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Arif Rafiq

STRONG CONVERGENCE OF AN IMPLICIT ITERATION PROCESS FOR A FINITE FAMILY OF QUASI-CONTRACTIVE OPERATORS

Abstract. The purpose of this paper is to establish a strong convergence of an implicit iteration process to a common fixed point for a finite family of quasi-contractive operators. Our theorems give an affirmative response to a question raised by [Xu and Ori, Numer. Funct. Anal. Optim. 22 (2001) 767-773].

1. Introduction and Preliminaries

We recall the following definitions in a metric space (X, d). A mapping $T : X \to X$ is called an a-contraction if

$$d(Tx, Ty) \leq ad(x, y) \text{ for all } x, y \in X, \tag{1.1}$$

where $a \in (0, 1)$.

The map T is called Kannan mapping [9] if there exists $b \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X. \tag{1.2}$$

A similar definition is due to Chatterjea [5]: there exists a $c \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)] \text{ for all } x, y \in X. \tag{1.3}$$

Combining these three definitions, Zamfirescu [16] proved the following important result.

THEOREM 1. *Let (X, d) be a complete metric space and $T : X \to X$ a mapping for which there exists the real numbers a, b and c satisfying $a \in (0, 1)$, $b, c \in (0, \frac{1}{2})$ such that for each pair $x, y \in X$, at least one of the following conditions holds:*

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(z_1) $d(Tx,Ty) \leq ad(x,y)$,
(z_2) $d(Tx,Ty) \leq b[d(x,Tx) + d(y,Ty)]$,
(z_3) $d(Tx,Ty) \leq c[d(x,Ty) + d(y,Tx)]$,

Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, \quad n \in \mathbb{N}$$

converges to p for any arbitrary but fixed $x_1 \in X$.

One of the most general contraction condition for which the unique fixed point can be approximated by means of Picard iteration, has been obtained by Ciric [7] in 1974: there exists $0 < h < 1$ such that

(QC) $\quad d(Tx,Ty) \leq h\{d(x,y),\ d(x,Tx),\ d(y,Ty),\ d(x,Ty),\ d(y,Tx)\}$,
$\forall x,\ y \in X$.

Remarks. 1. A mapping satisfying (QC) is commonly called quasi contraction. It is obvious that each of the conditions $(1.1 - 1.3)$ and $(z_1 - z_3)$ implies (QC).

2. An operator T satisfying the contractive conditions $(z_1 - z_3)$ in the above theorem is called $Z-$operator.

Let C be a nonempty closed convex subset of a normed space E_1.

In 2001, Xu and Ori [15] introduced the following implicit iteration process for a finite family of nonexpansive mappings $\{T_i : i \in I\}$ (here $I = \{1, 2, \ldots, N\}$), with $\{\alpha_n\}$ a real sequence in $(0,1)$, and an initial point $x_0 \in C$:

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + (1 - \alpha_1)T_1 x_1, \\
x_2 &= \alpha_2 x_1 + (1 - \alpha_2)T_2 x_2, \\
&\ \ \vdots \\
x_N &= \alpha_N x_{N-1} + (1 - \alpha_N)T_N x_N, \\
x_{N+1} &= \alpha_{N+1} x_N + (1 - \alpha_{N+1})T_1 x_{N+1}, \\
&\ \ \vdots
\end{aligned}$$

which can written in the following compact form:

(1.4) $$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T_n x_n, \ \forall n \geq 1,$$

where $T_n = T_{n(\mathrm{mod}\, N)}$ (here the $modN$ function takes values in I). Xu and Ori proved the weak convergence of this process to a common fixed point of the finite family defined in a Hilbert space. They further remarked that it is yet unclear what assumptions on the mappings and/or the parameters $\{\alpha_n\}$ are sufficient to guarantee the strong convergence of the sequence $\{x_n\}$.

In [17], Zhou and Chang studied the weak and strong convergence of this implicit process to a common fixed point for a finite family of nonexpansive mappings. More precisely, they proved the following result.

THEOREM 2. *([17, Theorem 3]) Let E be a uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $\{T_i : i \in I\}$ be N semi-compact nonexpansive self-mappings of K with $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$ (here $F(T_i)$ denotes the set of fixed points of T_i). Suppose that $x_0 \in K$ and $\{\alpha_n\} \subset (b, c)$ for some b, $c \in (0, 1)$. Then the sequence $\{x_n\}$ defined by the implicit iteration process (1.4) converges strongly to a common fixed point in F.*

In [6], Chidume and Shahzad studied the strong convergence of the implicit process (1.4) to a common fixed point for a finite family of nonexpansive mappings. They proved the following result.

THEOREM 3. *([6, Theorem 3.3]) Let E be a uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $\{T_i : i \in I\}$ be N nonexpansive self-mappings of K with $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$. Suppose that one of the mappings in $\{T_i : i \in I\}$ is semi-compact. Let $\{\alpha_n\}_{n\geq 1} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1)$. From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process (1.4). Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in I\}$.*

Remark. It is worth mentioning here that, the theorem 1 of Zhou and Chang in [17] is "for convergence of modified implicit iteration process for a finite family of asymptotically nonexpansive mappings in uniformly convex Banach spaces".

The purpose of this paper is to study the strong convergence of implicit iteration process (1.4) to a common fixed point for a finite family of quasi contractive operators in normed spaces. The results presented in this paper extend and improve among others the corresponding results of Refs. [1-3, 15, 17].

