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PSEUDO BCK-SEMILATTICES

Abstract. Pseudo BCK-algebras are algebras $(A, \rightarrow, \rightsquigarrow, 1)$ of type $\langle 2,2,0\rangle$ which generalize BCK-algebras in such a way that if the operations $\rightarrow$ and $\rightsquigarrow$ coincide then $(A, \rightarrow, 1)$ is a BCK-algebra. They can be also viewed as $\{\rightarrow, \rightsquigarrow, 1\}$-subreducts of non-commutative integral residuated lattices. In the paper, we study pseudo BCK-algebras whose underlying posets are semilattices or lattices; we call them pseudo BCK-join-semilattices, pseudo BCK-meet-semilattices and pseudo BCK-lattices, respectively. After describing their congruence properties we deal mainly with prime deductive systems of pseudo BCK-join-semilattices.

1. Preliminaries

In the last years there appeared a number of algebraic structures which are non-commutative generalizations of known algebras related to logic such as pseudo MV-algebras, pseudo BL-algebras, pseudo MTL-algebras (also called weak pseudo BL-algebras), non-commutative residuated lattices, etc. In the logical context this means that the strong conjunction is not commutative and the implication splits into two ones. Accordingly, G. Georgescu and A. Iorgulescu [6] introduced pseudo BCK-algebras as an extension of BCK-algebras:

DEFINITION 1.1. A structure $(A, \leq, \rightarrow, \rightsquigarrow, 1)$, where $\leq$ is a binary relation on A, $\rightarrow$ and $\rightsquigarrow$ are binary operations on A, and 1 is a distinguished element of A, is called a *pseudo BCK-algebra* (pedantically, a *reversed left pseudo BCK-algebra* [10]) if it satisfies the following axioms, for all $x, y, z \in A$:

(I) $x \rightarrow y \leq (y \rightarrow z) \rightsquigarrow (x \rightarrow z)$, $x \rightsquigarrow y \leq (y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)$,
(II) $x \leq (x \rightarrow y) \rightsquigarrow y$, $x \leq (x \rightsquigarrow y) \rightarrow y$,
(III) $x \leq x$,

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(IV) $x \leq 1$,
(V) $x \leq y$ and $y \leq x$ imply $x = y$,
(VI) $x \leq y$ iff $x \to y = 1$ iff $x \rightsquigarrow y = 1$.

This definition is adopted from [10]. It is clear that pseudo BCK-algebras can be treated as pure algebras with binary operations $\to$ and $\rightsquigarrow$, and a constant 1, since the relation $\leq$, which is always a partial order with 1 as a greatest element, can be retrieved by (VI). Namely, if $(A, \leq, \to, \rightsquigarrow, 1)$ is a pseudo BCK-algebra then the algebra $(A, \to, \rightsquigarrow, 1)$ satisfies the following identities and quasi-identity:

$$(x \to y) \rightsquigarrow [(y \to z) \rightsquigarrow (x \to z)] = 1, \tag{1.1}$$

$$(x \rightsquigarrow y) \to [(y \rightsquigarrow z) \to (x \rightsquigarrow z)] = 1, \tag{1.2}$$

$$1 \to x = x, \tag{1.3}$$

$$1 \rightsquigarrow x = x, \tag{1.4}$$

$$x \to 1 = 1, \tag{1.5}$$

$$(x \to y = 1 \;\&\; y \to x = 1) \;\Rightarrow\; x = y. \tag{1.6}$$

Conversely, if $(A, \to, \rightsquigarrow, 1)$ is an algebra of type $\langle 2, 2, 0 \rangle$ satisfying (1.1)–(1.6) then the relation defined by $x \leq y$ iff $x \to y = 1$ (iff $x \rightsquigarrow y = 1$) is a partial order on A which makes $(A, \leq, \to, \rightsquigarrow, 1)$ a pseudo BCK-algebra.

Thus the class of all pseudo BCK-algebras—considered as algebras of type $\langle 2, 2, 0 \rangle$—is a quasi-variety. Since BCK-algebras agree with pseudo BCK-algebras satisfying $\to = \rightsquigarrow$, and BCK-algebras are not closed under homomorphic images, it follows that neither are pseudo BCK-algebras, and hence this quasi-variety is not a variety.

By a *bounded pseudo BCK-algebra* we mean an algebra $(A, \to, \rightsquigarrow, 0, 1)$ such that $(A, \to, \rightsquigarrow, 1)$ is a pseudo BCK-algebra the least element of which is 0.

The partial order $\leq$ given by (VI) has no particular properties because an arbitrary poset $(P, \leq)$ with a greatest element 1 becomes a BCK-algebra by setting $x \to y = 1$ for $x \leq y$, and $x \to y = y$ otherwise. Nevertheless, it may happen that the underlying poset of a given pseudo BCK-algebra is a semilattice or even a lattice which is the case that we are interested in.

A *pseudo BCK-join-semilattice* is an algebra $(A, \vee, \to, \rightsquigarrow, 1)$ such that $(A, \vee)$ is a join-semilattice and $(A, \to, \rightsquigarrow, 1)$ is a pseudo BCK-algebra, where $x \to y = 1$ iff $x \vee y = y$. It can be easily seen that an algebra $(A, \vee, \to, \rightsquigarrow, 1)$ of type $\langle 2, 2, 2, 0 \rangle$ is a pseudo BCK-join-semilattice if and only if $(A, \vee)$ is a join-semilattice and it satisfies the identities (1.1)–(1.5) and

$$x \vee [(x \to y) \rightsquigarrow y] = (x \to y) \rightsquigarrow y, \tag{1.7}$$

$$x \to (x \vee y) = 1. \tag{1.8}$$

Therefore, the class of all pseudo BCK-join-semilattices forms a variety.

Pseudo BCK-algebras and pseudo BCK-join-semilattices are strongly related to residuated lattices (see [12], [13]). Actually, every pseudo BCK-algebra is isomorphic to a $\{\to, \rightsquigarrow, 1\}$-subreduct of some (bounded integral) residuated lattice, where also existing finite joins are preserved, and hence every pseudo BCK-join-semilattice arises as a $\{\vee, \to, \rightsquigarrow, 1\}$-subreduct of a residuated lattice.

We say that a pseudo BCK-algebra $(A, \to, \rightsquigarrow, 1)$ is *commutative* if it satisfies the identities

$$(x \to y) \rightsquigarrow y = (y \to x) \rightsquigarrow x, \tag{1.9}$$

$$(x \rightsquigarrow y) \to y = (y \rightsquigarrow x) \to x. \tag{1.10}$$

The underlying poset $(A, \leq)$ is then a join-semilattice with

$$x \vee y = (x \to y) \rightsquigarrow y = (x \rightsquigarrow y) \to y, \tag{1.11}$$

so that commutative pseudo BCK-algebras are a particular case of pseudo BCK-join-semilattices. Moreover, for each $a \in A$, the interval $[a, 1]$ is a distributive lattice in which $x \wedge_a y = ((x \rightsquigarrow a) \vee (y \rightsquigarrow a)) \to a = ((x \to a) \vee (y \to a)) \rightsquigarrow a$.

The name *commutative* may seem to be misleading since pseudo BCK-algebras are non-commutative generalizations of BCK-algebras, but we use it as an obvious counterpart of well-known commutative BCK-algebras.

It was proved in [6] that bounded commutative pseudo BCK-algebras (called here *lattice-ordered* pseudo BCK-algebras) are termwise equivalent to *pseudo MV-algebras*—non-commutative generalizations of MV-algebras introduced by G. Georgescu and A. Iorgulescu [5] and independently by J. Rachůnek [16]. The equivalence with the standard signature $\{\oplus, ^-, ^\sim, 0, 1\}$ is given as follows: if $(A, \to, \rightsquigarrow, 0, 1)$ is a bounded commutative pseudo BCK-algebra and we put $x \oplus y = (x \rightsquigarrow 0) \to y = (y \to 0) \rightsquigarrow x$, $x^- = x \to 0$ and $x^\sim = x \rightsquigarrow 0$, then $(A, \oplus, ^-, ^\sim, 0, 1)$ is a pseudo MV-algebra, and the reverse passage from $(A, \oplus, ^-, ^\sim, 0, 1)$ to $(A, \to, \rightsquigarrow, 0, 1)$ is given by $x \to y = x^- \oplus y$ and $x \rightsquigarrow y = y \oplus x^\sim$.

Another equivalent of bounded commutative pseudo BCK-algebras represent R. Ceterchi's *pseudo Wajsberg algebras* (see [2]) which are algebras of signature $\{\to, \rightsquigarrow, ^-, ^\sim, 1\}$. As pseudo MV- and pseudo Wajsberg algebras are termwise equivalent, one readily sees that whenever $(A, \to, \rightsquigarrow, 0, 1)$ is a bounded commutative pseudo BCK-algebra then $(A, \to, \rightsquigarrow, ^-, ^\sim, 1)$—where $x^- = x \to 0$ and $x^\sim = x \rightsquigarrow 0$—is a pseudo Wajsberg algebra, and conversely, if $(A, \to, \rightsquigarrow, ^-, ^\sim, 1)$ is a pseudo Wajsberg algebra then $(A, \to, \rightsquigarrow, 0, 1)$ is a bounded commutative pseudo BCK-algebra with $0 = 1^- = 1^\sim$.

