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HOMOGENEOUS NON-SYMMETRIC MEANS OF TWO VARIABLES

Abstract. Let $f, g : I \to \mathbb{R}$ be given continuous functions on the interval I such that $g \neq 0$, and $h := \frac{f}{g}$ is strictly monotonic (thus invertible) on I. Taking an increasing nonconstant function μ on $[0,1]$

$$M_{f,g,\mu}(x,y) := h^{-1}\left(\frac{\int_0^1 f(tx+(1-t)y)d\mu(t)}{\int_0^1 g(tx+(1-t)y)d\mu(t)}\right) \quad (x, y \in I)$$

is a mean value of $x, y \in I$. Here we solve the homogeneity equation

$$M_{f,g,\mu}(tx, ty) = tM_{f,g,\mu}(x,y) \quad (x, y \in I, t \in I_x \cap I_y)$$

assuming that $I \subset]0, \infty[$ is open, $1 \in I$, f, g are three times continuously differentiable functions with $h' \neq 0$ and the moments of μ satisfy two conditions (which do not hold for symmetric means).

1. Introduction

Let I be an interval $f, g : I \to \mathbb{R}$ be given continuous functions such that $g(x) \neq 0$ for $x \in I$ and $h(x) := \frac{f(x)}{g(x)}$ $(x \in I)$ is strictly monotonic (thus invertible) on I. Let further μ be an increasing nonconstant function on $[0,1]$ and

$$M_{f,g,\mu}(x,y) := h^{-1}\left(\frac{\int_0^1 f(tx+(1-t)y)d\mu(t)}{\int_0^1 g(tx+(1-t)y)d\mu(t)}\right) \quad (x, y \in I)$$

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where the integrals are Riemann-Stieltjes ones. From the mean value theorem it follows that

$$\min\{x,y\} \leq M_{f,g,\mu}(x,y) \leq \max\{x,y\}$$

i.e. $M_{f,g,\mu}$ is a two-variable mean on I. As $M_{f,g,c\mu}(x,y) = M_{f,g,\mu}(x,y)$ for any positive c we may assume without restricting the generality that

$$\int_0^1 d\mu(t) = 1. \tag{1}$$

We remark that $M_{f,g,\mu}$ could be defined not just for two, but for several variables analogously (see [3]).

Páles [6] noticed that with suitable choice of μ both the quasi-arithmetic means weighted with a weight function and the Cauchy (and also several other) mean can be obtained. Taking namely

$$\mu(t) = \mu_1(t) = \begin{cases} 0 & \text{if } t = 0 \\ A & \text{if } 0 < t < \frac{1}{2} \\ A + B & \text{if } \frac{1}{2} \leq t < 1 \\ A + B + C & \text{if } t = 1 \end{cases}$$

where $A, C > 0, B \geq 0$. Condition (1) holds for μ_1 if and only if $A + B + C = 1$, thus we get that

$$M_{f,g,\mu_1}(x,y) = h^{-1}\left(\frac{Cf(x) + (1-A-C)f\left(\frac{x+y}{2}\right) + Af(y)}{Cg(x) + (1-A-C)g\left(\frac{x+y}{2}\right) + Ag(y)}\right) \quad (x, y \in I)$$

where $A, C > 0, A + C \leq 1$. If $A = C = 1/2$ then this mean is the quasi-arithmetic mean with mapping function $h = f/g$ weighted with the weight function g. Concerning these means, and the homogeneity of their multivariable version see Aczél-Daróczy [1].