2. Main results

THEOREM 4. *Let C be a nonempty closed convex subset of a normed space E_1. Let $\{T_1, T_2, \ldots, T_N\} : C \to C$ be N operators satisfying conditions*

$(z_1 - z_3)$ with $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$. From arbitrary $x_0 \in C$, define the sequence $\{x_n\}$ by the implicit iteration process (1.4) satisfying $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, \ldots, T_N\}$.

Proof. By Theorem 1, we know that any of the operators $\{T_1, T_2, \ldots, T_N\}$ have a unique fixed point, i.e., $F(T_i) \neq \phi$, $\forall i = 1, 2, \ldots, N$ and hence, in view $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$, the operators $\{T_1, T_2, \ldots, T_N\}$ have a (unique) common fixed point say w. Consider x, $y \in C$. Since each $T_i : i \in I$ is a $Z-$operator, at least one of the conditions (z_1), (z_2) and (z_3) is satisfied. Following Berinde [1-3], if (z_2) holds, then

$$\begin{aligned}\|T_i x - T_i y\| &\leq b\left[\|x - T_i x\| + \|y - T_i y\|\right] \\ &\leq b\left[\|x - T_i x\| + \|y - x\| + \|x - T_i x\| + \|T_i x - T_i y\|\right],\end{aligned}$$

implies

$$(1-b)\|T_i x - T_i y\| \leq b\|x-y\| + 2b\|x - T_i x\|,$$

which yields (using the fact that $0 \leq b < \frac{1}{2}$)

$$\|T_i x - T_i y\| \leq \frac{b}{1-b}\|x-y\| + \frac{2b}{1-b}\|x - T_i x\|. \tag{2.1}$$

If (z_3) holds, then similarly we obtain

$$\|T_i x - T_i y\| \leq \frac{c}{1-c}\|x-y\| + \frac{2c}{1-c}\|x - T_i x\|. \tag{2.2}$$

Denote

$$\delta = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}. \tag{2.3}$$

Then we have $0 \leq \delta < 1$ and, in view of (z_1), (2.1-2.3) it results that the inequality

$$\|T_i x - T_i y\| \leq \delta\|x-y\| + 2\delta\|x - T_i x\| \tag{AR}$$

holds for all $x, y \in C$.

Using (1.4), we have

$$\begin{aligned}\|x_n - w\| &= \|\alpha_n x_{n-1} + (1-\alpha_n) T_n x_n - w\| \\ &= \|\alpha_n (x_{n-1} - w) + (1-\alpha_n)(T_n x_n - w)\| \\ &\leq \alpha_n\|x_{n-1} - w\| + (1-\alpha_n)\|T_n x_n - w\|.\end{aligned} \tag{2.4}$$

Now for $y = x_n$ and $x = w$, (AR) gives

$$\|Tx_n - w\| \leq \delta\|x_n - w\|, \tag{2.5}$$

and hence, by (2.4-2.5) we obtain

$$\|x_n - w\| \leq \frac{\alpha_n}{1-\delta(1-\alpha_n)} \|x_{n-1} - w\| . \tag{2.6}$$

Let $A_n = \alpha_n$, $B_n = 1 - \delta(1-\alpha_n)$ and consider

$$\begin{aligned}\beta_n &= 1 - \frac{A_n}{B_n} = 1 - \frac{\alpha_n}{1-\delta(1-\alpha_n)} \\ &= \frac{(1-\delta)(1-\alpha_n)}{1-\delta(1-\alpha_n)} \geq (1-\delta)(1-\alpha_n).\end{aligned}$$

Indeed

$$1 - \delta \leq 1 - \delta(1-\alpha_n) \leq 1,$$

implies

$$\frac{A_n}{B_n} \leq 1 - (1-\delta)(1-\alpha_n).$$

Thus from (2.6), we get

$$\|x_n - w\| \leq [1 - (1-\delta)(1-\alpha_n)] \|x_{n-1} - w\| , \ n = 1, 2, \ \dots \ . \tag{2.7}$$

By (2.7) we inductively obtain

$$\|x_n - w\| \leq \prod_{k=1}^{n} [1 - (1-\delta)(1-\alpha_k)] \|x_0 - w\| , \ n = 1, 2, \dots \ . \tag{2.8}$$

Using the fact that $0 \leq \delta < 1$, $0 < \alpha_n < 1$, and $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$, it results that

$$\lim_{n\to\infty} \prod_{k=1}^{n} [1 - (1-\delta)(1-\alpha_k)] = 0,$$

which by (2.8) implies

$$\lim_{n\to\infty} \|x_n - w\| = 0.$$

Consequently $x_n \to w \in F$ and this completes the proof. □

Remarks. 1. The Chatterjea's and the Kannan's contractive conditions (1.3) *and* (1.2) are both included in the class of Zamfirescu operators.

2. Recently concerning the convergence problems of an implicit (or non-implicit) iterative process to a common fixed point of finite family of nonexpansive mappings in Hilbert spaces have been considered by several authors (see for example [4, 8, 12–15, 17].

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DEPARTMENT OF MATHEMATICS
COMSATS INSTITUTE OF INFORMATION TECHNOLOGY
ISLAMABAD, PAKISTAN
30, H-8/1
e-mail: arafiq@comsats.edu.pk

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