A *pseudo BCK-meet-semilattice* is an algebra $(A, \wedge, \to, \rightsquigarrow, 1)$ such that $(A, \wedge)$ is a meet-semilattice, $(A, \to, \rightsquigarrow, 1)$ is a pseudo BCK-algebra and $x \to$

$y = 1$ iff $x \wedge y = x$. It is not hard to show that an algebra $(A, \wedge, \rightarrow, \rightsquigarrow, 1)$ of type $\langle 2, 2, 2, 0 \rangle$ is a pseudo BCK-meet-semilattice if and only if $(A, \wedge)$ is a meet-semilattice and it satisfies the identities (1.1)–(1.5) and

$$x \wedge [(x \rightarrow y) \rightsquigarrow y] = x, \tag{1.12}$$

$$(x \wedge y) \rightarrow y = 1. \tag{1.13}$$

As a particular kind of these pseudo BCK-algebras we can mention *hoops* and *pseudo hoops* (see [1], [7]) that are naturally ordered integral residuated partially ordered monoids. Indeed, given a pseudo hoop $(A, \odot, \rightarrow, \rightsquigarrow, 1)$, then $(A, \wedge, \rightarrow, \rightsquigarrow, 1)$ is a pseudo BCK-meet-semilattice in which $x \wedge y = x \odot (x \rightsquigarrow y) = (x \rightarrow y) \odot x$. Note that this is a pseudo BCK-algebra with the condition (pP) in the sense of [10], i.e., $x \odot y = \min\{a \in A : x \leq y \rightarrow a\} = \min\{a \in A : y \leq x \rightsquigarrow a\}$ for all $x, y \in A$.

Finally, an algebra $(A, \vee, \wedge, \rightarrow, \rightsquigarrow, 1)$ is called a *pseudo BCK-lattice* if $(A, \vee, \wedge)$ is a lattice, $(A, \rightarrow, \rightsquigarrow, 1)$ is a pseudo BCK-algebra and $x \rightarrow y = 1$ iff $x \vee y = y$ (iff $x \wedge y = x$). Pseudo BCK-lattices form a variety that is axiomatized by the identities (1.1)–(1.5), (1.7) and (1.8), or (1.1)–(1.5), (1.12) and (1.13), respectively, and by the identities of lattices.

Of course, any pseudo MV-algebra is a (bounded commutative) pseudo BCK-lattice. Also pseudo hoops can provide an example of pseudo BCK-lattices: By a *Wajsberg pseudo hoop* [7] we mean a pseudo hoop satisfying the equations (1.9) and (1.10). The $\{\rightarrow, \rightsquigarrow, 1\}$-reduct of a Wajsberg pseudo hoop is a commutative pseudo BCK-algebra and, consequently, every Wajsberg pseudo hoop is a distributive lattice in which (1.11) holds for all x, y.

In the lemma below we list some basic properties of pseudo BCK-algebras that can be easily derived and will be used without explicit references:

LEMMA 1.2. *The following hold in every pseudo BCK-algebra:*

(1) $x \rightarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \rightarrow z)$,
(2) $y \leq x \rightarrow y$, $y \leq x \rightsquigarrow y$,
(3) $1 \rightarrow x = x$, $1 \rightsquigarrow x = x$,
(4) $[(x \rightarrow y) \rightsquigarrow y] \rightarrow y = x \rightarrow y$, $[(x \rightsquigarrow y) \rightarrow y] \rightsquigarrow y = x \rightsquigarrow y$,
(5) $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$, $x \rightsquigarrow y \leq (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y)$,
(6) $x \leq y$ *implies* $z \rightarrow x \leq z \rightarrow y$ *and* $y \rightarrow z \leq x \rightarrow z$, *the same for* $\rightsquigarrow$,
(7) *if* $\bigvee_{i \in I} x_i$ *exists then so does* $\bigwedge_{i \in I}(x_i \rightarrow y)$ *and*

$$\left(\bigvee_{i \in I} x_i\right) \rightarrow y = \bigwedge_{i \in I}(x_i \rightarrow y),$$

and the same holds for $\rightsquigarrow$.

2. Deductive systems and congruence kernels

Deductive systems play an important role in the study of BCK-algebras. The analogue for pseudo BCK-algebras was introduced in [9]:

Let $(A, \to, \rightsquigarrow, 1)$ be a pseudo BCK-algebra. We call $D \subseteq A$ a *deductive system* if

(DS1) $1 \in D$,
(DS2) for all $a, b \in D$, if $a \in D$ and $a \to b \in D$, then $b \in D$.

The condition (DS2) is equivalent to saying that $a \in D$ and $a \rightsquigarrow b \in D$ together imply $b \in D$. Moreover, every deductive system D of $(A, \to, \rightsquigarrow, 1)$ is an order-filter in $(A, \leq)$, i.e., D contains with any a also all $b \geq a$.

The set $\mathcal{DS}(A)$ of all deductive systems of $(A, \to, \rightsquigarrow, 1)$, partially ordered by inclusion, is an algebraic distributive lattice in which infima coincide with set-theoretical intersections. For any $\varnothing \neq X \subseteq A$, the set

$$D(X) = \{a \in A : x_1 \to (\ldots \to (x_n \to a) \ldots) = 1 \text{ for some } x_1, \ldots, x_n \in X \text{ and } n \in \mathbb{N}\}$$

is the smallest deductive system containing X. We write $D(x_1, \ldots, x_n)$ for $D(X)$ when $X = \{x_1, \ldots, x_n\}$.

For any $x, y \in A$ and $n \in \mathbb{N}_0$, we define $x \to^n y$ inductively as follows:

$$x \to^0 y = y, \quad x \to^{n+1} y = x \to (x \to^n y);$$

$x \rightsquigarrow^n y$ is defined analogously.

Hence for every $x \in A$,

$$D(x) = \{a \in A : x \to^n a = 1 \text{ for some } n \in \mathbb{N}\}.$$

A deductive system D of a pseudo BCK-algebra $(A, \to, \rightsquigarrow, 1)$ is said to be *compatible* provided

(DS3) for all $a, b \in A$, $a \to b \in D$ iff $a \rightsquigarrow b \in D$.

The compatible deductive systems agree with the congruence kernels. As a matter of fact, if D is a compatible deductive system then the relation Θ_D given by

$$(2.1) \qquad (a, b) \in \Theta_D \quad \text{iff} \quad a \to b \in D \text{ and } b \to a \in D$$

is a congruence whose kernel is D, i.e., $[1]_{\Theta_D} = \{a \in A : (a, 1) \in \Theta_D\} = D$. Conversely, the kernel $[1]_\Theta = \{a \in A : (a, 1) \in \Theta\}$ of every congruence Θ certainly is a compatible deductive system, however, it may occur that Θ is not determined by $[1]_\Theta$, i.e., $\Theta \neq \Theta_{[1]_\Theta}$.

Therefore, the lattice $\mathcal{CK}(A)$ of all compatible deductive systems (= congruence kernels) of a pseudo BCK-algebra $(A, \to, \rightsquigarrow, 1)$, in general, is not isomorphic to the congruence lattice $\mathrm{Con}(A)$.

2.1. Pseudo BCK-join-semilattices. We begin with recalling several universal algebraic notions (see e.g. [3]):

An algebra A from a variety $\mathscr{K}$ with a constant 1 is *weakly regular* if, for every $\Theta, \Phi \in \mathrm{Con}(A)$, $[1]_\Theta = [1]_\Phi$ implies $\Theta = \Phi$. $\mathscr{K}$ is weakly regular if and only if there exist binary terms $d_1, \dots, d_n$ for some $n \in \mathbb{N}$ such that $d_1(x,y) = \dots = d_n(x,y)$ is equivalent to $x = y$.

We say that A is *permutable at* 1 if $[1]_{\Theta\circ\Phi} = [1]_{\Phi\circ\Theta}$ for all $\Theta, \Phi \in \mathrm{Con}(A)$. It is known that A is permutable at 1 if and only if $[1]_{\Theta\vee\Phi} = [1]_{\Theta\circ\Phi}$ for all $\Theta, \Phi \in \mathrm{Con}(A)$.

An algebra A is *distributive at* 1 if $[1]_{\Theta\cap(\Phi\vee\Psi)} = [1]_{(\Theta\cap\Phi)\vee(\Theta\cap\Psi)}$ for all $\Theta, \Phi, \Psi \in \mathrm{Con}(A)$, and A is *arithmetical at* 1 if it is both permutable at 1 and distributive at 1. A variety $\mathscr{K}$ is arithmetical at 1 if and only if there exists a binary term t satisfying $t(x,x) = t(1,x) = 1$ and $t(x,1) = x$.