Let

$$\mu(t) = \mu_2(t) = t$$

and for continuous functions f denote the integral $\int_{x_0}^{x} f(s)ds \quad (x_0, x \in I)$ by $\tilde{f}(x)$. Then

$$M_{f,g,\mu_2}(x,y) = \left(\frac{\tilde{f}'}{\tilde{g}'}\right)^{-1}\left(\frac{\int_0^1 \tilde{f}'(tx+(1-t)y)dt}{\int_0^1 \tilde{g}'(tx+(1-t)y)dt}\right)$$
$$= \left(\frac{\tilde{f}'}{\tilde{g}'}\right)^{-1}\left(\frac{\tilde{f}(x)-\tilde{f}(y)}{\tilde{g}(x)-\tilde{g}(y)}\right) \quad (x,y\in I),$$

is the Cauchy or difference mean. The homogeneity problem for the $n \geq 3$ variable version of these means has been solved in [3].

Several other means, among others some non-symmetric ones, can be obtained by suitable choices of μ. The aim of this paper is to determine the homogeneous non-symmetric $M_{f,g,\mu}$ means for a large class of μ.

2. Some remarks on homogeneous functions

In the sequel $I \subset]0,\infty[$ will be an *open interval containing the point* 1. The mean $M_{f,g,\mu}$ is called *homogeneous* (in the generalized sense) if the equation (called homogeneity equation)

$$(2) \qquad M_{f,g,\mu}(tx,ty) = tM_{f,g,\mu}(x,y) \quad (x,y\in I, t\in I_x\cap I_y)$$

holds, where, for any $x\in I$, $I_x := \{t\in\mathbb{R} : tx\in I\}$.

Concerning (2) we can pose two problems:

(i) find all functions f,g satisfying the functional equation (2);

(ii) find all means $M_{f,g,\mu}$ which satisfy (2).

Clearly from the solution of (i) one can get the solutions of (ii). We shall see that problem (i) has more solutions than (ii) as the same mean can be built up from several pairs f,g (see [2], [4] for the equality problem of two variable means).

In each class of means the *homogeneous* ones form a very important subclass.

In the sequel we shall use the next two lemmas.

LEMMA 1. *If* $f : I \to \mathbb{R}$ *satisfies*

$$(3) \qquad f(tx) = t^m f(x) \quad (x\in I, t\in I_x = \{s\in\mathbb{R} : sx\in I\})$$

then $f(x) = f(1)x^m$.

To prove this substitute $x=1$ in (3) and observe that $I_1 = I$.

LEMMA 2. *Let $F : I^2 \to \mathbb{R}$ be a $k+l$-times continuously differentiable homogeneous function of degree m i.e*

$$F(tx, ty) = t^m F(x, y) \quad (x, y \in I, t \in I_x \cap I_y).$$

Then the derivative

$$\frac{\partial^{k+l} F(x, y)}{\partial x^k \partial y^l}$$

is homogeneous of degree $m - (k + l)$.

To justify this differentiate the homogeneity equation, first with respect to y l-times and after that with respect to x k-times.

To determine the unknown functions f, g in the homogeneity equation (2) we need two independent equations. For this purpose we use the partial derivatives of $M_{f,g,\mu}$ taken at the point (x, x). The second and third derivatives

$$\frac{\partial^2 M_{f,g,\mu}(x, x)}{\partial x \partial y}, \text{ and } \frac{\partial^3 M_{f,g,\mu}(x, x)}{\partial x^2 \partial y}$$

are by Lemma 2 homogeneous functions of degree -1 and -2 respectively thus by Lemma 1 these derivatives are equal to constant$/x$ and constant$/x^2$ respectively. For non-symmetric means under some additional conditions these derivatives supply the two equations needed in the solution.

3. Differential equations for h, f, g

Suppose now that $f, g : I \to \mathbb{R}$ are three times continuously differentiable functions $g(x) \neq 0\,(x \in I)$, $h(x) = f(x)/g(x)$ $(x \in I)$ has non-vanishing derivative on I and $M_{f,g,\mu}$ is homogeneous.

