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A NUMERICAL VIEW ON TOPOLOGICAL TRANSITIVITY AND MIXING FOR SYMBOLIC DYNAMICS

Abstract. The aim of this paper is to give some algorithms detecting topological transitivity, mixing and other properties of subshifts of finite type.

1. Introduction

The symbolic dynamics plays an important role in the theory of dynamical systems. It can be applied to detection of chaos in dynamical systems generated by differential equations [2], [3]. Symbolic dynamics has also many applications in modelling of processes in many fields of science (e.g. coding theory, automata theory, genetics and biotechnology [7]). In this paper a construction of algorithms detecting dynamical properties of subshifts of finite type like topological transitivity and mixing is considered. Topologically transitive subshifts are chaotic in the sense of Devaney [2]. Topological methods are particulary applicable in the case of chaotic dynamical systems. In this situation one is unable to compute true orbits, even increasing the precision of calculations, as the system has sensitive dependence on initial conditions.

The main part of the paper is Section 5, where algorithms are effectively checking whether a subshift of finite type is nonempty, topologically transitive or mixing. An algorithm transforming a graph to an essential graph is also given. Using graph theory there are constructed algorithms detecting properties described above with linear complexity depending on the number of vertices and edges. However, detecting topological mixing requires $\mathcal{O}(n^3)$ operations, as this property is more complex than the others.

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2. Basic definitions and properties

Let $\mathcal{A}$ be a finite set, *an alphabet*. The *(full)* $\mathcal{A}$*-shift* is the product space $\mathcal{A}^{\mathbb{Z}}$ with a shift map $\sigma : \mathcal{A}^{\mathbb{Z}} \ni (x_i)_{i\in\mathbb{Z}} \to (x_{i+1})_{i\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ and the metric d on $\mathcal{A}^{\mathbb{Z}}$ given by $d((x_i),(y_i)) = 2^{-j}$, where $j \in \mathbb{N}$ is the smallest integer such that $x_j \neq y_j$ or $x_{-j} \neq y_{-j}$. A compact subset X of $\mathcal{A}^{\mathbb{Z}}$ invariant under σ is called a *subshift*.

Elements of the set $W_n(\mathcal{A}) = \mathcal{A}^n$ are called *n-words* (*n-blocks*) over $\mathcal{A}$. A *word* (*block*) over $\mathcal{A}$ is an element of the set $W(\mathcal{A}) = \bigcup_{n\in\mathbb{N}} W_n(\mathcal{A})$. The *length of word*, $|w|$, $x \in W(\mathcal{A})$ is N such that $x \in W_N(\mathcal{A})$ (i.e. $|x| = N$).

A word $w \in W(\mathcal{A})$ is *allowed* for a subshift $X \subset \mathcal{A}^{\mathbb{Z}}$ if there exists $x \in X$ such that $w = x_i x_{i+1} \dots x_{i+n-1} = x_{[i,i+n-1]}$ for some $i \in \mathbb{Z}$, where $n = |w|$.

Let $W_n(X)$ denotes the set of all n-words allowed for X and let $W(X)$ be the set of all words allowed for X.

Each subshift may be defined by a collection of *forbidden words* $\mathcal{F} \subset W(\mathcal{A})$. A *subshift of finite type* can be determined by a finite set of forbidden words. A subshift of finite type is *M-step* if it can be defined by a collection of forbidden words all of length M+1. It can be easily shown that every M-step subshift of finite type is also (M+1)-step.

We call $G = (V, E, i, t)$ a *graph* if V and E are finite sets, $V \neq \emptyset$ and i, t are maps from E to V. Vertex $i(e)$ is an *initial state* of edge e, and $t(e)$ is a *terminal state.*

DEFINITION 1. *Let G be a graph. The* vertex shift *over alphabet $\mathcal{A} = V_G$ is the shift space specified by*

$$X_G = \{x = (x_j)_{j\in\mathbb{Z}} : x_j \in V_G, t(e_j) = x_j, i(e_j) = x_{j+1}, e_j \in E_G \text{ for all } j \in \mathbb{Z}\}$$

and the edge shift *over $\mathcal{A} = E_G$ is the shift space defined by*

$$\widehat{X_G} = \{\xi = (\xi_j)_{j\in\mathbb{Z}} : \xi_j \in E_G, t(\xi_j) = i(\xi_{j+1}) \text{ for all } j \in \mathbb{Z}\}.$$

