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FIXED POINT THEOREMS FOR MORE GENERALIZED CONTRACTIONS IN COMPLETE METRIC SPACES

Abstract. We generalize Suzuki's fixed point theorem for asymptotic contractions of Meir-Keeler type in complete metric spaces.

1. Introduction

It is well known that the Banach contraction principle [1] is very useful in nonlinear analysis. Also, this principle has many generalizations; see [2–4, 6, 8, 11–13] and others. For example, Meir and Keeler [11] proved the following very interesting fixed point theorem.

THEOREM 1 (Meir and Keeler [11]). *Let (X, d) be a complete metric space and let T be a Meir-Keeler contraction (MKC, for short) on X, i.e., for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$d(x,y) < \varepsilon + \delta \quad \textit{implies} \quad d(Tx,Ty) < \varepsilon$$

for all $x, y \in X$. Then T has a unique fixed point.

Theorem 1 is also a generalization of Edelstein's fixed point theorem in [5]. Very recently, motivated by Theorem 1 and Kirk's fixed point theorem for asymptotic contractions [9], Suzuki [18] proved the following fixed point theorem.

THEOREM 2 ([18]). *Let (X, d) be a complete metric space and let T be a continuous mapping on X. Assume that T is an asymptotic contraction of Meir-Keeler type (ACMK, for short), i.e., there exists a sequence $\{\varphi_n\}$ of functions from $[0, \infty)$ into itself satisfying the following:*

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(A1) $\limsup_n \varphi_n(\varepsilon) \leq \varepsilon$ *for all* $\varepsilon \geq 0$.
(A2) *For each* $\varepsilon > 0$, *there exist* $\delta > 0$ *and* $\nu \in \mathbb{N}$ *such that* $\varphi_\nu(t) \leq \varepsilon$ *for all* $t \in [\varepsilon, \varepsilon + \delta]$.
(A3) $d(T^n x, T^n y) < \varphi_n\big(d(x, y)\big)$ *for all* $n \in \mathbb{N}$ *and* $x, y \in X$ *with* $x \neq y$.

Then there exists a unique fixed point $z \in X$. *Moreover* $\lim_n T^n x = z$ *for all* $x \in X$.

Using Lim's characterization (Proposition 1), we can prove that ACMK is an asymptotic version of MKC. That is, Theorem 2 is a generalization of Theorem 1.

In this paper, using the notion of τ-distances, we shall prove fixed point theorems for more generalized contractions in complete metric spaces.

2. Preliminaries

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers.

In this section, we give some preliminaries. Lim [10] introduced the notion of L-functions and characterized MKC. See also [19].

DEFINITION 1 (Lim [10]). *A function* φ *from* $[0, \infty)$ *into itself is called an L-function if* $\varphi(0) = 0$, $\varphi(s) > 0$ *for* $s \in (0, \infty)$, *and for every* $s \in (0, \infty)$ *there exists* $\delta > 0$ *such that* $\varphi(t) \leq s$ *for all* $t \in [s, s + \delta]$.

PROPOSITION 1 (Lim [10]). *Let* (X, d) *be a metric space and let* T *be a mapping on* X. *Then the following are equivalent:*

(i) T *is an MKC.*
(ii) There exists an L-function φ *such that*

$$x, y \in X,\ x \neq y \quad \text{implies} \quad d(Tx, Ty) < \varphi\big(d(x, y)\big). \tag{1}$$

(iii) There exists a nondecreasing, Lipschitz continuous L-function φ *satisfying* (1).

In 2001, Suzuki introduced the notion of τ-distances in order to generalize results in Kada, Suzuki and Takahashi [7], Tataru [20], Zhong [21, 22] and others.

