join for $B \subseteq A$ iff a is an upper bound for B and $Ea = \bigvee_{b \in B} Eb$.* ∎

A presheaf A is a *sheaf* if every compatible family has a unique join where $B \subseteq A$ is compatible if $Eb \lfloor b' = Eb \,\&\, Eb' \lfloor b$ for all $b, b' \in B$.

We write *Sh(Q)* and *QSC* $\urcorner$B*-categories* respectively for the categories, sheaves on Q and morphisms, and quasi-symmetric skeletal Cauchy-complete $\urcorner$B-categories and $\urcorner$B-functors.

5. Sheaves as $\urcorner$B-categories

To a sheaf A we assign a $\urcorner$B-category LA with A as underlying set and

$$e(a) = Ea, \ d(a, a') = Ea \,\& \bigvee \{p \in Q \mid p \lfloor a = p \lfloor a'\}.$$

Note that each of $D(a, a') \lfloor a$ and $D(a, a') \lfloor a'$ is join for the compatible family

$$\{Ea \,\&\, p \lfloor a \mid p \lfloor a = p \lfloor a'\}$$

which is same as

$$\{E\ a \,\&\, p \lfloor a' \mid p \lfloor a = p \lfloor a'\}.$$

Thus

$$D(a, a') \lfloor a = D(a, a') \lfloor a'.$$

The $\urcorner$B-category defined above is quasi-symmetric, we have

PROPOSITION. *LA is Cauchy-complete and skeletal.*

Proof. If $\varphi : X \nrightarrow LA$ then for any $x \in X$ the family $\{\Phi(x,a) \downharpoonright a \mid a \in A\}$ is compatible. Let $\bar{x}$ be its join so that

$$E\bar{x} = \bigvee_{a\in A} \Phi(x,a) \quad \text{and} \quad \Phi(x,a) \downharpoonright a \prec \bar{x} \qquad \text{for all } a \in A.$$

Now

$$I_{e(x)} \leq d(x,x) \leq \bigvee_{a\in A} \varphi^*(a,x) \circ \varphi(x,a)$$

thus

$$Ex = \bigvee_{a\in A} \Phi(x,a) = E\bar{x} \quad \text{and} \quad \varphi(x,a) \leq d(\bar{x},a).$$

Further

$$\begin{aligned} d(\bar{x},a) &= d(\bar{x},a) \circ I_{e(\bar{x})} \\ &\leq \bigvee_{a'\in A} d(\bar{x},a) \circ \varphi^*(a',x) \circ \varphi(x,a') \\ &\leq \bigvee_{a'\in A} d(\bar{x},a) \circ d(a',\bar{x}) \circ \varphi(x,a') \quad \text{as } \varphi(x,a') \leq d(\bar{x},a') \\ &\leq \bigvee_{a'\in A} d(a',a)\ o\varphi(x,a') \\ &\leq \varphi(x,a). \end{aligned}$$

Thus $\varphi(x,a) = d(\bar{x},a)$ and therefore φ arises from $\urcorner$B-functor

$$X \longrightarrow LA; \qquad x \longmapsto \bar{x}.$$

To see that LA is skeletal, suppose

$$e(a) = e(a') = w$$

and

$$I_w \leq d(a,a'), I_w \leq d(a',a).$$

Then

$$Ea = D(a,a') \quad \text{and} \quad Ea \downharpoonright a = D(a,a') \downharpoonright a,$$

hence

$$a = D(a,a') \downharpoonright a' = Ea \downharpoonright a'.$$

So that $a \prec a'$. Similarly $a' \prec a$. ∎

On the other hand underlying set of quasi-symmetric skeletal Cauchy-complete $\urcorner$B-category X can be provided with presheaf structure as follows.

For $p \in Q$ and $x \in X$ consider, $u = p\ \&\ Ex$ and define a $\urcorner$B-category $\hat{u}$ with $\{*\}$ as underlying set and $e(*) = u, d(*,*) = u$. We obtain a self-adjoint bimodule $\varphi : \hat{u} \longrightarrow X$ given by

$$\varphi(*,x') = p\ \&\ D(x,x')$$

and hence a $\urcorner$B-functor $F : \hat{u} \longrightarrow X$ such that

$$p \,\&\, D(x, x') = d(F(*), x') \text{ for all } x' \in X.$$

Uniqueness of $F(*)$ follows from the fact that X is skeletal.