A *bi-infinite path* in G is $\xi = (\xi_i)_{i\in\mathbb{Z}}$ such that $\xi_i \in E_G$ and $t(\xi_i) = i(\xi_{i+1})$ for all $i \in \mathbb{Z}$. A vertex $v \in V$ is *stranded* if either the set $\{e \mid i(e) = v\}$ or $\{e \mid t(e) = v\}$ is empty. A graph is *essential* if it has no stranded vertices. If ξ is a bi-infinite path on G then for every $i \in \mathbb{Z}$ vertices $i(\xi_i)$ and $t(\xi_i)$ are not stranded.

In this paper it is assumed that all graphs are essential. Transformation of a graph to essential form does not change presented subshift (the set of all bi-infinite paths is the same for both graphs).

DEFINITION 2. *Let X be a subshift over alphabet $\mathcal{A}$ and $\mathcal{A}^{[N]} = W_N(\mathcal{A})$. Define a* higher N-block code $\beta_N : X \to (\mathcal{A}^{[N]})^{\mathbb{Z}}$ *by* $\beta_N(x)_{[i]} = x_{[i,i+N-1]}$. *The* higher N-block presentation *of X, denoted by $X^{[N]}$, is the image of β_N.*

REMARK 3. A function β_N is a topological conjugacy of a subshift X and its Nth higher block $X^{[N]}$.

Let X be a compact metric space. A homeomorphism $f : X \to X$ is topologically transitive iff for all open nonempty sets $U, V \subset X$ exists a positive integer m such that $f^{-m}(U) \cap V \neq \emptyset$ and f is topologically mixing iff for all open nonempty sets $U, V \subset X$ exists a positive integer m_0 such that $f^{-m}(U) \cap V \neq \emptyset$ for every integer $m > m_0$.

A subshift X is topologically transitive (mixing) if the map $\sigma|_X$ is topologically transitive (mixing).

REMARK 4. Topological transitivity and mixing are topological conjugacy invariants, so instead checking them for X it can be done for higher block presentation $X^{[N]}$. A subshift conjugate to a nonempty subshift is also nonempty.

PROPOSITION 5 (see [7, Thm. 2.3.2]). *Let X be an M-step subshift of finite type. Then there is a graph G such that $X^{[M]} = X_G$ and $X^{[M+1]} = \widehat{X_G}$.*

In this article, there are given results for vertex shifts only, as the theory for edge shifts is equivalent. This implies that algorithms considered in Section 5 are the same in the case of edge shifts (i.e. the same test checks the same properties in the case of the edge and the vertex shift).

Let $\mathcal{M}_n$ be a set of all $n \times n$ matrices with nonnegative integer entries.

DEFINITION 6. *The oriented graph $G(A)$ (or just G) associated with a matrix $A \in \mathcal{M}_n$ consists of n vertices. There are k edges in G from the i-th to the j-th vertex iff $a_{ij} = k$, where $A = [a_{ij}]_{i,j=1,\dots,n}$. A cycle is a path in the graph having the same initial and terminal vertex.*

The matrix A is called a transition (adjacency) matrix *for graph G.*

Let X_A and $\widehat{X_A}$ denote the vertex shift and the edge shift in the graph associated with the matrix A. For a given oriented graph $\mathcal{G}$ with at most one edge between two vertices, there exists a unique, square matrix which is its transition matrix.

Let $\mathbb{N}_0$ be a set of all nonnegative integer. Let us define:

$$\mathcal{E}_n := \{B = [a_{ij}]_{i,j=1\dots n} : a_{ij} \in \mathbb{N}_0, \forall_{i \in \{1,\dots,n\}} \exists_{j \in \{1,\dots,n\}}\ a_{ij} > 0\}.$$

THEOREM 7 (see for instance [4]). *Let $A = [a_{ij}]_{i,j=1,\dots,n} \in \mathcal{M}_n$. Then the following conditions are equivalent:*

(*i*) $X_A \neq \emptyset$; $\widehat{X_A} \neq \emptyset$;

(*ii*) *there exist* $k \in \{1, \ldots, n\}$ *and pairwise distinct integers* $b_1, \ldots, b_k \in \{1, \ldots, n\}$ *such that* $D = [a_{b_i b_j}]_{i,j=1,\ldots,k} \in \mathcal{E}_k$;

(*iii*) $\mathrm{tr}(A + A^2 + \ldots + A^n) > 0$.