DEFINITION 2 ([14]). *Let* (X, d) *be a metric space. Then a function* p *from* $X \times X$ *into* $[0, \infty)$ *is called a* τ*-distance on* X *if there exists a function* η *from* $X \times [0, \infty)$ *into* $[0, \infty)$ *and the following are satisfied:*

(τ1) $p(x, z) \leq p(x, y) + p(y, z)$ *for all* $x, y, z \in X$;
(τ2) $\eta(x, 0) = 0$ *and* $\eta(x, t) \geq t$ *for all* $x \in X$ *and* $t \in [0, \infty)$, *and* η *is concave and continuous in its second variable;*
(τ3) $\lim_n x_n = x$ *and* $\lim_n \sup\big\{\eta\big(z_n, p(z_n, x_m)\big) : m \geq n\big\} = 0$ *imply* $p(w, x) \leq \liminf_n p(w, x_n)$ *for all* $w \in X$;

(τ4) $\lim_n \sup\{p(x_n, y_m) : m \geq n\} = 0$ *and* $\lim_n \eta(x_n, t_n) = 0$ *imply* $\lim_n \eta(y_n, t_n) = 0$;

(τ5) $\lim_n \eta\big(z_n, p(z_n, x_n)\big) = 0$ *and* $\lim_n \eta\big(z_n, p(z_n, y_n)\big) = 0$ *imply* $\lim_n d(x_n, y_n) = 0$.

The metric d is a τ-distance on X. Many useful examples and propositions are stated in [7, 14–17] and others. The following is Lemma 2 in [14].

LEMMA 1 ([14]). *Let X be a metric space with a τ-distance p. Then $p(z, x) = 0$ and $p(z, y) = 0$ imply $x = y$.*

Let (X, d) be a metric space and let p be a τ-distance on X. Then a sequence $\{x_n\}$ in X is called *p-Cauchy* if there exist a function η from $X \times [0, \infty)$ into $[0, \infty)$ satisfying (τ2) – (τ5) and a sequence $\{z_n\}$ in X such that $\lim_n \sup\big\{\eta\big(z_n, p(z_n, x_m)\big) : m \geq n\big\} = 0$. We know the following.

LEMMA 2 ([14]). *Let X be a metric space with a τ-distance p. If $\{x_n\}$ is a p-Cauchy sequence, then $\{x_n\}$ is a Cauchy sequence in the usual sense.*

LEMMA 3 ([14]). *Let (X, d) be a metric space with a τ-distance p. If a sequence $\{x_n\}$ in X satisfies $\lim_n \sup\{p(x_n, x_m) : m > n\} = 0$, then $\{x_n\}$ is a p-Cauchy sequence. Moreover if a sequence $\{y_n\}$ in X satisfies $\lim_n p(x_n, y_n) = 0$, then $\{y_n\}$ is also a p-Cauchy sequence and $\lim_n d(x_n, y_n) = 0$.*

The following is the τ-distance version of Theorem 1.

THEOREM 3 ([16]). *Let X be a complete metric space with a τ-distance p, and let T be a mapping on X. Suppose that T is a Meir-Keeler contraction with respect to p (p-MKC, for short), i.e., for any $\varepsilon > 0$, there exists $\delta > 0$ such that for every $x, y \in X$,*

$$p(x, y) < \varepsilon + \delta \quad \textit{implies} \quad p(Tx, Ty) < \varepsilon.$$

Then T has a unique fixed point z in X. Further such z satisfies $p(z, z) = 0$.

We can easily modify Lim's characterization for the τ-distance version of it as follows.

PROPOSITION 2 ([19]). *Let X be a metric space with a τ-distance p, and let T be a mapping on X. Then T is a p-MKC if and only if there exists a (nondecreasing, Lipschitz continuous) L-function φ satisfying the following:*

(i) If $p(x, y) = 0$, then $p(Tx, Ty) = 0$.
(ii) If $p(x, y) > 0$, then $p(Tx, Ty) < \varphi\big(p(x, y)\big)$.