In [9] we proved that every variety of pseudo BCK-algebras is weakly regular and arithmetical at 1, and hence congruence distributive. The terms we used in [9] are $d_1(x,y) = x \to y$ and $d_2(x,y) = t(x,y) = y \to x$, thus also pseudo BCK-join-semilattices enjoy the mentioned properties:

THEOREM 2.1. *The variety $\mathscr{J}$ of all pseudo BCK-join-semilattices is weakly regular, arithmetical at* 1 *and congruence distributive.*

Although the join operation $\vee$ in pseudo BCK-join-semilattices is not a term operation in $\to$ and $\rightsquigarrow$, it turns out that the congruence kernels still are precisely the compatible deductive systems. In addition, since the variety $\mathscr{J}$ is weakly regular, there is a one-to-one correspondence between the congruence relations and the compatible deductive systems:

THEOREM 2.2. *Let $(A, \vee, \to, \rightsquigarrow, 1)$ be a pseudo BCK-join-semilattice. If D is a compatible deductive system then the relation Θ_D defined via* (2.1) *is a congruence on $(A, \vee, \to, \rightsquigarrow, 1)$ such that $[1]_{\Theta_D} = D$. Conversely, for every $\Theta \in \mathrm{Con}(A)$, $[1]_\Theta$ is a compatible deductive system and $\Theta_{[1]_\Theta} = \Theta$.*

Proof. Let D be a compatible deductive system of A. By [9], Θ_D is an equivalence relation which is compatible with $\to$ and $\rightsquigarrow$, and $[1]_{\Theta_D} = D$. It remains to show that Θ_D is compatible with $\vee$, too.

Suppose that $(a,b) \in \Theta_D$, i.e., $a \to b \in D$ and $b \to a \in D$. For any $c \in A$, we have

$$\begin{aligned}(a \vee c) \to (b \vee c) &= (a \to (b \vee c)) \wedge (c \to (b \vee c)) \\ &= (a \to (b \vee c)) \wedge 1 \\ &= a \to (b \vee c) \\ &\geq a \to b,\end{aligned}$$

so that $(a \vee c) \to (b \vee c) \in D$ since $a \to b \in D$. Analogously, we get $(b \vee c) \to (a \vee c) \in D$, and hence $(a \vee c, b \vee c) \in \Theta_D$ proving $\Theta_D \in \mathrm{Con}(A)$.

Conversely, if $\Theta \in \mathrm{Con}(A)$ then certainly $[1]_\Theta$ is a compatible deductive system. Since $\mathscr{J}$ is weakly regular and $[1]_\Theta$ is the kernel of both Θ and $\Theta_{[1]_\Theta}$, it follows that $\Theta = \Theta_{[1]_\Theta}$. ∎

Thus, for every pseudo BCK-join-semilattice $(A, \vee, \to, \rightsquigarrow, 1)$, the lattice $\mathcal{CK}(A)$ of all compatible deductive systems and the congruence lattice $\mathrm{Con}(A)$ are isomophic under the inverse mappings $D \mapsto \Theta_D$ and $\Theta \mapsto [1]_\Theta$.

PROPOSITION 2.3. *Let $(A, \vee, \to, \rightsquigarrow, 1)$ be a pseudo BCK-join-semilattice. Then $\mathcal{CK}(A)$ is a complete sublattice of $\mathcal{DS}(A)$.*

Proof. We start with proving that $\mathcal{CK}(A)$ is a sublattice of $\mathcal{DS}(A)$, i.e., we show that

$$[1]_\Theta \vee_{\mathcal{CK}} [1]_\Phi = [1]_\Theta \vee_{\mathcal{DS}} [1]_\Phi$$

for every $\Theta, \Phi \in \mathrm{Con}(A)$. Clearly, $[1]_\Theta \vee_{\mathcal{DS}} [1]_\Phi \subseteq [1]_\Theta \vee_{\mathcal{CK}} [1]_\Phi$. Conversely, take any $a \in [1]_\Theta \vee_{\mathcal{CK}} [1]_\Phi = [1]_{\Theta\vee\Phi} = [1]_{\Theta\circ\Phi}$. Then there exists $b \in A$ such that $(a, b) \in \Theta$ and $(b, 1) \in \Phi$. Since $(a, b) \in \Theta$ yields $(b \to a, 1) \in \Theta$, we have $b \to a \in [1]_\Theta$ and $b \in [1]_\Phi$ which along with $b \rightsquigarrow ((b \to a) \rightsquigarrow a) = 1$ means that $a \in D([1]_\Theta \cup [1]_\Phi) = [1]_\Theta \vee_{\mathcal{DS}} [1]_\Phi$.

Now, let $\{\Theta_i : i \in I\}$ be an arbitrary family of congruences on A. First of all, note that for every $n \in \mathbb{N}$, $n \geq 2$, we have $[1]_{\Theta_1\vee\cdots\vee\Theta_n} = [1]_{\Theta_1\circ\cdots\circ\Theta_n}$; this easily follows by induction on n from the permutability at 1.

Put $\Theta = \bigvee_{\mathrm{Con}}\{\Theta_i : i \in I\}$. It is clear that $\bigvee_{\mathcal{DS}}\{[1]_{\Theta_i} : i \in I\} \subseteq \bigvee_{\mathcal{CK}}\{[1]_{\Theta_i} : i \in I\} = [1]_\Theta$. Conversely, let $a \in [1]_\Theta$. Then $a \in [1]_{\Theta_{i_1}\circ\cdots\circ\Theta_{i_n}} = [1]_{\Theta_{i_1}\vee\cdots\vee\Theta_{i_n}}$ for some $i_1, \ldots, i_n \in I$, $n \in \mathbb{N}$. But we already know that $\mathcal{CK}(A)$ is a sublattice of $\mathcal{DS}(A)$, therefore $a \in [1]_{\Theta_{i_1}\vee\cdots\vee\Theta_{i_n}} = [1]_{\Theta_{i_1}} \vee_{\mathcal{CK}} \cdots \vee_{\mathcal{CK}} [1]_{\Theta_{i_n}} = [1]_{\Theta_{i_1}} \vee_{\mathcal{DS}} \cdots \vee_{\mathcal{DS}} [1]_{\Theta_{i_n}}$. This proves $[1]_\Theta \subseteq \bigvee_{\mathcal{DS}}\{[1]_{\Theta_i} : i \in I\}$. ∎

Let $(A, \to, \rightsquigarrow, 1)$ be an arbitrary pseudo BCK-algebra and $\varnothing \neq X \subseteq A$. The set

$$\langle X \rangle = \{a \in A : a \to x = x \text{ for all } x \in X\}$$

is called the *annihilator of* X. We proved in [9] that $\langle X \rangle \in \mathcal{DS}(A)$ and, moreover, if $D \in \mathcal{DS}(A)$ then $\langle D \rangle$ is the pseudocomplement of D in the lattice $\mathcal{DS}(A)$.

We show next that in case of pseudo BCK-join-semilattices, the pseudocomplements in $\mathcal{DS}(A)$ can alternatively be characterized as the so-called polars:

Given a pseudo BCK-join-semilattice $(A, \vee, \to, \rightsquigarrow, 1)$, by the *polar of* $\varnothing \neq X \subseteq A$ we mean the set

$$X^\delta = \{a \in A : a \vee x = 1 \text{ for all } x \in X\}.$$

We write x^δ instead of $\{x\}^\delta$. It is easily seen that $X^\delta = \bigcap\{x^\delta : x \in X\}$; other obvious properties are:

(a) $X \subseteq X^{\delta\delta}$,
(b) $X \subseteq Y$ implies $X^\delta \supseteq Y^\delta$,
(c) $X^{\delta\delta\delta} = X^\delta$.

PROPOSITION 2.4. *Let A be a pseudo BCK-join-semilattice. For every $\varnothing \neq X \subseteq A$, we have $X^\delta \in \mathcal{DS}(A)$ and $X^\delta = D(X)^\delta$. In addition, $D^\delta = \langle D \rangle$ whenever $D \in \mathcal{DS}(A)$.*

Proof. Take $x \in X$ and assume that $a \in x^\delta$ and $a \to b \in x^\delta$. Then $a \to b \leq a \to (b \vee x)$ implies $1 = (a \to b) \vee x \leq (a \to (b \vee x)) \vee x$, so $(a \to (b \vee x)) \vee x = 1$. But $a \to (b \vee x) \geq b \vee x \geq x$, and hence $a \to (b \vee x) = 1$, i.e., $a \leq b \vee x$. This yields $1 = a \vee x \leq b \vee x$ and $b \vee x = 1$ proving $b \in x^\delta$. Thus $x^\delta \in \mathcal{DS}(A)$, and consequently, $X = \bigcap\{x^\delta : x \in X\} \in \mathcal{DS}(A)$.