Differentiating $M_{f,g,\mu}(x, y)$ and substituting $y = x$ we obtain that

$$\frac{\partial^2 M_{f,g,\mu}(x, x)}{\partial x \partial y} = (m_1^2 - m_2)(2G + H), \tag{4}$$

$$\begin{aligned} \frac{\partial^3 M_{f,g,\mu}(x, x)}{\partial x^2 \partial y} &= (-2m_1^3 + 3m_1 m_2 - m_3)(3G^2 + 3GH + H^2) \\ &\quad + (m_1^3 - m_1^2 + m_2 - m_3)H' \\ &\quad + (-2m_1^2 + 3m_1 m_2 + 2m_2 - 3m_3)G', \end{aligned} \tag{5}$$

where

$$G = G(x) := \frac{g'(x)}{g(x)}, \quad H = H(x) := \frac{h''(x)}{h'(x)} \quad (x \in I),$$

and

$$m_k = \int_0^1 t^k d\mu(t) \quad (k = 0, 1 \ldots)$$

are the kth moments of μ and by (1) $m_0 = 1$.

In finding the third derivative and also in some other elementary but tedious calculations (substituting long expressions into others etc.) the software package Maple V was used.

The above equations can be used for our purpose only if

$$(6) \qquad \begin{aligned} m_1^2 - m_2 &\neq 0 \\ -2m_1^3 + 3m_1m_2 - m_3 &\neq 0. \end{aligned}$$

The negation of the *first condition* means that

$$\int_0^1 t d\mu(t) = m_1 = \sqrt{m_2} = \left(\int_0^1 t^2 d\mu(t)\right)^{\frac{1}{2}}$$

which holds, if and only if, the measure given by μ is concentrated on a single point of the interval $[0, 1]$, i.e. the increasing function μ has one single jump. Thus, the first condition of (6) holds, if and only if, the measure given by μ is not concentrated on a single point.

The left hand side of the *second condition* in (6) can be rewritten as

$$-2m_1^3 + 3m_1m_2 - m_3 = m_1^3 - m_3 + 3m_1(m_2 - m_1^2).$$

Since $\sqrt{m_2} \geq m_1$, $\sqrt[3]{m_3} \geq m_1$ (as the nth power integral mean $\sqrt[n]{m_n}$ is an increasing function of n) the first term here is nonpositive the second is nonnegative (moreover positive, if the first condition of (6) holds). This shows that non-symmetry of the mean, in general, does not guarantee the second condition. At the same time, for symmetric means (like the Cauchy mean and the quasi arithmetic mean weighted by a weight function) the second condition of (6) does not hold. For this (two *particular* symmetric) means the solution of the homogeneity problem is much more complicated, one has to use in addition to (4) the fourth and sixth derivatives of the mean at the point (x, x) see [5].

Unfortunately we cannot characterize the functions μ that satisfy the second condition of (6). Calculating other third order derivatives than (5) leads to the same equations/conditions. Using the fourth derivative (similarly to [5]) leads to Riccati equations whose solutions cannot be obtained by help of quadratures. Thus the only possibilities are the use of the fifth or sixth derivatives. These however depend on the moments $m_1, \ldots, m_6$ which makes the system of differential equations unmanageable (at least for the time being). On the other hand the (nonsymmetric) examples of the last section show, that equality in the second condition of (6) in the class of non-symmetric means is rather exceptional.

REMARK (added in proof). Páles [6] noticed that

$$\int_0^1 (t-m_1)^3 d\mu(t) = \int_0^1 (t^3 - 3t^2 m_1 + 3tm_1^2 - m_1^3) d\mu(t) = m_3 - 3m_2 m_1 + 2m_1^3.$$

Thus *the second condition of* (6) *means* that the integral on the left hand side is non-zero, or *that the measure generated by the function* μ *has non-vanishing third central moment.*