Let a_{ij}^m denotes i, j-th entry of m−th power of A.

DEFINITION 8. *A matrix A is irreducible iff for every $i, j \in \{1, \ldots, n\}$ there is a positive integer m such that $a_{ij}^m > 0$.*

A matrix A is primitive iff for every $i, j \in \{1, \ldots, n\}$ there is a positive integer m_0 such that $a_{ij}^m > 0$ for every integer $m > m_0$.

DEFINITION 9. *A graph G is irreducible (strongly connected) iff for any two vertices $I, J \in V_G$ there exists a path $\pi = e_1 \ldots e_m$ in the graph G such that $I = i(e_1)$ and $J = t(e_m)$. An irreducible component of the graph G is every maximal irreducible subgraph of G.*

REMARK 10. A matrix A is irreducible iff it is a transition matrix for an irreducible graph.

THEOREM 11 (see [6], [8]).

For a given transition matrix A the following properties are satisfied:

(1) *The subshift X_A $(\widehat{X_A})$ is topologically transitive iff A is irreducible.*

(2) *The subshift X_A $(\widehat{X_A})$ is topologically mixing iff A is irreducible and*

$$1 \in \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{Z} \cdot j \cdot \mathrm{sgn}\,(a_{ii}^j),$$

where $\mathbb{Z}$ is the set of integers. The above condition can be stated in the following equivalent form:

$$\exists\ \{c_{ij}\}_{i,j \in \{1,\ldots,n\}} \subset \mathbb{Z}\ :\ 1 = \sum_{i,j=1}^{n} c_{ij} \cdot j \cdot \mathrm{sgn}(a_{ii}^j).$$

Let $\mathrm{GCD}(m_1, \ldots, m_k)$ be the *greatest common divisor* of positive integers $m_1, \ldots, m_k$. Since there exist integers $t_1, \ldots, t_k$ such that

$$\mathrm{GCD}(m_1, \ldots, m_k) = t_1 m_1 + t_2 m_2 + \ldots + t_k m_k,$$

the following corollary is satisfied

COROLLARY 12. *Let A be a transition matrix. Then the subshift X_A $(\widehat{X_A})$ is topologically mixing iff A is irreducible and there exist cycles $\pi_1, \ldots, \pi_k$ of lengths $m_1, \ldots, m_k \leq n$ in the graph $\mathcal{G}(A)$ such that $GCD(m_1, \ldots, m_k) = 1$.*

To verify whether a given subshift is mixing it is sufficient, by Corollary 12, to find all numbers $m_k \leq n$ representing cycles lengths (i.e.

$tr(A^{m_k}) > 0$) and then check whether $\mathrm{GCD}(m_1, \ldots, m_k) = 1$. Unfortunately, multiplication of two matrices needs $\mathcal{O}(n^3)$ operations so in order to find all numbers m_k it is required $\mathcal{O}(n^4)$ operations. Therefore it has been proposed another criteria allowing to verify topological mixing faster.

Let G be a graph with n vertices. For $v \in V_G$ it is defined $I_v^3 \subset \{1, \ldots, 3n\}$ in the following way: $j \in I_v^3$ if there exists a cycle π starting in v and $|\pi| = j \leq 3n$. $\mathrm{GCD}(B)$ means the greatest number which divides all elements of a finite set $B \subset \mathbb{N}$. Obviously, if A, B are any two finite sets such that $A \subset B$, then $\mathrm{GCD}(B)$ divides $\mathrm{GCD}(A)$.

