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A KNESER-TYPE THEOREM FOR AN INTEGRAL EQUATION IN LOCALLY CONVEX SPACES

Abstract. We shall give sufficient conditions for the existence of solutions of the integral equation (1) in locally convex spaces. We also prove that the set of these solutions is a continuum.

Let E be a quasicomplete locally convex topological vector space, and let P be a family of continuous seminorms generating the topology of E.

Denote by Ω the family of all open, balanced and convex neighbourhoods of 0 in E. Assume that $I = [0, a]$ and $B = \{x \in E : p_i(x) \le b, i = 1, \dots, k\}$, where $p_1, \dots, p_k \in P$.

In this paper we investigate the existence of solutions and the structure of the solutions set of the integral equation

$$x(t) = \int_0^t K(t,s) f(s, x(s)) ds, \tag{1}$$

where

1° $f : I \times B \mapsto E$ is a bounded continuous function;

2° $K(t,s) = \frac{A(t,s)}{(t-s)^r}$, $0 < r < 1$, where A is a real continuous function.

Put $M = \sup\{p_i(f(t,x)) : t \in I, x \in B, i = 1, \dots, k\}$ and $c = \max_{t,s \in I} |A(t,s)|$. Choose a positive number d such that $d \le a$ and $M \cdot c \cdot \frac{d^{1-r}}{1-r} < b$.

Let $J = [0, d]$. Denote by $C = C(J, E)$ the space of continuous functions $J \mapsto E$ endowed with the topology of uniform convergence.

For any bounded subset D of E and $p \in P$ we denote by $\beta_p(D)$ the infimum of all $\varepsilon > 0$ for which there exists a finite subset $\{x_1, x_2, \dots, x_n\}$ of E such that $D \subset \{x_1, x_2, \dots, x_n\} + B_p(\varepsilon)$, where $B_p(\varepsilon) = \{x \in E : p(x) \le \varepsilon\}$. The family $(\beta_p(D))_{p \in P}$ is called the measure of noncompactness of D.

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It is known [6] that
1) X is relatively compact in $E \;\Leftrightarrow\; \beta_p(X) = 0$ for every $p \in P$;
2) $X \subset Y \;\Rightarrow\; \beta_p(X) \le \beta_p(Y)$;
3) $\beta_p(X \cup Y) = \max\{\beta_p(X), \beta_p(Y)\}$;
4) $\beta_p(X + Y) \le \beta_p(X) + \beta_p(Y)$;
5) $\beta_p(\lambda X) = |\lambda|\beta_p(X) \quad (\lambda \in \mathbb{R})$;
6) $\beta_p(\bar{X}) = \beta_p(X)$;
7) $\beta_p(\mathrm{conv X}) = \beta_\mathrm{p}(\mathrm{X})$;
8) $\beta_p(\bigcup_{0 \le \lambda \le h} \lambda X) = h\beta_p(X)$.

The following has been proved in [7].

LEMMA 1. *Let H be a bounded countable subset of C. For each $t \in J$ put $H(t) = \{u(t) : u \in H\}$. If the space E is separable, then for each $p \in P$ the function $t \mapsto \beta_p(H(t))$ is integrable and*

$$\beta_p\Big(\Big\{\int_J u(s)ds : u \in H\Big\}\Big) \le \int_J \beta_p(H(s))ds.$$

In what follows we shall need the following result of W. Mydlarczyk given in [5].

THEOREM 1. *Let $\alpha > 0$ and let $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a nondecreasing function such that $g(0) = 0$, $g(t) > 0$ for $t > 0$. Then the equation*

$$u(t) = \int_0^t (t-s)^{\alpha-1} g(u(s))ds \quad (t \ge 0)$$

has a nontrivial continuous solution if and only if

$$\int_0^\delta \frac{1}{s}\left[\frac{s}{g(s)}\right]^{\frac{1}{\alpha}} ds < \infty \qquad (\delta > 0).$$

We can now formulate our main result.

THEOREM 2. *Suppose that for each $p \in P$ there exists a continuous nondecreasing function $w_p : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $w_p(0) = 0$, $w_p(t) > 0$ for $t > 0$ and*

$$\int_0^\delta \frac{1}{s}\left[\frac{s}{w_p(s)}\right]^{\frac{1}{1-r}} ds = \infty \qquad (\delta > 0). \tag{2}$$

If $1^\circ - 2^\circ$ hold and

$$\beta_p(f(t, X)) \le w_p(\beta_p(X)) \tag{3}$$

for $p \in P$, $t \in I$ and bounded subsets X of E, then the set S of all solutions of (1) defined on J is nonempty, compact and connected in $C(J, E)$.