REMARK 1. *We note that $x = y$ does not necessarily imply $p(x, y) = 0$, and $p(x, y) = 0$ does not necessarily imply $x = y$.*

3. p-ACMK*

In this section, we shall introduce a notion which is a generalization of both ACMK and p-MKC.

DEFINITION 3. *Let X be a metric space with a τ-distance p. Then a mapping T on X is said to be a p-ACMK* if there exists a sequence $\{\varphi_n\}$ of functions from $[0,\infty)$ into $[0,\infty]$ satisfying the following:*

(B1) $\lim_{t\to 0}\limsup_{n\to\infty}\varphi_n(t)=0.$

(B2) For each $\varepsilon>0$, there exist $\delta>0$ and $\nu\in\mathbb{N}$ such that $\varphi_\nu(t)\le\varepsilon$ for all $t\in[\varepsilon,\varepsilon+\delta]$.

(B3) If $\varphi_n\big(p(x,y)\big)=0$, then $p(T^nx,T^ny)=0$.

(B4) If $\varphi_n\big(p(x,y)\big)>0$, then $p(T^nx,T^ny)<\varphi_n\big(p(x,y)\big)$.

It is obvious that p-ACMK* is a weaker notion than d-ACMK*, which is a slightly weaker notion than ACMK.

PROPOSITION 3. *Let X be a metric space with a τ-distance p, and let T be a p-MKC on X. Then T is a p-ACMK* on X.*

Proof. By Proposition 2, there exists an L-function φ from $[0,\infty)$ into itself satisfying (i) and (ii) in Proposition 2. Define a sequence $\{\varphi_n\}$ of functions by $\varphi_n=\varphi$ for all $n\in\mathbb{N}$. Then we have

$$\lim_{t\to 0}\limsup_{n\to\infty}\varphi_n(t)=\lim_{t\to 0}\varphi(t)\le\lim_{t\to 0}t=0$$

and hence (B1) holds. It is obvious that $\{\varphi_n\}$ satisfies (B2). We note

$$p(Tx,Ty)\le p(x,y)$$

for all $x,y\in X$ because $p(x,y)=0$ implies $p(Tx,Ty)=0$, and $p(x,y)>0$ implies

$$p(Tx,Ty)<\varphi\big(p(x,y)\big)\le p(x,y).$$

Fix $x,y\in X$ and $n\in\mathbb{N}$. In the case of $\varphi_n\big(p(x,y)\big)=0$, we have $p(x,y)=0$ because $\varphi_n=\varphi$ is an L-function. So we obtain

$$p(T^nx,T^ny)\le\cdots\le p(T^2x,T^2y)\le p(Tx,Ty)\le p(x,y)=0.$$

That is, $p(T^nx,T^ny)=0$. We have shown (B3). In the case of $\varphi_n\big(p(x,y)\big)>0$, we have $p(x,y)>0$. By Proposition 2 (ii), we have

$$p(T^nx,T^ny)\le\cdots\le p(Tx,Ty)<\varphi\big(p(x,y)\big)=\varphi_n\big(p(x,y)\big).$$

This implies (B4). This completes the proof. □

4. Fixed Point Theorems

In this section, we prove fixed point theorems. We first prove the following, which is a generalization of both Theorems 2 and 3.

THEOREM 4. *Let X be a complete metric space with a τ-distance p. Let T be a p-ACMK* on X with some $\{\varphi_n\}$ in Definition 3. Assume that either of the following holds:*

(i) T^ℓ is continuous for some $\ell \in \mathbb{N}$.
(ii) $\lim_{t\to 0} \varphi_\ell(t) = 0$ for some $\ell \in \mathbb{N}$.

Then there exists a unique fixed point $z \in X$. Moreover such z satisfies

$$p(z,z) = \lim_{n\to\infty} p(z, T^n x) = \lim_{n\to\infty} p(T^n x, z) = 0 \quad \text{and} \quad \lim_{n\to\infty} T^n x = z$$

for all $x \in X$.