THEOREM 13. *Let G be an irreducible graph. If the subshift X_G is topologically mixing, then $GCD(I_v^3) = 1$ for every $v \in V_G$. If there exists $v \in V_G$ such that $GCD(I_v^3) = 1$, then X_G is topologically mixing.*

Proof. Suppose that the subshift X_G is topologically mixing and take any $v \in V_G$. By Corollary 12, there exist vertices $v_1, \ldots v_k$ and cycles $\pi_1, \ldots, \pi_k$ through that vertices such that $GCD(|\pi_1|, \ldots, |\pi_k|) = 1$ and $|\pi_i| \leq n$ for $i = 1, \ldots, k$. Graph G is irreducible, so there exist paths from v to v_i and from v_i to v. Furthermore, there exist such paths with length at most n. Thus, we have cycles ξ_i going through vertices v and v_i with length less than $2n$. Combining cycles π_i and ξ_i together it is obtained a cycle going through v with length $|\pi_i| + |\xi_i|$. By the definition of I_v^3, the following property is satisfied

$$\{|\xi_1|, \ldots, |\xi_k|, |\pi_1| + |\xi_1|, \ldots, |\pi_k| + |\xi_k|\} \subset I_v^3.$$

Because $GCD(I_v^3)$ divides $|\xi_i|$ and $|\pi_i| + |\xi_i|$, then $GCD(I_v^3)$ divides $|\pi_i|$. Thus

$$1 \leq GCD(I_v^3) \leq GCD(\pi_1, \ldots, \pi_k) = 1.$$

Conversely, if there exist cycles in the graph with the greatest common divisor equal to 1 and the graph is irreducible, then the subshift X_G is topologically mixing. The proof of this statement is the same as the proof of Theorem 11 (see [6]). □

3. Subshifts generated by a finite sum of matrices

Subshifts of this type were introduced in [5]. Let n, k be positive integers and let $A_1, \ldots, A_k \in \mathcal{M}_n$ and let k be a nonnegative integer. Let us consider the subshift of finite type X_C ($\widehat{X_C}$) with

$$C = [c_{ij}], \quad c_{ij} = \mathrm{sgn}\,((a_1)_{ij} + \ldots + (a_k)_{ij}),$$

where $(a_l)_{ij}$ is the i,j–entry of matrix A_l.

REMARK 14. $X_{A_1} \cup \ldots \cup X_{A_k} \subset X_C$.

REMARK 15. Let $A_1, \ldots, A_k$ be transition matrices as above. If one of the subshifts $X_{A_1}, \ldots, X_{A_k}$ is topologically transitive (mixing), then the subshift X_C is topologically transitive (mixing).

For $A \in \mathcal{M}_n$:

$$A > 0 \Leftrightarrow a_{ij} > 0 \text{ for all } i, j \in \{1, \ldots, n\}.$$

COROLLARY 16. *If the condition*

$$\left(A_1 + A_1^2 + \ldots + A_1^n\right) + \left(A_2 + A_2^2 + \ldots + A_2^n\right) + \ldots + \left(A_k + A_k^2 + \ldots + A_k^n\right) > 0$$

is satisfied then the subshift X_C is topologically transitive.

COROLLARY 17. *If one of the transition matrices $A_1, \ldots, A_k$ is irreducible and the condition* $\operatorname{tr}(A_1 + A_2 + \ldots + A_k) > 0$ *is satisfied, then the subshift X_C is topologically mixing.*

COROLLARY 18. *If one of the transition matrices $A_1, \ldots, A_k$ is irreducible and for some $i \in \{1, \ldots, k\}$ there exist cycles $\pi_1, \ldots, \pi_k$ of lengths $m_1, \ldots, m_k \leq n$ in the graph $G(A_i)$ such that $GCD(m_1, \ldots, m_k) = 1$, then the subshift X_C is topologically mixing.*

4. Examples

EXAMPLE 19. Let us consider a subshift defined by a finite sum of matrices. Topological transitivity or mixing of the subshift generated by a finite sum of matrices can be tested by checking irreducibility or other properties for the sum of a given matrix A. As the number of matrices can be large so it might be better to check separately some of the properties for matrices $A_1, \ldots, A_k$.

The subshift in Figure 1 is generated by the sum of two matrices, one of which is irreducible and the other with a cycle of length 1. It does not matter how many additional matrices we will add to that sum, it will remain mixing.