Suppose now that (6) holds. Then from (4) we get that

$$2G + H = \frac{2c}{x}, \tag{7}$$

where c is a constant. Substituting $G = \frac{1}{2}\left(\frac{2c}{x} - H\right)$, $G' = \frac{1}{2}\left(-\frac{2c}{x^2} - H'\right)$ into (5) we obtain

$$\frac{\partial^3 M_{f,g,\mu}(x,x)}{\partial x^2 \partial y} = \left(m_1^3 - \frac{3}{2}m_1 m_2 + \frac{1}{2}m_3\right)\left(H' - \frac{1}{2}H^2 - \frac{6c^2}{x^2}\right) + \frac{(2m_1^2 - 3m_1 m_2 - 2m_2 + 3m_3)c}{x^2}$$

and hence

$$H' - \frac{1}{2}H^2 = \frac{2d}{x^2}, \tag{8}$$

where d is a constant.

Integrating (7) we get that

$$h'(x)g(x)^2 = \beta x^{2c}, \tag{9}$$

where β is an arbitrary constant.

The Riccati equation (8) transforms to the Euler differential equation

$$z'' + \frac{d}{x^2}z = 0 \tag{10}$$

with

$$z(x) = e^{-\frac{1}{2}\int H(x)\,dx} (> 0), \qquad H(x) = -2\frac{d}{dx}(\ln z(x)). \tag{11}$$

The nonzero solutions of (10) are

$$(12)\qquad \begin{aligned} &z = R\sqrt{x}\cos(\ln x^a + D), && \text{if } -(2a)^2 = 1-4d < 0\\ &z = R\sqrt{x}\cosh(\ln x^a + D), && \text{if } (2a)^2 = 1-4d > 0\\ &z = R\sqrt{x}\sinh(\ln x^a + D), && \text{if } (2a)^2 = 1-4d > 0\\ &z = R\sqrt{x}\exp(\pm\ln x^a), && \text{if } (2a)^2 = 1-4d > 0\\ &z = R\sqrt{x}, && \text{if } 1-4d = 0\\ &z = R\sqrt{x}(\ln x + D), && \text{if } 1-4d = 0, \end{aligned}$$

where $R \neq 0, a \neq 0, D$ are constants (which should be chosen such that $z(x) > 0$).

From (11)

$$\left(\ln|h'|\right)' = \frac{h''}{h'} = H = -2\frac{z'}{z} = \left(\ln z^{-2}\right)',$$

hence, by integrating, we obtain

$$(13)\qquad h' = K_1 z^{-2}, \qquad h = K_1 \textstyle\int z^{-2} + L,$$

and by (9), (13)

$$(14)\qquad g = K_2 x^c z, \quad f = g\,h,$$

where $K_1 \neq 0, K_2 \neq 0, L$ are arbitrary constants.

Performing the integrations with the solutions z given by (12) and introducing some new constants if necessary we get the following solutions for h, g and f

(15)

h	g	f
$K\tan(\ln x^a + D) + L$	$Mx^c\cos(\ln x^a + D)$	$Mx^c[K\sin(\ln x^a + D) + L\cos(\ln x^a + D)]$
$K\tanh(\ln x^a + D) + L$	$Mx^c\cosh(\ln x^a + D)$	$Mx^c[K\sinh(\ln x^a + D) + L\cosh(\ln x^a + D)]$
$K\coth(\ln x^a + D) + L$	$Mx^c\sinh(\ln x^a + D)$	$Mx^c[K\cosh(\ln x^a + D) + L\sinh(\ln x^a + D)]$
$Kx^{\mp 2a} + L$	$Mx^c x^{\pm 2a}$	$Mx^c\left[Kx^{\mp 2a} + Lx^{\pm 2a}\right]$
$K(\ln x + D)^{-1} + L$	$Mx^c\sqrt{x}(\ln x + D)$	$Mx^c\sqrt{x}[K + L(\ln x + D)]$
$K\ln x + L$	$Mx^c\sqrt{x}$	$Mx^c\sqrt{x}[K\ln x + L]$,

where $a, c, D, K, L, M \in \mathbb{R}$ are arbitrary constants apart from the restrictions $aKM \neq 0$.