Proof. Define a sequence $\{\psi_n\}$ of functions from $[0,\infty)$ into $[0,\infty]$ by

$$\psi_n(t) = \max\{\varphi_n(t),\ t/2\}$$

for $n \in \mathbb{N}$ and $t \in [0,\infty)$. Then such $\{\psi_n\}$ satisfies $(\mathrm{B1})_\psi$ - $(\mathrm{B4})_\psi$ and $\psi_n(t) > 0$ for all $n \in \mathbb{N}$ and $t > 0$. We note

$$p(T^n x, T^n y) \le \psi_n(p(x,y))$$

for all $n \in \mathbb{N}$ and $x, y \in X$. We first show

$$\lim_{n\to\infty} p(T^n x, T^n y) = 0 \tag{2}$$

for all $x, y \in X$. In the case of $p(T^j x, T^j y) = 0$ for some $j \in \mathbb{N}$, we have

$$p(T^{n+j}x, T^{n+j}y) \le \psi_n(p(T^j x, T^j y)) = \psi_n(0)$$

for $n \in \mathbb{N}$ and hence

$$\lim_{n\to\infty} p(T^n x, T^n y) = \lim_{n\to\infty} p(T^{n+j}x, T^{n+j}y) \le \lim_{n\to\infty} \psi_n(0) = 0$$

by $(\mathrm{B1})_\psi$. In the other case of $p(T^j x, T^j y) > 0$ for all $j \in \mathbb{N}$, we put $\alpha := \liminf_n p(T^n x, T^n y)$. For $i \in \mathbb{N}$, since $p(T^i x, T^i y) > 0$, there exists $\nu_1 \in \mathbb{N}$ such that $\psi_{\nu_1}(p(T^i x, T^i y)) \le p(T^i x, T^i y)$ by $(\mathrm{B2})_\psi$. We have

$$p(T^{i+\nu_1}x, T^{i+\nu_1}y) < \psi_{\nu_1}(p(T^i x, T^i y)) \le p(T^i x, T^i y).$$

That is, for each $i \in \mathbb{N}$, there exists $j > i$ such that $p(T^j x, T^j y) < p(T^i x, T^i y)$. This implies $\alpha < p(T^j x, T^j y)$ for all $j \in \mathbb{N}$. Arguing by contradiction, we assume $\alpha > 0$. By $(\mathrm{B2})_\psi$, there exist $\delta_2 > 0$ and $\nu_2 \in \mathbb{N}$ such that $\psi_{\nu_2}(t) \le \alpha$ for all $t \in [\alpha, \alpha + \delta_2]$. Taking $j \in \mathbb{N}$ with $p(T^j x, T^j y) < \alpha + \delta_2$, we have

$$p(T^{\nu_2+j}x, T^{\nu_2+j}y) < \psi_{\nu_2}(p(T^j x, T^j y)) \le \alpha.$$

This contradicts $\alpha < p(T^{\nu_2+j}x, T^{\nu_2+j}y)$. Therefore $\alpha = 0$. For each $\varepsilon > 0$, there exists $\delta_3 > 0$ such that $\limsup_n \psi_n(t) < \varepsilon$ for $t \in [0, \delta_3]$. Taking $j \in \mathbb{N}$ with $p(T^j x, T^j y) < \delta_3$, we have

$$\limsup_{n\to\infty} p(T^{n+j}x, T^{n+j}y) \le \limsup_{n\to\infty} \psi_n(p(T^j x, T^j y)) < \varepsilon$$

and hence (2) holds. Fix $x \in X$ and define a sequence $\{x_n\}$ in X by $x_n := T^n x$ for $n \in \mathbb{N}$. We shall show that