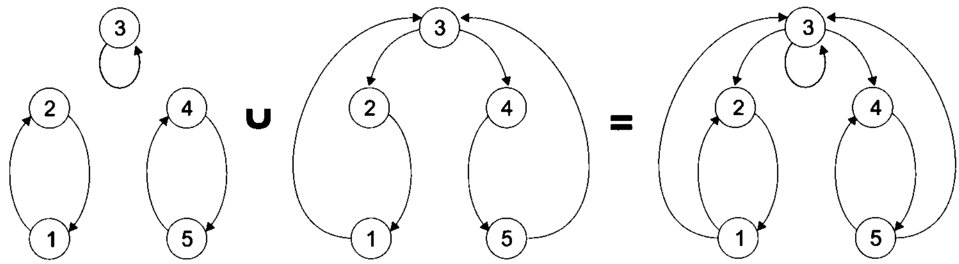

Fig. 1. Reducible + Irreducible = Mixing

EXAMPLE 20. Let

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

The matrix A is irreducible and $\operatorname{tr}(A^5) > 0$, $\operatorname{tr}(A^6) > 0$. Thus the subshift X_A is topologically transitive and mixing.

5. Algorithms

In this section it is assumed that G is the graph with n vertices and m edges presented by adjacency lists. As it is possible that there is more than one edge between two vertices, sometimes $m > n^2$. When m differs slightly from n^2 it is better to use transition (adjacency) matrix representation. In this case m should be changed into n^2 in the complexity description, however to change one presentation to another, we may need $\mathcal{O}(n+m)$ operations. In presented algorithms it is only required to know if there exists any edge between two vertices, so we may use boolean matrices as adjacency matrices. However, the number of edges between two vertices is important information in case we would like to calculate the entropy of shift space or check other dynamical properties.

THEOREM 21. *Let G be a graph with n vertices and m edges. There exists an algorithm checking if X_G and $\widehat{X_G}$ are non-empty subshifts. Furthermore, the algorithm complexity is equal to $\mathcal{O}(n+m)$.*

Proof. To check if X_G and $\widehat{X_G}$ are non-empty it is enough to check if there can be constructed any bi-infinite path in G. If there is a vertex with self-loop, then there exists a bi-infinite path in G and the process is stopped. If there is no vertex with a self-loop (this can be checked in n steps), then the next part of the algorithm is performed.

Let us find all connected components of the graph. It can be done, by the algorithm STRONGLY-CONNECTED-COMPONENTS, presented in [1], with complexity $\mathcal{O}(n+m)$. This algorithm will also label the vertices in the same connected component of the graph with the same color, so we can count the number of vertices in each of the components. If there exists a component which consists of at least two vertices then there is a cycle in G and a bi-infinite path can be constructed.

Conversely, let us suppose that each of the components contains only one vertex. Let us observe that there is no self-loops and every two vertices lay in different connected components, so there is no cycle in G. This means that there is no bi-infinite path in the graph and thus the shift spaces X_G and $\widehat{X_G}$ are empty. □

THEOREM 22. *Let G be a graph with n vertices and m edges. There exists an algorithm changing G into essential graph and the algorithm complexity is $\mathcal{O}(n+m)$.*

Proof. Let us take the graph G and its inversion G^T (graph G^T can be produced in $\mathcal{O}(n+m)$ operations). Let us also construct tables I and O, in the way that $I(v)$ denotes the number of edges terminating in vertex v and $O(v)$ denotes the number of edges initializing in v. The tables I, O and C can be filled using $\mathcal{O}(m+n)$ operations. Vertex $v \in V_G$ is stranded if $I(v)$ or $O(v)$ is equal to zero. At the beginning we set the color of each vertex as white.

Stranded vertices are removed recursively in the following way: if there exists a vertex such that $I(v) = 0$ or $O(v) = 0$ and its color is white then this vertex is grayed. Next the values in the table I are decreased by one for all vertices reachable from v in the graph G and do the same with the values of the table O for vertices reachable from v in G^T (we virtually remove all edges going in and out of v). If one of these numbers reached 0 and a vertex is white then this vertex is considered in the next step, and its color is gray. The process is stopped if there is no vertex satisfying previous criteria (there is no gray vertex, which was not considered). As any vertex may be grayed at most one time, and numbers in the tables I, O may be changed at most $2m$ times, so the complexity of this part of the algorithm is $\mathcal{O}(n+m)$.