Now, let $a \in X^\delta$. Then $a^\delta \supseteq X^{\delta\delta} \supseteq X$ whence it follows $a^\delta \supseteq D(X)$ as a^δ is a deductive system. Therefore $a \in a^{\delta\delta} \subseteq D(X)^\delta$ showing $X^\delta \subseteq D(X)^\delta$. The other inclusion is a consequence of $X \subseteq D(X)$.

Finally, assume that D is a deductive system of A. Let $a \in D^\delta$. Then $x = 1 \to x = (a \vee x) \to x = a \to x$ for every $x \in D$, so that $a \in \langle D \rangle$. Conversely, if $a \in \langle D \rangle$ then for each $x \in D$ we have $a \vee x \in D$, and hence $1 = a \to (a \vee x) = a \vee x$, so $a \in D^\delta$. ∎

Observe that for a non-empty subset X which is not a deductive system we have $X^\delta \subseteq \langle X \rangle$ since $a \vee x = 1$ yields $x = (a \vee x) \to x = a \to x$, but the polar X^δ can differ from the annihilator $\langle X \rangle$:

EXAMPLE 2.5. The set $A = \{0, a, b, 1\}$ equipped with the operations $\to$ and $\rightsquigarrow$ given by the following tables is a pseudo BCK-join-semilattice where $0 < a < b < 1$:

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	a	a	1	1
1	0	a	b	1

$\rightsquigarrow$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	0	a	1	1
1	0	a	b	1

One readily sees that $a^\delta = \{1\}$, while $\langle a \rangle = \{b, 1\}$.

2.2. Pseudo BCK-meet-semilattices and pseudo BCK-lattices. This subsection is devoted to deductive systems of pseudo BCK-algebras which are meet-semilattices or lattices, respectively. The presence of the meet as a fundamental operation brings new congruence properties of these two varieties:

THEOREM 2.6. *The variety $\mathscr{M}$ of all pseudo BCK-meet-semilattices and the variety $\mathscr{L}$ of all pseudo BCK-lattices are weakly regular and arithmetical.*

Proof. The term

$$m(x,y,z) = ((x \to y) \rightsquigarrow z) \wedge ((z \to y) \rightsquigarrow x) \wedge ((x \to z) \rightsquigarrow z)$$

is the Pixley term for $\mathscr{M}$ as well as for $\mathscr{L}$. Indeed, we have

$$\begin{aligned} m(x,y,y) &= ((x \to y) \rightsquigarrow y) \wedge ((y \to y) \rightsquigarrow x) \wedge ((x \to y) \rightsquigarrow y) \\ &= ((x \to y) \rightsquigarrow y) \wedge x \\ &= x, \end{aligned}$$

$$\begin{aligned} m(x,y,x) &= ((x \to y) \rightsquigarrow x) \wedge ((x \to y) \rightsquigarrow x) \wedge ((x \to x) \rightsquigarrow x) \\ &= ((x \to y) \rightsquigarrow x) \wedge x \\ &= x \end{aligned}$$

and

$$\begin{aligned} m(y,y,x) &= ((y \to y) \rightsquigarrow x) \wedge ((x \to y) \rightsquigarrow y) \wedge ((y \to x) \rightsquigarrow x) \\ &= x \wedge ((x \to y) \rightsquigarrow y) \wedge ((y \to x) \rightsquigarrow x) \\ &= x. \ \blacksquare \end{aligned}$$

The description of congruence kernels in the varieties $\mathscr{M}$ and $\mathscr{L}$ is slightly more complicated than in case of pseudo BCK-join-semilattices. Specifically, a compatible deductive system is not necessarily a filter in the underlying meet-semilattice and hence not all compatible deductive systems are congruence kernels, and on the other hand, a compatible deductive system which is a filter need not be a congruence kernel:

EXAMPLE 2.7. Consider the BCK-lattice $(A, \vee, \wedge, \to, 1)$ from Figure 2.1(a) with the operation $\to$ given as follows:

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	c	1	1
b	0	a	1	1	1
c	0	a	c	1	1
1	0	a	b	c	1

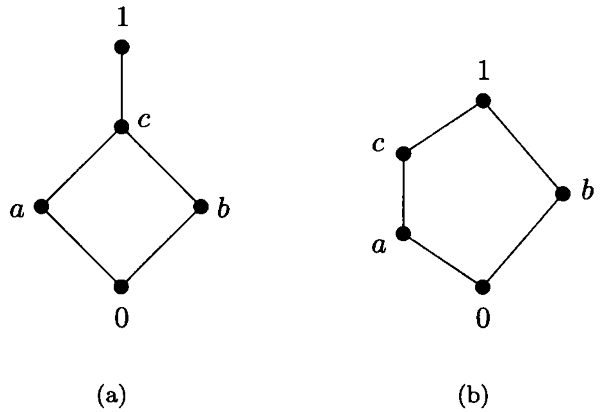

Fig. 2.1.

Then the equivalence relation Θ with the partition $\{0\}$, $\{a, b, c, 1\}$ is a congruence on $(A, \vee, \to, 1)$, but Θ is not a lattice congruence and its kernel $\{a, b, c, 1\}$ is not a lattice filter.

EXAMPLE 2.8. Let $(A, \vee, \wedge, \to, 1)$ be the BCK-lattice as shown in Figure 2.1(b), where $x \to y = 1$ if $x \leq y$, and $x \to y = y$ otherwise, i.e., $\to$ is given by the table

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	1	1
b	0	a	1	c	1
c	0	a	b	1	1
1	0	a	b	c	1

The set $D = \{c, 1\}$ is a filter and also a compatible deductive system, but it is not a congruence kernel. Indeed, suppose that $D = [1]_\Theta$ for some congruence Θ on $(A, \vee, \wedge, \to, 1)$. Then $(0, b) = (c \wedge b, 1 \wedge b) \in \Theta$ which yields $(0, 1) = (b \to 0, b \to b) \in \Theta$, a contradiction.

Note that the congruence Θ_D on $(A, \vee, \to, 1)$ defined via (2.1) has the partition $\{0\}$, $\{a\}$, $\{b\}$, $\{c, 1\}$.

THEOREM 2.9. *Let $(A, \wedge, \to, \rightsquigarrow, 1)$ be a pseudo BCK-meet-semilattice. Let D be a compatible deductive system satisfying the following additional condition:*

(DS4) *for all $a, b, c \in A$, if $c \to a \in D$ and $c \to b \in D$, then $c \to (a \wedge b) \in D$.*

Then the relation Φ_D *defined by*

$$(2.2) \qquad (a,b) \in \Phi_D \quad \textit{iff} \quad (a \to b) \wedge (b \to a) \in D$$

is a congruence on $(A, \wedge, \to, \rightsquigarrow, 1)$ *with* $[1]_{\Phi_D} = D$.

Conversely, for every congruence Θ *on* $(A, \wedge, \to, \rightsquigarrow, 1)$*, the kernel* $[1]_\Theta$ *is a compatible deductive system satisfying* (DS4) *and we have* $\Phi_{[1]_\Theta} = \Theta$.

Proof. Let D be a compatible deductive system that fulfils (DS4). First of all, when putting $c = 1$ in (DS4) we get that D is a filter. Therefore $(a,b) \in \Phi_D$ iff $(a \to b) \wedge (b \to a) \in D$ iff $a \to b, b \to a \in D$ iff $(a,b) \in \Theta_D$, so that $\Phi_D = \Theta_D$ and hence Φ_D is a congruence relation on $(A, \to, \rightsquigarrow, 1)$ the kernel of which is D. We show that Φ_D is compatible with $\wedge$.

Let $(a,b) \in \Phi_D$, $c \in A$. We have $(a \wedge c) \to c = 1 \in D$ and $(a \wedge c) \to b \in D$ since $(a \wedge c) \to b \geq a \to b \in D$. By (DS4) this implies $(a \wedge c) \to (b \wedge c) \in D$. Analogously, $(b \wedge c) \to (a \wedge c) \in D$, and so $(a \wedge c, b \wedge c) \in \Phi_D$.

Conversely, let $\Theta \in \mathrm{Con}(A)$. It is clear that $[1]_\Theta$ is a compatible deductive system. We prove that it enjoys the property (DS4). Assume that $(c \to a, 1) \in \Theta$ and $(c \to b, 1) \in \Theta$. Then $((c \to a) \rightsquigarrow a, a) \in \Theta$ whence $(c, c \wedge a) = (c \wedge ((c \to a) \rightsquigarrow a), c \wedge a) \in \Theta$, and similarly, $(c, c \wedge b) \in \Theta$. Thus $(c, a \wedge b \wedge c) \in \Theta$ which entails $(c \to (a \wedge b), 1) = (c \to (a \wedge b), (a \wedge b \wedge c) \to (a \wedge b)) \in \Theta$.