4. Homogeneous non-symmetric means

THEOREM 1. *Let $I \subset]0,\infty[$ be an open interval containing the point 1, $f, g : I \to \mathbb{R}$ be three times continuously differentiable functions, $g(x) \neq 0\,(x \in I)$, $h'(x) = (f(x)/g(x))' \neq 0$ $(x \in I)$ and suppose that $\mu : I \to \mathbb{R}$ is an increasing nonconstant function whose moments satisfy (6). If $M_{f,g,\mu}$ is homogeneous i.e. the equation (2) is satisfied, then for all $x \in I$ the functions g, f are given by (15), where $a, c, D, K, L, M \in \mathbb{R}$ are constants* *such that $aKM \neq 0$. Further, (in order that h be defined and $g(x) \neq 0$ on I) we have to assume that*

$$\ln x^a + D \in \left]\frac{(2k-1)\pi}{2}, \frac{(2k+1)\pi}{2}\right[\quad (x \in I) \tag{16}$$

holds with some $k \in \mathbb{Z}$ if the first line of (15) is valid,

$$\ln x^a + D > 0 \text{ or } \ln x^a + D < 0 \quad (x \in I) \tag{17}$$

holds if the third line of (15) is valid,

$$\ln x + D > 0 \text{ or } \ln x + D < 0 \quad (x \in I) \tag{18}$$

holds if the fifth line of (15) is valid.

Proof. One can easily see that $g(x) \neq 0$ $(x \in I)$ if and only if (16) or (17) or (18) holds (depending on which line of (15) is taken into consideration). Calculating the derivatives h' it turns out that for each line of (15) the conditions $g(x) \neq 0$ $(x \in I)$ and $h'(x) \neq 0$ $(x \in I)$ are satisfied simultaneously. □

It is easy to check that

$$M_{f_1,g_1,\mu}(x,y) = M_{f_2,g_2,\mu}(x,y) \quad (x,y \in I)$$

if

$$\begin{aligned} f_2(x) &= pf_1(x) + qg_1(x) \quad (x \in I) \\ g_2(x) &= rf_1(x) + sg_1(x) \quad (x \in I), \end{aligned} \tag{19}$$

where $p,q,r,s \in \mathbb{R}$ are constants for which $ps - qr \neq 0$ (where it is assumed that $f_1, g_1, f_2, g_2 : I \to \mathbb{R}$, $g_1, g_2 \neq 0$, $h_1 = f_1/g_1, h_2 = f_2/g_2$ are continuous and strictly monotonic on I). Let us call the pairs (f_1, g_1) and (f_2, g_2) *equivalent* (with respect to the mean $M_{f,g,\mu}$) if (19) holds. Equivalent pairs generate the same mean.

By the addition formulae of sin, cos, sinh, cosh the pairs (f,g) listed below are equivalent to the pairs given in (15).

$$\begin{array}{lll} g & f & \text{condition} \\ x^c\cos(\ln x^a) & x^c\sin(\ln x^a) & (16) \text{ with } D=0 \\ x^c\cosh(\ln x^a) & x^c\sinh(\ln x^a) & \text{none} \\ x^c\sinh(\ln x^a) & x^c\cosh(\ln x^a) & (17) \text{ with } D=0 \\ x^{c+a} & x^{c-a} & \text{none} \\ x^c & x^c\ln x & (18) \text{ with } D=0, \end{array} \tag{20}$$

where $a \neq 0, c, d \neq c$ are arbitrary constants (for the uniform appearance of the constants in (20) we replaced $2a$ by a in the fourth line of (15) and we replaced $c+\frac{1}{2}$ by c in lines five and six).

Next we find the means corresponding to the functions (15) or (20).