Now define $p \downharpoonleft x = F(*)$ and put $E = e$. We denote this presheaf by TX.

Note that $D(x, x')$ & $D(x, x'')$ and $d(x', x'') \circ d(x, x')$ determine same arrow in $\urcorner$B $(e(x), e(x''))$ for all $x, x', x'' \in X$.

Thus $D(x, x') \downharpoonleft x = D(x, x') \downharpoonleft x'$ and hence

$$D(x, x') = Ex \,\&\, \bigvee\{p \in Q \mid p \downharpoonleft x = p \downharpoonleft x'\}.$$

Therefore LTX is just X. Further if b, b' are in a compatible family $B \subseteq X$ then Eb & $Eb' = D(b, b')$, we have

PROPOSITION. *TX is a sheaf.*

Proof. Let $B \subseteq X$ be compatible. Consider

$$u = \bigvee_{b \in B} Eb, \text{ then } \varphi : \hat{u} \longrightarrow X \text{ given by } \varphi(*, x) = \bigvee\nolimits_{b \in B} D(b, x)$$

is self-adjoint bimodule.

Let $F : \hat{u} \longrightarrow X$ be corresponding $\urcorner$B-functor. We assert that $F(*)$is unique join for B. Then

$$\bigvee_{b \in B} Eb = E(F(*))$$

is clear. Since $Eb \in \urcorner$B $(Eb, F(*))$ for any $b \in B$, therefore

$$\begin{aligned} d(Eb \downharpoonleft F(*), x) &= d(F(*), x) \circ Eb \\ &= \bigvee_{b' \in B} d(b', x) \circ (Eb \,\&\, Eb') \\ &= \bigvee_{b' \in B} d(b', x) \circ d(b, b') \\ &\leq d(b, x). \end{aligned}$$

Thus $b = Eb \downharpoonleft F(*)$ because X is skeletal. Therefore $b \prec F(*)$. Hence $F(*)$ is join for B (Lemma). For uniqueness suppose c is another join for B then

$$\begin{aligned} d(c, x) &= d(c, x) \circ I_{e(c)} \\ &= \bigvee_{b \in B} d(c, x) \circ d(b, c) \\ &= \bigvee_{b \in B} d(b, x). \ \blacksquare \end{aligned}$$

The assignments L and T are functorial where for $f : A \longrightarrow B$ in Sh(Q), $Lf : LA \longrightarrow LB$; $a \longmapsto f(a)$; and to a ⌉B-functor $F : X \longrightarrow Y$ we assign $TF : TX \longrightarrow TY$; $x \longmapsto G(*)$ where $G : \hat{u} \longrightarrow TY$ is given by self- adjoint bimodule $\hat{u} \nrightarrow TY; (*, y) \longmapsto \bar{F}(x, y)$ (here $u = Ex$). We have already noted that $LTX = X$, it is also clear that T re-determines $\downharpoonleft$ of A for LA where A is a sheaf, so that $TLA = A$. Thus we have proved that

THEOREM *The Categories* Sh(Q) *and* QSC ⌉B*-categories are isomorphic.* ∎

Acknowledgements. Author is grateful to the referee for several fruitful suggestions that have made significant impact on the final presentation of this paper. Author is also grateful to Prof. Bernhard Banaschewski for inviting him to McMaster University Canada during last quarter of 1985 as research fellow funded by his NSERC operating grant. First draft of material on quantales and ⌉B-categories was written during that period. At that time very few people knew about Quantales and the term "quantaloid" had not been coined yet. Now that many people are working on quantales and the subject has taken new turn by introduction of concept of quantaloid, it is appropriate time to submit this work for interest of new researchers in the origin of the study of sheaves on a quantale. Author is grateful to his daughter Sadaf Abdul Rauf for preparing the tex format of this paper.