Consequently, since $[1]_\Theta$ is the kernel of Θ as well as of $\Phi_{[1]_\Theta}$, it follows that $\Theta = \Phi_{[1]_\Theta}$. ∎

The same result holds for pseudo BCK-lattices:

THEOREM 2.10. *Let* $(A, \vee, \wedge, \to, \rightsquigarrow, 1)$ *be a pseudo BCK-lattice. If* D *is a compatible deductive system satisfying the condition* (DS4) *then the relation* Φ_D *defined by* (2.2) *is a congruence on* $(A, \vee, \wedge, \to, \rightsquigarrow, 1)$ *such that* $[1]_{\Phi_D} = D$.

Conversely, for every congruence Θ *on* $(A, \vee, \wedge, \to, \rightsquigarrow, 1)$*, the kernel* $[1]_\Theta$ *is a compatible deductive system satisfying* (DS4) *and we have* $\Phi_{[1]_\Theta} = \Theta$.

3. Prime deductive systems

In this section we are concerned with those deductive systems of pseudo BCK-join-semilattices which are meet-prime elements of the lattice of deductive systems.

Let $(A, \vee, \to, \rightsquigarrow, 1)$ be a pseudo BCK-join-semilattice. We say that a deductive system P of A is *prime* if for every $X, Y \in \mathcal{DS}(A)$, $X \cap Y \subseteq P$ implies $X \subseteq P$ or $Y \subseteq P$.

Because of the distributivity of the lattice $\mathcal{DS}(A)$, the meet-primeness coincides with the meet-irreducibility, so that $P \in \mathcal{DS}(A)$ is prime if and only if, for all $X, Y \in \mathcal{DS}(A)$, $P = X$ or $P = Y$ whenever $P = X \cap Y$.

THEOREM 3.1. *Let $(A, \vee, \rightarrow, \rightsquigarrow, 1)$ be a pseudo BCK-join-semilattice. Let $D \in \mathcal{DS}(A)$ and let I be an ideal in the join-semilattice $(A, \vee)$ such that $D \cap I = \varnothing$. Then there exists a prime deductive system P satisfying $D \subseteq P$ and $I \cap P = \varnothing$.*

Proof. A routine application of Zorn's lemma yields that the set of all deductive systems having the required properties has a maximal element, say P. Assume that $P = X \cap Y$ for $X, Y \in \mathcal{DS}(A)$ with $P \subset X$ and $P \subset Y$. Then $I \cap X \neq \varnothing$ and $I \cap Y \neq \varnothing$ in view of the maximality of P, so there exist $x \in I \cap X$ and $y \in I \cap Y$ whence it follows $x \vee y \in I \cap X \cap Y = I \cap P = \varnothing$, a contradiction. Hence P is a prime deductive system. ∎

COROLLARY 3.2. *Let $(A, \vee, \rightarrow, \rightsquigarrow, 1)$ be a pseudo BCK-join-semilattice.*

(1) *If $D \in \mathcal{DS}(A)$ and $a \in A \setminus D$ then $D \subseteq P$ and $a \notin P$ for some prime deductive system P of A.*
(2) *Every deductive system of A is the intersection of all prime deductive systems containing it.*

Proof. (1) If $a \notin D$ then $D \cap (a] = \varnothing$, where $(a] = \{x \in A : x \leq a\}$ is an ideal in $(A, \vee)$, hence there exists a prime deductive system P such that $D \subseteq P$ and $P \cap (a] = \varnothing$, i.e. $a \notin P$.

(2) This follows easily from (1). ∎

The following technical lemma comes in useful:

LEMMA 3.3. *Let A be a pseudo BCK-join-semilattice. If $x \rightarrow^m a = 1$ and $y \rightarrow^n a = 1$ for $m, n \in \mathbb{N}$, then $(x \vee y) \rightarrow^r a = 1$ for some $r \in \mathbb{N}$. The same holds also for $\rightsquigarrow$.*

Proof. First note that $m \leq n$ entails $x \rightarrow^m a \leq x \rightarrow^n a$, and hence we may assume that $m = n$.

By induction on $n \in \mathbb{N}$. For $n = 1$ we have $x \rightarrow a = y \rightarrow a = 1$, so $x, y \leq a$ whence $x \vee y \leq a$ which is equivalent to $(x \vee y) \rightarrow a = 1$. Thus $r = 1$. Suppose that the statement holds for all $k \in \mathbb{N}$ with $k \leq n$. Let $x \rightarrow^{n+1} a = y \rightarrow^{n+1} a = 1$. From $y \rightarrow^{n+1} a = 1$ we obtain $y \rightarrow (y \rightsquigarrow^n a) = 1$, hence

$$y \rightarrow (x \rightsquigarrow^n (y \rightsquigarrow^n a)) = x \rightsquigarrow^n (y \rightarrow (y \rightsquigarrow^n a)) = x \rightsquigarrow^n 1 = 1. \tag{3.1}$$

From $y \rightsquigarrow^n a \geq a$ it follows $x \rightsquigarrow^{n+1} (y \rightsquigarrow^n a) \geq x \rightsquigarrow^{n+1} a = 1$, so that $x \rightsquigarrow^{n+1} (y \rightsquigarrow^n a) = 1$ which yields

$$x \rightarrow (x \rightsquigarrow^n (y \rightsquigarrow^n a)) = 1. \tag{3.2}$$

Now, by the first induction step and by (3.1), (3.2), we conclude that

$$x \rightsquigarrow^n (y \rightsquigarrow^n ((x \vee y) \to a)) = (x \vee y) \to (x \rightsquigarrow^n (y \rightsquigarrow^n a)) = 1,$$

and therefore

$$x \to (x \rightsquigarrow^{n-1} (y \rightsquigarrow^n ((x \vee y) \to a))) = 1. \tag{3.3}$$

Further, $(x \vee y) \to a \geq a$ entails $y \to^{n+1} ((x \vee y) \to a) \geq y \to^{n+1} a = 1$, so that $y \to^{n+1} ((x \vee y) \to a) = 1$ which is equivalent to $y \to (y \rightsquigarrow^n ((x \vee y) \to a)) = 1$. Hence

$$\begin{aligned}(3.4)\quad y \to (x \rightsquigarrow^{n-1} (y \rightsquigarrow^n ((x \vee y) \to a))) &= \\ = x \rightsquigarrow^{n-1} (y \to (y \rightsquigarrow^n ((x \vee y) \to a))) &= x \rightsquigarrow^{n-1} 1 = 1.\end{aligned}$$

Again by the first induction step, from (3.3) and (3.4) we obtain

$$\begin{aligned}x \rightsquigarrow^{n-1} (y \rightsquigarrow^n ((x \vee y) \to^2 a)) &= \\ = (x \vee y) \to (x \rightsquigarrow^{n-1} (y \rightsquigarrow^n ((x \vee y) \to a))) &= 1,\end{aligned}$$

whence

$$x \to (x \rightsquigarrow^{n-2} (y \rightsquigarrow^n ((x \vee y) \to^2 a))) = 1. \tag{3.5}$$

Analogously, $(x \vee y) \to^2 a \geq a$ implies $y \to^{n+1} ((x \vee y) \to^2 a) \geq y \to^{n+1} a = 1$, and consequently, $y \to (y \rightsquigarrow^n ((x \vee y) \to^2 a)) = 1$. This yields

$$\begin{aligned}(3.6)\quad y \to (x \rightsquigarrow^{n-2} (y \rightsquigarrow^n ((x \vee y) \to^2 a))) &= \\ = x \rightsquigarrow^{n-2} (y \to (y \rightsquigarrow^n ((x \vee y) \to^2 a))) &= x \rightsquigarrow^{n-2} 1 = 1.\end{aligned}$$

By (3.5) and (3.6),

$$\begin{aligned}x \rightsquigarrow^{n-2} (y \rightsquigarrow^n ((x \vee y) \to^3 a)) &= \\ = (x \vee y) \to (x \rightsquigarrow^{n-2} (y \rightsquigarrow^n ((x \vee y) \to^2 a))) &= 1.\end{aligned}$$

By repeating this procedure we gain

$$y \rightsquigarrow^n ((x \vee y) \to^{n+1} a) = 1,$$

and equivalently,

$$y \to^n ((x \vee y) \to^{n+1} a) = 1. \tag{3.7}$$

When interchanging x and y we have

$$x \to^n ((x \vee y) \to^{n+1} a) = 1. \tag{3.8}$$

Now we can apply the induction hypothesis to (3.7) and (3.8), so there exists $s \in \mathbb{N}$ such that $(x \vee y) \to^s ((x \vee y) \to^{n+1} a) = 1$ and hence $(x \vee y) \to^{s+n+1} a = 1$, i.e., $r = s + n + 1$. ∎

Proposition 3.4. *Let* $(A, \vee, \rightarrow, \rightsquigarrow, 1)$ *be a pseudo BCK-join-semilattice. Then*

$$D(x) \cap D(y) = D(x \vee y)$$

for every $x, y \in A$.