Proof. **1.** Put

$$r(x) = \begin{cases} x & \text{for } x \in B \\ \frac{x}{K(x)} & \text{for } x \in E \setminus B \end{cases}$$

and $g(t,x) = f(t, r(x))$ for $(t,x) \in J \times E$, where K is the Minkowski functional of B. As B is a closed, balanced and convex neighbourhood of 0, from the known properties of Minkowski's functional it follows that r is a continuous function from E into B and

$$r(X) \subset \bigcup_{0 \le \lambda \le 1} \lambda X \quad \text{for any subset } X \text{ of } E.$$

Thus $\beta_p(r(X)) \le \beta_p(X)$ for any $p \in P$ and any bounded subset X of E.

Consequently, g is a bounded continuous function from $J \times E$ into E such that

$$\beta_p(g(t,X)) \le w_p(\beta_p(X)) \tag{3'}$$

for $p \in P$, $t \in J$ and bounded subsets X of E, and

$$p_i(g(t,x)) \le b \quad \text{for } i = 1, \ldots, k, \quad t \in J \text{ and } x \in E. \tag{4}$$

We introduce a mapping F defined by

$$F(x)(t) = \int_0^t K(t,s) g(s, x(s)) ds, \quad (x \in B,\ t \in J).$$

Arguing similarly as in [3, p. 132-133] we can prove that the set $F(C)$ is equicontinuous and bounded. On the other hand, from the following Krasnoselskii type

LEMMA 2. *For any $u \in C$ and $U \in \Omega$ there exists a V in Ω such that*

$$f(t, x(t)) - f(t, u(t)) \in U \quad \textit{for every } t \in J$$

whenever $x \in C$ and $x(t) - u(t) \in V$ for every $t \in J$. (cf. [8]).

It follows that F is a continuous mapping from C into itself.

It is clear from (1) and (4) that if $x = F(x)$, then

$$p_i(x(t)) \le \int_0^t \frac{|A(t,s)|}{(t-s)^r} M ds \le \frac{d^{1-r}}{1-r} \cdot M \cdot c < b,$$

so $x(t) \in B$ for $t \in J$. Therefore, a function $x \in C$ is a solution of (1) iff $x = F(x)$.

2. For any $n \in N$ put

$$u_n(t) = \begin{cases} 0 & \text{if } 0 \le t \le \frac{d}{n} \\ \int_0^{t-\frac{d}{n}} \frac{A(t,s)}{(t-s)^r} g(s, u_n(s)) ds & \text{if } \frac{d}{n} \le t \le d. \end{cases}$$

Then u_n is a continuous function $J \mapsto B$ and

$$\lim_{n\to\infty}(u_n(t) - F(u_n)(t)) = 0 \tag{5}$$

uniformly for $t \in J$. Let $V = \{u_n : n \in N\}$. From (5) it follows that the set $\{u_n - F(u_n) : n \in N\}$ is relatively compact in C. Since

$$V \subset \{u_n - F(u_n) : n \in N\} + F(V) \tag{6}$$

and the set $F(V)$ is bounded and equicontinuous, we conclude that set V is also bounded and equicontinuous. Hence for each $p \in P$ the function $t \mapsto v(t) = \beta_p(V(t))$ is continuous on J.

Denote by H a closed separable subspace of E such that

$$g(s, u_n(s)) \in H \quad \text{for } s \in J,\ n \in N.$$

Let $(\beta_p^H)_{p\in P}$ be the measure of noncompactness in H. Fix $t \in J$ and $p \in P$. From (3′) we have

$$\beta_p^H(g(s, V(s))) \le 2\beta_p(g(s, V(s))) \le 2w_p(\beta_p(V(s))) \ \text{ for } s \in [0, t].$$

By Lemma 1, we get

$$\begin{aligned}\beta_p(F(V)(t)) &= \beta_p(\{\int_0^t \frac{A(t,s)}{(t-s)^r} g(s, u_n(s))ds : n \in N\}) \\ &\le \beta_p^H(\{\int_0^t \frac{A(t,s)}{(t-s)^r} g(s, u_n(s))ds : n \in N\}) \\ &\le \int_0^t \beta_p^H(\{\frac{A(t,s)}{(t-s)^r} g(s, u_n(s)) : n \in N\})ds \\ &= \int_0^t \frac{|A(t,s)|}{(t-s)^r}\beta_p^H(g(s, V(s))ds \le 2\int_0^t \frac{|A(t,s)|}{(t-s)^r} w_p(\beta_p(V(s))ds.\end{aligned}$$