References

[1] J. Benabou, *Introduction to bicategoires*, Reports of the Midwest Category Seminar, Lecture Notes in Mathematics, Springer, Berlin, 1967.

[2] G. Birkhoff, J. V. Neumann, *The logic of quantum mechanics*, Ann. Math. 37, No. 4, October, 1936.

[3] F. Borceux, *About quantales and quantic spaces*, The University of Sydney, Sydney Category Seminar Reports, 1984.

[4] F. Borceux, I. Stubbe, *Short Introduction to Enriched Categories*, in: Current Research in Operational Quantum Logic, Kluwer Academic Publishers, Dordrecht, 2000, pp. 167–194.

[5] F. Borecux and G. Van Den Bossche, *Quantales and their sheaves*, Order 3, 61 (1986).

[6] M. P. Fourman and D. S. Scott, *Sheaves and Logic*, Applications of Sheaves, Lecture Notes in Mathematics, Springer, Heidelberg, 1979.

[7] D. Kruml, J . W. Palleteir, P. Resende, J. Rosicky, *On quantales and spectras of C^*-algebras*, Applied Categorical Structures Volume 11, Number 6, December 2003, pp. 543–560.

[8] F. W. Lawvere, *Metric spaces, generalized logic and closed categories*, Rend. Sem. Mat. Fuis. Milano 43, pp. 135–166.

[9] F. Miraglia, U. Solitro, *Sheaves on the right sided idempotent quantales*, Logic Journal of the IGPL Vol. No. 4, pp. 545–600, 1998.

[10] C. J. Mulvey, M. Nawaz, *Quantales: Quantal sets*, Theory and Decision Library, Series B: Mathematical and Statistical Methods Volume 32, Non-Classical Logics

And Their Applications To Fuzzy Subsets, edited by Ulrich Hohle and Eric Peter Klement, Kluwer Academic Publishers, Netherlands. pp. 159–217, 1995.

[11] C. J. Mulvey, &, Suppl. Rend. Circ. Mat. Palermo 12, pp. 99–104, 1986.

[12] C. J. Mulvey, J. W. Palletier, *On quantization of points*, J. Pure Appl. Algebra 175 (2001), 289–325.

[13] C. J. Mulvey, P. Resende, *A noncommutatice theory of Penrose tilings*, Intern. J. Theoret. Phys., vol. 44, Number 6, June 2005, pp. 655–689.

[14] M. Nawaz, *Quantales: Quantal Sets*, D. Phil. thesis, University of Sussex, September 1985.

[15] K. I. Rosenthal, *The Theory of Quantaloids*, Pitman Research Notes in Mathematics Series. Longman, Harlow 1996.

[16] I. Stubbe, *Categorical structures enriched in a quantaloid: categories and semicategories*, thesis, Universite de Catholique de Louvain, November, 2003.

[17] I. Stubbe, *Categorical structures enriched in a quantaloid: categories, distributors and functors*, Theory Appl. Categ. 14 (2005), 1–45.

[18] I. Stubbe, *Categorical structures enriched in a quantaloid: tensored and cotensored categories*, Theory Appl. Categ. 16 (2006), 283–306.

[19] I. Stubbe, *Categorical structures enriched in a quantaloid: regular presheaves, regular semicategories*, Cahiers Topologie Géom. Différentielle Catégoriques 46 (2005), 99–121.

[20] I. Stubbe, *Categorical structures enriched in a quantaloid: orders and ideals over a base quantaloid*, Applied Categorical Structures 13 (2005), 235–255.

[21] R. F. C. Walters, *Sheaves and Cauchy-complete categories*, Cahiers Topologie Géom. Différentielle vol. XXII-3 (1981), 283–286.

BALOCHISTAN UNIVERSITY OF INFORMATION TECHNOLOGY
AND MANAGEMENT SCIENCES (BUITMS)
Samungli Road
JINNAH TOWN, QUETTA, PAKISTAN
e-mail: nawaz_arain@hotmail.com

Received January 26, 2006; revised version December 16, 2006.