Proof. It is plain that $D(x \vee y) \subseteq D(x) \cap D(y)$ since $x \vee y \in D(x) \cap D(y)$. Conversely, $a \in D(x) \cap D(y)$ if and only if $x \rightarrow^m a = 1$ and $y \rightarrow^n a = 1$ for some $m, n \in \mathbb{N}$. But by the previous lemma there is $r \in \mathbb{N}$ such that $(x \vee y) \rightarrow^r a = 1$, so $a \in D(x \vee y)$. Hence $D(x) \cap D(y) \subseteq D(x \vee y)$. ∎

Proposition 3.5. *For any pseudo BCK-join-semilattice* A*, the compact elements of* $\mathcal{DS}(A)$ *form a sublattice of* $\mathcal{DS}(A)$.

Proof. Let X, Y be two compact elements of the lattice $\mathcal{DS}(A)$, i.e., $X = D(x_1, \ldots, x_m)$ and $Y = D(y_1, \ldots, y_n)$ for some $x_i, y_j \in A$, $m, n \in \mathbb{N}$. Due to the distributivity of $\mathcal{DS}(A)$ and using Proposition 3.4 we have

$$\begin{aligned} X \cap Y &= D(x_1, \ldots, x_n) \cap D(y_1, \ldots, y_n) \\ &= (D(x_1) \vee \cdots \vee D(x_m)) \cap (D(y_1) \vee \cdots \vee D(y_n)) \\ &= (D(x_1) \cap D(y_1)) \vee \cdots \vee (D(x_m) \cap D(y_n)) \\ &= D(x_1 \vee y_1) \vee \cdots \vee D(x_m \vee y_n) \\ &= D(x_1 \vee y_1, \ldots, x_m \vee y_n) \end{aligned}$$

which is a compact element of $\mathcal{DS}(A)$. Thus the finitely generated deductive systems of A form a sublattice of $\mathcal{DS}(A)$. ∎

Proposition 3.6. *Let* A *be a pseudo BCK-join-semilattice and* $P \in \mathcal{DS}(A)$. *Then* P *is prime if and only if, for all* $x, y \in A$,

$$x \vee y \in P \text{ implies } x \in P \text{ or } y \in P. \tag{3.9}$$

Proof. Assume that P is a prime deductive system and let $x \vee y \in P$. Applying Proposition 3.4 we have $D(x) \cap D(y) = D(x \vee y) \subseteq P$ which entails $D(x) \subseteq P$ or $D(y) \subseteq P$, thus $x \in P$ or $y \in P$.

Conversely, assume that P satisfies the condition (3.9). If $P = X \cap Y$, where both X and Y are deductive systems distinct from P, then there exist $x \in X \setminus P$ and $y \in Y \setminus P$. Obviously, $x \vee y \in X \cap Y = P$ which by (3.9) yields $x \in P$ or $y \in P$, a contradiction. ∎

A proper prime deductive system P of a pseudo BCK-join-semilattice A is called *minimal prime* if there is no prime deductive system Q of A such that $Q \subset P$.

Corollary 3.7. *Let* $(A, \vee, \rightarrow, \rightsquigarrow, 1)$ *be a pseudo BCK-join-semilattice and let* $\{P_i : i \in I\}$ *be any chain of prime deductive systems of* A. *Then* $P =$

$\bigcap_{i\in I} P_i$ is prime. Consequently, every prime deductive system of A contains a minimal prime deductive system.

Proof. Let $x \vee y \in P$. Take an arbitrary $i \in I$ and suppose that $x \notin P_i$. Then necessarily $y \in P_i$, whence we conclude $y \in P_j$ for every $j \in I$ with $P_i \subseteq P_j$. If $P_k \subseteq P_i$, $k \in I$, then $x \notin P_k$ (as otherwise $x \in P_i$) and hence $y \in P_k$. This means that $y \in P$. ∎

The minimal prime deductive systems are related to the polars of deductive systems:

PROPOSITION 3.8. *Let A be a pseudo BCK-join-semilattice.*

(1) *$P \in \mathcal{DS}(A)$ is minimal prime if and only if*

$$P = \bigvee_{\mathcal{DS}} \{D^\delta : D \in \mathcal{DS}(A), D \text{ is compact and } D \nsubseteq P\}.$$

(2) *For any $D \in \mathcal{DS}(A)$,*

$$D^\delta = \bigcap \{P \in \mathcal{DS}(A) : P \text{ is minimal prime and } D \nsubseteq P\}.$$

Proof. Since $\mathcal{DS}(A)$ is an algebraic distributive lattice whose compact elements (= finitely generated deductive systems) form a sublattice and since the polar D^δ is the pseudocomplement of $D \in \mathcal{DS}(A)$, the statements follow directly from [17], Lemma 2.3 and Lemma 2.4. See also [14], Corollary 2.5.1. ∎

THEOREM 3.9. *Let $(A, \vee, \rightarrow, \rightsquigarrow, 1)$ be a pseudo BCK-join-semilattice satisfying the identities*

$$(3.10)\qquad \begin{gathered}(x \rightarrow y) \vee (y \rightarrow x) = 1,\\ (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1.\end{gathered}$$

Then for any $P \in \mathcal{DS}(A)$, the following are equivalent:

(i) *P is prime;*
(ii) *for all $x, y \in A$, if $x \vee y \in P$ then $x \in P$ or $y \in P$;*
(iii) *for all $x, y \in A$, if $x \vee y = 1$ then $x \in P$ or $y \in P$;*
(iv) *for all $x, y \in A$, $x \rightarrow y \in P$ or $y \rightarrow x \in P$;*
(v) *for all $x, y \in A$, $x \rightsquigarrow y \in P$ or $y \rightsquigarrow x \in P$;*
(vi) *the set of all deductive systems containing P is a chain (under inclusion).*

Proof. (i) and (ii) are equivalent by Proposition 3.6. Obviously, (ii) implies (iii), and (iii) along with (3.10) implies (iv). Likewise, (vi) yields (i) by Corollary 3.2 (2) and Corollary 3.7. It remains to show that (iv) or (v) implies (vi). For that purpose, let $P \subseteq X$ and $P \subseteq Y$ for $X, Y \in \mathcal{DS}(A)$. Suppose that X, Y are incomparable, i.e., $X \nsubseteq Y$ and $Y \nsubseteq X$. Then there exist $x \in X \setminus Y$ and $y \in Y \setminus X$. But we have $x \to y \in P \subseteq X \cap Y$ or $y \to x \in P \subseteq X \cap Y$, whence it follows $y \in X \cap Y$ or $x \in X \cap Y$, a contradiction. ■

By (iv) and (vi):

COROLLARY 3.10. *Let A be a pseudo BCK-join-semilattice that satisfies* (3.10).

(1) *The poset of all prime deductive systems of A forms a root-system;*
(2) *A is linearly ordered if and only if so is $\mathcal{DS}(A)$.*

Proof. The assertion (1) is plain. By (iv), A is linearly ordered if and only if the deductive system $\{1\}$ is prime, and hence (2) easily follows from (v). ■

4. Spectral topology

Let $(A, \vee, \to, \rightsquigarrow, 1)$ be a pseudo BCK-join-semilattice. We denote by $\mathcal{P}(A)$ and $\mathcal{M}(A)$ the set of all proper prime deductive systems of A and the set of all maximal deductive systems of A, respectively. We have $\mathcal{M}(A) \subseteq \mathcal{P}(A)$. For any $X \subseteq A$, we put

$$\mathcal{O}(X) := \{P \in \mathcal{P}(A) : X \nsubseteq P\}$$

and

$$\mathcal{C}(X) := \{P \in \mathcal{P}(A) : X \subseteq P\}.$$

We write $\mathcal{O}(a) = \mathcal{O}(\{a\})$ and $\mathcal{C}(a) = \mathcal{C}(\{a\})$ for $a \in A$. It is easily seen that for any $X \subseteq A$ we have

(A) $\mathcal{O}(X) = \mathcal{O}(D(X))$ and $\mathcal{C}(X) = \mathcal{C}(D(X))$,

so that we may restrict ourselves to the case when $X \in \mathcal{DS}(A)$. Further,

(B) $\mathcal{O}(A) = \mathcal{P}(A)$ and $\mathcal{O}(1) = \varnothing$;
(C) given any $X, Y \in \mathcal{DS}(A)$, $\mathcal{O}(X \cap Y) = \mathcal{O}(X) \cap \mathcal{O}(Y)$;
(D) for any family $\{X_i : i \in I\}$ of deductive systems of A,

$$\mathcal{O}\Big(\bigvee_{i \in I} X_i\Big) = \bigcup_{i \in I} \mathcal{O}(X_i).$$

In particular, (C) entails

(E) $\mathcal{O}(a \vee b) = \mathcal{O}(a) \cap \mathcal{O}(b)$ for every $a, b \in A$.