The last part of the algorithm consists in removing all grayed vertices and all edges having such a vertex in one of the ends. It can be done using $\mathcal{O}(n+m)$ operations. Let us observe that if any vertex v is not grayed then $I(v) > 0$ and $O(v) > 0$ (it is not stranded in the graph containing only white vertices). The conclusion is that it is needed at most $\mathcal{O}(n+m)$ operations to change š given graph to an essential graph presenting the same shift space. □

THEOREM 23. *Let G be a graph with n vertices and m edges. There exists an algorithm checking if $\widehat{X_G}$ and X_G are topologically transitive subshifts. Furthermore it can by done in $\mathcal{O}(n+m)$ operations.*

Proof. First of all the graph has to be changed to an essential form. Next, due to previous observations, it is sufficient to check if the graph is irreducible. So by Remark 10 we must check if there is a path from some chosen vertex v to any $u \in V_G \backslash \{u\}$ in the graph G and its inversion G^T. It can be done by a slight modification of the algorithm STRONGLY-CONNECTED-COMPONENTS having complexity $\mathcal{O}(n+m)$. □

THEOREM 24. *Let G be a graph with n vertices and m edges. There exists an algorithm checking if $\widehat{X_G}$ and X_G are topologically mixing subshifts. Furthermore it can done using $\mathcal{O}(n \cdot m)$ operations.*

Step 1. It must be checked if X_G is topologically transitive. We can do it, by Theorem 23, in $\mathcal{O}(n+m)$ operations. If X_G is not topologically transitive, then it is also not mixing, and the process is stopped.

Step 2. Let us take any vertex $v \in V_G$. We define the family of sets $\{P_k\}_{k=1,\dots,3n}$ as follows:

$$P_0 = \{v\} \quad , \quad P_{i+1} = \{v \in V_G \mid \exists\, e \in E_G \;:\; t(e) = v \;,\; i(e) \in P_i\}.$$

Observe that $k \in I_v^3$ iff $v \in P_k$, so it is sufficient to make recursive construction, checking if $v \in P_k$. In every step i we must remember only the set P_{i-1} and elements of I_v^3. After $3n$ steps we have the set I_v^3, and the whole process needs at most $3n \cdot m + n$ operations.

Step 3. By Theorem 13 it is sufficient to check if $GCD(I_v^3) = 1$. We can find GCD(a,b) by the well known EUCLID algorithm described e.g. in [1], and it can be done in $\mathcal{O}(log(b))$ operations, where $a > b > 0$. It is easy to see that to find $GCD(I_v^3)$ we need at most $\mathcal{O}(3n \cdot log(3n))$ operations. If $GCD(I_v^3) = 1$, then X_G is topologically mixing, otherwise it is not.

The complexity of all steps is bounded by $\mathcal{O}(n \cdot m)$ so it is also the complexity of the whole algorithm. □

References

[1] T. H. Cormen, C. E. Leiserson and R. L. Rivest, *Introduction to Algorithms*, Cambridge, 1991.

[2] R. L. Devaney, *An Introduction to Chaotic Dynamical Systems*, Addison-Wesley, New York, 1989.

[3] Z. Galias, P. Zgliczyński, *Abundance of homoclinic and heteroclinic orbits and rigorous bounds for the topological entropy for the Henon map*, Nonlinearity 14 (2001), no. 5, 909–932.

[4] D. Jabłoński, *Numerical conditions for some dynamical properties of subshifts of finite type*, Demonstratio Math. 34 (2001), no. 4, 969–974.

[5] D. Jabłoński, *Dynamical properties of finite type subshifts generated by a sum of $(0,1)$-square matrices*, Demonstratio Math. 35 (2002), no. 4, 891–897.

[6] D. Jabłoński, M. Kulczycki, *Topological transitivity, mixing and nonwandering set of subshifts of finite type—a numerical approach*, Int. J. Comput. Math. 80 (2003), no. 6, 671–677.
[7] D. Lind, B. Marcus, *An Introduction to Symbolic Dynamics and Coding*, Cambridge University Press, 1995.
[8] R. Mañé, *Ergodic Theory and Differentiable Dynamics*, Springer Verlag, Berlin, 1987.

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