$$\lim_{n\to\infty} \sup_{m>n} p(x_n, x_m) = 0. \tag{3}$$

Let $\varepsilon > 0$ be fixed. Then there exist $\delta_4 \in (0, \varepsilon)$ and $\nu_4 \in \mathbb{N}$ such that $\psi_{\nu_4}(t) \le \varepsilon$ for all $t \in [\varepsilon, \varepsilon + \delta_4]$. From (2), we can choose $N \in \mathbb{N}$ such that $p(x_n, x_{n+1}) < \delta_4/\nu_4$ for every $n \ge N$. Fix $L \in \mathbb{N}$ with $L \ge N$. Using induction, we shall show

$$p(x_L, x_{L+n}) \le \varepsilon + \delta_4 \tag{4}$$

for all $n \in \mathbb{N}$. For every $n \in \{1, 2, \cdots, \nu_4\}$, we have

$$p(x_L, x_{L+n}) \le \sum_{j=0}^{n-1} p(x_{L+j}, x_{L+j+1}) \le n\,\delta_4 \,/\, \nu_4 \le \delta_4 < \varepsilon + \delta_4.$$

For $m \in \mathbb{N}$ with $m > \nu_4$, we assume (4) holds for every $n \in \mathbb{N}$ with $n < m$. In particular, $p(x_L, x_{L+m-\nu_4}) \le \varepsilon + \delta_4$. In the case of $p(x_L, x_{L+m-\nu_4}) \le \varepsilon$, we have

$$\begin{aligned} p(x_L, x_{L+m}) &\le p(x_L, x_{L+m-\nu_4}) + \sum_{j=1}^{\nu_4} p(x_{L+m-j}, x_{L+m-j+1}) \\ &\le \varepsilon + \nu_4\,\delta_4 \,/\, \nu_4 = \varepsilon + \delta_4. \end{aligned}$$

In the other case of $\varepsilon < p(x_L, x_{L+m-\nu_4}) \le \varepsilon + \delta_4$, we have

$$\begin{aligned} p(x_L, x_{L+m}) &\le p(x_L, x_{L+\nu_4}) + p(x_{L+\nu_4}, x_{L+m}) \\ &\le \delta_4 + \psi_{\nu_4}\big(p(x_L, x_{L+m-\nu_4})\big) \le \delta_4 + \varepsilon. \end{aligned}$$

Therefore (4) holds when $n = m$. Thus, by induction, we obtain (4) for all $n \in \mathbb{N}$, which implies (3). By Lemma 3, $\{x_n\}$ is p-Cauchy. So $\{x_n\}$ is also a Cauchy sequence by Lemma 2. From the completeness of X, $\{x_n\}$ converges to some $z \in X$. In the case of (i), we have

$$z = \lim_{n\to\infty} T^{\ell+n} x = \lim_{n\to\infty} T^\ell x_n = T^\ell \left(\lim_{n\to\infty} x_n \right) = T^\ell z.$$

That is, z is a fixed point of T^ℓ. By (2), we have

$$p(z, z) = \lim_{n\to\infty} p(T^{n\ell} z, T^{n\ell} z) = 0$$

and

$$p(z, Tz) = \lim_{n\to\infty} p(T^{n\ell} z, T \circ T^{n\ell} z) = 0.$$

By Lemma 1, we have $Tz = z$. We assume that y is a fixed point of T. Then

$$p(z, y) = \lim_{n\to\infty} p(T^n z, T^n y) = 0.$$

By Lemma 1 again, we have $z = y$. That is, the fixed point z is unique. By (2) again, we obtain $\lim_n p(z, T^n x) = \lim_n p(T^n x, z) = 0$. In the case of (ii), by ($\tau 3$), we have

$$\limsup_{n\to\infty} p(x_n, z) \le \limsup_{n\to\infty} \liminf_{m\to\infty} p(x_n, x_m) \le \limsup_{n\to\infty} \sup_{m>n} p(x_n, x_m) = 0.$$

Therefore $\lim_n \varphi_\ell\big(p(x_n, z)\big) = 0$. Since $p(x_n, T^\ell z) \le \varphi_\ell\big(p(x_{n-\ell}, z)\big)$ for $n \in \mathbb{N}$, we obtain

$$\limsup_{n\to\infty} p(x_n, T^\ell z) \le \limsup_{n\to\infty} \varphi_\ell\big(p(x_{n-\ell}, z)\big) = 0.$$

By Lemma 3, $\{x_n\}$ also converges to $T^\ell z$, which implies $T^\ell z = z$. As in the case of (i), we can obtain the desired result. □

We shall prove a generalization of Edelstein's fixed point theorem [4].