Indeed, $\mathcal{O}(a \vee b) = \mathcal{O}(D(a \vee b)) = \mathcal{O}(D(a) \cap D(b)) = \mathcal{O}(D(a)) \cap \mathcal{O}(D(b)) = \mathcal{O}(a) \cap \mathcal{O}(b)$.

PROPOSITION 4.1. *Let A be a pseudo BCK-join-semilattice, $X \in \mathcal{DS}(A)$. Then*

$$X = \bigcap \mathcal{C}(X) \quad \text{and} \quad X^\delta = \bigcap \mathcal{O}(X).$$

Proof. By Corollary 3.2 (2) it holds $X = \bigcap \mathcal{C}(X)$.

For every $P \in \mathcal{O}(X)$, since P is prime and $X \cap X^\delta = \{1\} \subseteq P$, we have $X^\delta \subseteq P$, which along with Proposition 3.8 (2) yields

$$X^\delta \subseteq \bigcap \mathcal{O}(X) \subseteq \bigcap \{P \in \mathcal{P}(A) : P \text{ is minimal and } X \nsubseteq P\} = X^\delta,$$

thus $X^\delta = \bigcap \mathcal{O}(X)$. ∎

The properties (B), (C) and (D) together mean that

$$\mathcal{T}_{\mathcal{P}(A)} = \{\mathcal{O}(X) : X \in \mathcal{DS}(A)\}$$

is a topology on $\mathcal{P}(A)$ whose basis is $\{\mathcal{O}(a) : a \in A\}$. Observe that the closed subsets are the sets $\mathcal{C}(X)$, $X \in \mathcal{DS}(A)$.

THEOREM 4.2. *For every pseudo BCK-join-semilattice A, the mapping*

$$\varphi : X \longmapsto \mathcal{O}(X)$$

is an isomorphism of the lattice $\mathcal{DS}(A)$ onto the lattice of all open subsets of $\mathcal{P}(A)$.

Proof. Clearly, φ is a surjective homomorphism. If X, Y are distinct deductive systems then $X \nsubseteq Y$ or $Y \nsubseteq X$, say $X \nsubseteq Y$, so there is $x \in X \setminus Y$, and consequently, $Y \subseteq P$ and $x \notin P$ for some $P \in \mathcal{P}(A)$. Thus $P \in \mathcal{O}(X)$ while $P \notin \mathcal{O}(Y)$. ∎

COROLLARY 4.3. *Let A be a pseudo BCK-join-semilattice. Then $\mathcal{P}(A)$ is a compact space if and only if there exist $a_1, \dots, a_n \in A$ ($n \in \mathbb{N}$) such that $A = D(a_1, \dots, a_n)$.*

Proof. It is obvious that $\mathcal{P}(A) = \mathcal{O}(A)$ is compact iff A is a compact element of the lattice $\mathcal{DS}(A)$, i.e., iff A is generated (as a deductive system) by a finite number of elements of A. ∎

THEOREM 4.4. *For any pseudo BCK-join-semilattice A, $\mathcal{P}(A)$ is a T_0-space. If $\mathcal{M}(A) \neq \varnothing$ then $\mathcal{M}(A)$—endowed with the relative topology—is a T_1-space. Moreover, if $\mathcal{M}(A) \neq \varnothing$ and A satisfies the identities* (3.10), *then $\mathcal{M}(A)$ is a T_2-space.*

Proof. Let $P, Q \in \mathcal{P}(A)$ with $P \neq Q$. If, e.g., $P \nsubseteq Q$ then $Q \in \mathcal{O}(P)$ and $P \notin \mathcal{O}(P)$. Since for any two distinct $P, Q \in \mathcal{M}(A)$ we have $P \nsubseteq Q$ and $Q \nsubseteq P$, it follows that $\mathcal{M}(A)$ is a T_1-space provided $\mathcal{M}(A) \neq \varnothing$.

Assume now that A fulfils (3.10) and $\mathcal{M}(A) \neq \varnothing$. Let $P, Q \in \mathcal{M}(A)$, $P \neq Q$. Then there exist $a \in P \setminus Q$ and $b \in Q \setminus P$. One readily sees that also $b \to a \in P \setminus Q$ and $a \to b \in Q \setminus P$. In addition, $\mathcal{O}(a \to b) \cap \mathcal{O}(b \to a) = \mathcal{O}((a \to b) \vee (b \to a)) = \mathcal{O}(1) = \varnothing$, so that $\mathcal{O}(a \to b)$ and $\mathcal{O}(b \to a)$ are disjoint neighbourhoods of P and Q, respectively. Thus $\mathcal{M}(A)$ is a T_2-space. ∎

An algebraic lattice L is said to be *archimedean* if for each compact element $c \in L$ the intersection of the elements which are maximal below c is 0. This notion was introduced by J. Martinez [14] and is motivated by the fact that an abelian ℓ-group G is archimedean (i.e., for all $0 < a, b \in G$, $na \nleq b$ for some $n \in \mathbb{N}$) if and only if the lattice $\mathcal{I}(G)$ of its ℓ-ideals is archimedean. Further, L is called *hyper-archimedean* if for every $x \in L$ the interval $[x, 1]$ is an archimedean lattice. Again, an abelian ℓ-group G is hyper-archimedean (i.e., the homomorphic images of G are archimedean ℓ-groups) exactly if $\mathcal{I}(A)$ is hyper-archimedean.

THEOREM 4.5. *Let A be a pseudo BCK-join-semilattice satisfying the identities* (3.10). *The following statements are equivalent:*

(i) *$\mathcal{P}(A)$ is a T_2-space,*
(ii) *$\mathcal{P}(A) = \mathcal{M}(A)$,*
(iii) *every prime deductive system is minimal prime,*
(iv) *$\mathcal{DS}(A)$ is a hyper-archimedean lattice,*
(v) *for every compact (= finitely generated) deductive system X, the polar X^δ is the complement of X in $\mathcal{DS}(A)$.*

With any of these conditions, the lattice $\mathcal{DS}(A)$ is isomorphic to the lattice $\mathcal{I}(G)$ of all ℓ-ideals of some hyper-archimedean ℓ-group G.

Proof. (i) ⇒ (ii). Let $\mathcal{P}(A)$ be a T_2-space. Let $P \in \mathcal{P}(A)$ and $X \in \mathcal{DS}(A)$ with $P \subseteq X$. For every $a \in A \setminus X$ there is $Q \in \mathcal{P}(A)$ such that $X \subseteq Q$ and $a \notin Q$. Then clearly $P \subseteq X \subseteq Q$. Suppose that $P \neq Q$. In this case there exist $x, y \in A$ such that $P \in \mathcal{O}(x)$, $Q \in \mathcal{O}(y)$ and $\mathcal{O}(x \vee y) = \mathcal{O}(x) \cap \mathcal{O}(y) = \varnothing$. The last equality entails $x \vee y = 1$, and hence $y \in P$ as $P \in \mathcal{O}(x)$. But this is also impossible since $P \subseteq Q$. Altogether, we have $P = X = Q$ proving that P is a maximal deductive system, so $\mathcal{P}(A) = \mathcal{M}(A)$.

(ii) ⇒ (i). By Theorem 4.4.

(ii) ⇔ (iii). Trivial.

(ii) ⇔ (iv). By [14], Theorem 1.6, the set of all meet-irreducible elements of a hyper-archimedean lattice L is trivially ordered, and if L is modular and its meet-irreducible elements are trivially ordered then L is hyper-archimedean. Since $\mathcal{DS}(A)$ is an algebraic distributive lattice, it follows that $\mathcal{DS}(A)$ is hyper-archimedean if and only if $\mathcal{P}(A)$ is trivially ordered, if and only if $\mathcal{P}(A) = \mathcal{M}(A)$.

(ii) $\Leftrightarrow$ (v). According to [14], Theorem 2.4, an algebraic distributive lattice L has the property that $c \vee c^* = 1$ for every compact element $c \in L$ (where c^* stands for the pseudocomplement of c) if and only if (a) the set of all compact elements of L is closed under finite meets, and (b) the set of all prime elements of L is trivially ordered. Therefore, in the light of Proposition 3.5, and since X^δ is the pseudocomplement of $X \in \mathcal{DS}(A)$, the equivalence of (ii) and (v) is clear.