On the other hand, from (5) and (6) we obtain

$$\beta_p(V(t)) \le \beta_p(F(V)(t)).$$

Hence

$$\beta_p(V(t)) \le 2\int_0^t \frac{|A(t,s)|}{(t-s)^r} w_p(\beta_p(V(s))ds \quad (t \in J, p \in P),$$

i.e.

$$v(t) \le 2c\int_0^t \frac{1}{(t-s)^r} w_p(v(s))ds \quad \text{for } t \in J.$$

Applying Theorem 1 with $\alpha = 1-r$ and theorem on integral inequalities ([2], Lemma 1) from this we deduce that $v(t) = 0$ for $t \in J$. Thus $\beta_p(V(t)) = 0$ for $t \in J$ and $p \in P$. Therefore for each $t \in J$ the set $V(t)$ is relatively compact in E. As the set V is equicontinuous, Ascoli's theorem proves that V is relatively compact in C. Hence the sequence (u_n) has a limit point u. As F is continuous from (5) we conclude that $u = F(u)$, i.e. u is a solution of (1). This proves that the set S is nonempty.

3. Let us first remark that the set S is compact in C.

Indeed, as $(I - F)(S) = \{0\}$, in the same way as in Step **2**, we can prove that S is relatively compact in C. Moreover, from the continuity of F it follows that S is closed in C.

Suppose that S is not connected. Thus there exist nonempty closed sets S_0, S_1 such that $S = S_0 \cup S_1$ and $S_0 \cap S_1 = \emptyset$. As S_0, S_1 are compact subsets of C and C is a Tichonov space, this implies ([4], §41, II, Remark 3) that there exists a continuous function $v : C \mapsto [0,1]$ such that $v(x) = 0$ for $x \in S_0$ and $v(x) = 1$ for $x \in S_1$.

Further, for any $n \in N$ we define a mapping F_n by

$$F_n(x)(t) = F(x)(r_n(t)) \quad (x \in C, t \in J),$$

where

$$r_n(t) = \begin{cases} 0 & \text{for } 0 \le t \le \frac{d}{n} \\ t - \frac{d}{n} & \text{for } \frac{d}{n} \le t \le d. \end{cases}$$

It can be easily verified (cf. [8]) that
(i) F_n ia a continuous mapping $C \mapsto C$;
(ii) $\lim_{n\to\infty} F_n(x) = F(x)$ uniformly for $x \in C$;
(iii) $I - F_n$ is a homeomorphism $C \mapsto C$.
Here I denotes the identity mapping.

Fix $u_0 \in S_0$, $u_1 \in S_1$ and $n \in N$. Put

$$e_n(\lambda) = \lambda(u_1 - F_n(u_1)) + (1 - \lambda)(u_0 - F_n(u_0)) \quad (0 \le \lambda \le 1).$$

Let $u_{n\lambda} = (I - F_n)^{-1}(e_n(\lambda))$. As $e_n(\lambda)$ depends continuosly on λ and $I - F_n$ is a homeomorphism, we see that the mapping $\lambda \mapsto v(u_{n\lambda})$ is continuous on $[0,1]$. Moreover, $u_{n0} = u_0$ and $u_{n1} = u_1$, so that $v(u_{n0}) = 0$ and $v(u_{n1}) = 1$. Thus there exists $\lambda_n \in [0,1]$ such that

$$v(u_{n\lambda_n}) = \frac{1}{2}. \tag{7}$$

For simplicity put $v_n = u_{n\lambda_n}$ and $V = \{v_n : n = 1, 2, \ldots\}$. Since $\lim_{n\to\infty} e_n(\lambda) = 0$ uniformly for $\lambda \in [0,1]$, we get

$$\lim_{n\to\infty} (v_n - F(v_n)) = \lim_{n\to\infty} (e_n(\lambda) + F_n(v_n) - F(v_n)) = 0, \tag{8}$$

and therefore the set $(I-F)(V)$ is relatively compact in C. Using now similar argument as in Step **2**, we can prove that the set V is relatively compact in C. Consequently the sequence (v_n) has a limit point z. In view of (8) and continuity of F, we infer that $z \in S$, so $v(z) = 0$ or $v(z) = 1$. On the other hand, from (7) it is clear that $v(z) = \frac{1}{2}$, which yields a contradiction. Thus S is connected.

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