Let X be a metric space with a τ-distance p. For $\theta \in (0, \infty)$, X is called *θ-chainable* with respect to p [4, 16] if for each $(x, y) \in X \times X$, there exists a finite sequence $\{u_0, u_1, u_2, \cdots, u_k\}$ in X such that $u_0 = x$, $u_k = y$ and $p(u_{i-1}, u_i) < \theta$ for $i = 1, 2, \cdots, k$.

DEFINITION 4. *Let X be a metric space with a τ-distance p. Then a mapping T on X is said to be a (p,θ)-ACMK* if there exists a sequence $\{\varphi_n\}$ of functions from $[0, \theta)$ into $[0, \infty]$ satisfying the following:*

(C1) $\lim_{t\to 0} \limsup_{n\to\infty} \varphi_n(t) = 0$.
(C2) For each $\varepsilon \in (0, \theta)$, there exist $\delta \in (0, \theta - \varepsilon)$ and $\nu \in \mathbb{N}$ such that $\varphi_\nu(t) \le \varepsilon$ for all $t \in [\varepsilon, \varepsilon + \delta]$.
(C3) If $p(x, y) < \theta$ and $\varphi_n\big(p(x, y)\big) = 0$, then $p(T^n x, T^n y) = 0$.
(C4) If $p(x, y) < \theta$ and $\varphi_n\big(p(x, y)\big) > 0$, then $p(T^n x, T^n y) < \varphi_n\big(p(x, y)\big)$.

The following is a generalization of Theorem 3.6 in [16].

THEOREM 5. *Let X be a complete metric space. Suppose that X is θ-chainable with respect to p for some $\theta \in (0, \infty)$ and for some τ-distance p on X. Let T be a (p,θ)-ACMK* on X with $\{\varphi_n\}$ in Definition 4. Assume that either of the following holds:*

(i) T^ℓ is continuous for some $\ell \in \mathbb{N}$.
(ii) $\lim_{t\to 0} \varphi_\ell(t) = 0$ for some $\ell \in \mathbb{N}$.

Then there exists a unique fixed point $z \in X$. Moreover such z satisfies

$$p(z, z) = \lim_{n\to\infty} p(z, T^n x) = \lim_{n\to\infty} p(T^n x, z) = 0 \quad \text{and} \quad \lim_{n\to\infty} T^n x = z$$

for all $x \in X$.

Proof. For $x, y \in X$, there exist $u_0, u_1, u_2, \cdots, u_k \in X$ such that $u_0 = x$, $u_k = y$ and $p(u_{i-1}, u_i) < \theta$ for $i = 1, 2, \cdots, k$. As in the proof of Theorem 4,

$$\lim_{n\to\infty} p(T^n u_{i-1}, T^n u_i) = 0$$

for all i. Hence we obtain

$$\lim_{n\to\infty} p(T^n x, T^n y) \le \lim_{n\to\infty} \sum_{i=1}^{k} p(T^n u_{i-1}, T^n u_i) = 0.$$

Fix $x \in X$ and define a sequence $\{x_n\}$ in X by $x_n := T^n x$ for $n \in \mathbb{N}$. Then we can prove that for every $\varepsilon \in (0, \theta/2)$, there exists $N \in \mathbb{N}$ such that $p(x_L, x_{L+n}) < 2\varepsilon$ for all $L, n \in \mathbb{N}$ with $L \ge N$. That is, (3) holds. We can prove the remainder as in the proof of Theorem 4. □

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