Finally, by [14], Theorem 3.2, the property $c \vee c^* = 1$ for every compact c is sufficient for an algebraic distributive lattice L to be isomorphic to the lattice $\mathcal{I}(G)$ for some hyper-archimedean ℓ-group G. Thus if A satisfies any of the conditions (i)–(v) then there is a hyper-archimedean ℓ-group G such that $\mathcal{DS}(A) \cong \mathcal{I}(G)$. ∎

PROPOSITION 4.6. *Let A be a pseudo BCK-join-semilattice. Then for every $\mathcal{S} \subseteq \mathcal{P}(A)$, $\overline{\mathcal{S}} = \mathcal{C}\left(\bigcap \mathcal{S}\right)$ is the closure of $\mathcal{S}$ in $\mathscr{T}_{\mathcal{P}(A)}$. In particular, for every $X \in \mathcal{DS}(A)$, we have $\overline{\mathcal{O}(X)} = \mathcal{C}(X^\delta)$.*

Proof. It is plain that $\mathcal{S} \subseteq \mathcal{C}(\bigcap \mathcal{S})$. Assume that $\mathcal{S} \subseteq \mathcal{C}(X)$ for some $X \in \mathcal{DS}(A)$. Then $X \subseteq \bigcap \mathcal{S}$ and $\mathcal{C}(\bigcap \mathcal{S}) \subseteq \mathcal{C}(X)$ proving that $\mathcal{C}(\bigcap \mathcal{S})$ is the smallest closed subset of $\mathcal{P}(A)$ that contains $\mathcal{S}$.

For the latter claim, given any $X \in \mathcal{DS}(A)$, then $\overline{\mathcal{O}(X)} = \mathcal{C}(\bigcap \mathcal{O}(X)) = \mathcal{C}(X^\delta)$. ∎

COROLLARY 4.7. *Let A be a pseudo BCK-join-semilattice and $X \in \mathcal{DS}(A)$. Then $\mathcal{O}(X)$ is clopen if and only if X^δ is the complement of X in $\mathcal{DS}(A)$.*

Proof. If $\mathcal{O}(X)$ is a clopen subset then $\mathcal{O}(X) = \overline{(\mathcal{O}(X))} = \mathcal{C}(X^\delta)$, hence $\mathcal{O}(X \vee X^\delta) = \mathcal{O}(X) \cup \mathcal{O}(X^\delta) = \mathcal{C}(X^\delta) \cup \mathcal{O}(X^\delta) = \mathcal{P}(A) = \mathcal{O}(A)$ which implies $X \vee X^\delta = A$.

Conversely, assume that $X \vee X^\delta = A$. Then $\mathcal{P}(A) = \mathcal{O}(X) \cup \mathcal{O}(X^\delta)$, whence it follows that $\mathcal{O}(X) = \mathcal{P}(A) \setminus \mathcal{O}(X^\delta) = \mathcal{C}(X^\delta)$ because $\mathcal{O}(X) \cap \mathcal{O}(X^\delta) = \varnothing$. ∎

COROLLARY 4.8. *Let A be a pseudo BCK-join-semilattice that satisfies (3.10). Then any of the conditions (i)–(v) of Theorem 4.5 is equivalent to the condition that $\mathcal{O}(X)$ is a clopen subset in $\mathcal{P}(A)$ for every compact $X \in \mathcal{DS}(A)$.*

5. Prime deductive systems of pseudo BCK-lattices

As we have seen, every deductive system D of any pseudo BCK-algebra A is an order-filter of the underlying poset, but if A is a pseudo BCK-lattice then D need not be a filter. Hence we consider the class of pseudo BCK-lattices satisfying certain simple identities that force deductive systems to be filters.

PROPOSITION 5.1. *Given a pseudo BCK-lattice* $(A, \vee, \wedge, \rightarrow, \rightsquigarrow, 1)$ *satisfying the identities*

$$(5.1) \qquad \begin{aligned} x \rightarrow y &= x \rightarrow (x \wedge y), \\ x \rightsquigarrow y &= x \rightsquigarrow (x \wedge y), \end{aligned}$$

then every $D \in \mathcal{DS}(A)$ *is a filter in the lattice* $(A, \vee, \wedge)$. *If, moreover, D is a prime deductive system then it is also a prime filter.*

Proof. Let $a, b \in D$. Then $a \rightarrow (b \rightarrow (b \wedge a)) = a \rightarrow (b \rightarrow a) = 1 \in D$ entails $a \wedge b \in D$. ■

A natural example of pseudo BCK-lattices satisfying (5.1) are commutative pseudo BCK-lattices.

Though the converse of Proposition 5.1 obviously fails to be true, we shall show below that the minimal prime filters of $(A, \vee, \wedge)$ coincide with the minimal prime deductive systems of $(A, \vee, \wedge, \rightarrow, \rightsquigarrow, 1)$.

PROPOSITION 5.2. *Let* $(A, \vee, \wedge, \rightarrow, \rightsquigarrow, 1)$ *be a pseudo BCK-lattice and let F be a proper filter of* $(A, \vee, \wedge)$. *Denote*

$$\mathfrak{D}(F, x) = \{a \in A : a \rightarrow x \notin F\} \qquad \textit{for } x \in A \setminus F,$$

and

$$\mathfrak{D}(F) = \bigcap \{\mathfrak{D}(F, x) : x \in A \setminus F\}.$$

Then $\mathfrak{D}(F)$ *is a deductive system such that* $\mathfrak{D}(F) \subseteq F$. *Moreover, if F is a prime filter then* $\mathfrak{D}(F)$ *is a prime deductive system.*

Proof. First, note that $\mathfrak{D}(F) \subseteq F$. Indeed, if $a \in \mathfrak{D}(F)$ and $a \notin F$, then $a \in \mathfrak{D}(F, a)$, so $1 = a \rightarrow a \notin F$, a contradiction.

Further, we show that $\mathfrak{D}(F) \in \mathcal{DS}(A)$. Clearly, $1 \in \mathfrak{D}(F)$ as $1 \rightarrow x = x \notin F$ for all $x \in A \setminus F$. Assume that $a, a \rightarrow b \in \mathfrak{D}(F)$, and take an arbitrary $x \in A \setminus F$. Then we have $a \rightarrow x \notin F$, whence $a \rightarrow b \in \mathfrak{D}(F, a \rightarrow x)$ and $(a \rightarrow b) \rightarrow (a \rightarrow x) \notin F$. But $(a \rightarrow b) \rightarrow (a \rightarrow x) \geq b \rightarrow x$, and consequently, $b \rightarrow x \notin F$. This means $b \in \mathfrak{D}(F)$.

Before proving that $\mathfrak{D}(F)$ is prime whenever F is a prime filter, observe that the following two properties hold:

(A) $x \leq y$ implies $\mathfrak{D}(F, y) \subseteq \mathfrak{D}(F, x)$;
(B) if $a \vee b \in \mathfrak{D}(F, x)$ then $a \in \mathfrak{D}(F, x)$ or $b \in \mathfrak{D}(F, x)$.

Indeed, $x \leq y$ yields $a \rightarrow x \leq a \rightarrow y$, so that if $a \rightarrow y \notin F$ then $a \rightarrow x \notin F$, which is (A). If $a \rightarrow x, b \rightarrow x \in F$ then $(a \vee b) \rightarrow x = (a \rightarrow x) \wedge (b \rightarrow x) \in F$ proving (B).

Now, assume that F is a prime filter of $(A, \vee, \wedge)$. Let $a \vee b \in \mathfrak{D}(F)$. If neither a nor b lies in $\mathfrak{D}(F)$, then $a \notin \mathfrak{D}(F, x)$ and $b \notin \mathfrak{D}(F, y)$ for some $x, y \in A \setminus F$. Since F is prime, we have $x \vee y \notin F$, so $a \vee b \in$

$\mathfrak{D}(F, x \vee y)$. However, $\mathfrak{D}(F, x \vee y) \subseteq \mathfrak{D}(F, x) \cap \mathfrak{D}(F, y)$ by (A), and therefore, $a \in \mathfrak{D}(F, x \vee y) \subseteq \mathfrak{D}(F, x) \cap \mathfrak{D}(F, y)$ or $b \in \mathfrak{D}(F, x \vee y) \subseteq \mathfrak{D}(F, x) \cap \mathfrak{D}(F, y)$ which contradicts to $a \notin \mathfrak{D}(F, x)$ and $b \notin \mathfrak{D}(F, y)$. Altogether, $a \vee b \in \mathfrak{D}(F)$ entails $a \in \mathfrak{D}(F)$ or $b \in \mathfrak{D}(F)$ as desired. ∎

REMARK 5.3. Observe that if a given filter F is a deductive system then $\mathfrak{D}(F) = F$. Indeed, for every $a \in F$ and $x \in A \setminus F$ we have $a \to x \notin F$, so $a \in \mathfrak{D}(F, x)$ yielding $F \subseteq \mathfrak{D}(F)$.

COROLLARY 5.4. *Let* $(A, \vee, \wedge, \to, \rightsquigarrow, 1)$ *be a pseudo BCK-lattice that fulfils* (5.1). *Then for any* $X \subset A$, X *is a minimal prime deductive system of* $(A, \vee, \wedge, \to, \rightsquigarrow, 1)$ *if and only if* X *is a minimal prime filter of* $(A, \vee, \wedge)$.

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