THEOREM 2. *Let $I \subset]0,\infty[$ be an open interval containing the point 1, $f,g : I \to \mathbb{R}$ be three times continuously differentiable functions, $g(x) \neq 0\,(x \in I)$, $h'(x) = (f(x)/g(x))' \neq 0$ $(x \in I)$ and suppose that $\mu : I \to \mathbb{R}$ is an increasing nonconstant function whose moments satisfy* (6). *The mean $M_{f,g,\mu}$ is homogeneous, i.e.* (2) *holds, if and only if it has one of the forms*

$$(21)\quad M_{f,g,\mu}(x,y) = \exp\left(\frac{1}{a}\arctan\frac{\int_0^1 (tx+(1-t)y)^c \sin(a\ln(tx+(1-t)y))d\mu(t)}{\int_0^1 (tx+(1-t)y)^c \cos(a\ln(tx+(1-t)y))d\mu(t)}\right),$$

$$(22)\quad M_{f,g,\mu}(x,y) = \left(\frac{\int_0^1 (tx+(1-t)y)^{c+a}d\mu(t)}{\int_0^1 (tx+(1-t)y)^{c-a}d\mu(t)}\right)^{\frac{1}{2a}},$$

$$(23)\quad M_{f,g,\mu}(x,y) = \exp\left(\frac{\int_0^1 (tx+(1-t)y)^c \ln(tx+(1-t)y)d\mu(t)}{\int_0^1 (tx+(1-t)y)^c d\mu(t)}\right),$$

where $x,y \in I$, and $a \neq 0, c \in \mathbb{R}$ are constants. In case of the first mean the maximal interval is $\left(\exp\left(-\frac{\pi}{2a}\right), \exp\left(\frac{\pi}{2a}\right)\right)$, in case of the last two means the maximal interval is $(0,\infty)$.

Proof. The first pair (f,g) of (20) gives the mean (21), and the maximal interval I as stated.

The second, third, fourth pairs (f,g) of (20) build up the same mean, as they are linear combinations of $x^c \exp a\ln x = x^{c+a}$ and $x^c \exp(-a\ln x) = x^{c-a}$. From them we get the mean (22), the maximal interval being $I = (0,\infty)$.

Finally the last pair (f,g) of (20) builds up the mean (23), with the maximal interval again $I = (0,\infty)$.

It is easy to check that the means (21), (22), (23) satisfy the homogeneity equation (2). □

5. Examples for families of non-symmetric homogeneous means

As our *first example* consider the mean

$$M_{f,g,\mu_1}(x,y)=h^{-1}\left(\frac{Cf(x)+(1-A-C)f\left(\frac{x+y}{2}\right)+Af(y)}{Cg(x)+(1-A-C)g\left(\frac{x+y}{2}\right)+Ag(y)}\right)\quad (x,y\in I),$$

where $A,C>0, A+C\le 1$ and the interval I and the functions f,g satisfy the assumptions of Theorem 2.

We have $m_i=(1-A-C)/2^i+C$ $(i=1,2,3)$ and

$$\begin{aligned}-2m_1^3+3m_1m_2-m_3&=\frac{A-C}{8}(2A^2-4AC-3A+1+2C^2-3C)\\ &=\frac{A-C}{4}\left[\left(A-C-\frac{3}{4}\right)^2-\left(3C+\frac{1}{16}\right)\right].\end{aligned}$$

This shows that condition (6) is not satisfied if $A=C$ (when M_{f,g,μ_1} is obviously symmetric), if $A\ne C$ it holds if and only if

$$A\ne C+3/4\pm\sqrt{3C+1/16}.$$

This holds if we take the positive sign in front of the square root, since then the right hand side is greater or equal to 1. It also holds if $C\ge 1/2$ (and for symmetry reasons if $A\ge 1/2$) since then the right hand side (with negative sign in front of the root) is less than or equal to 0.

Summarizing, (6) holds if and only if $A\ne C$ and

$$\begin{aligned}&0<A<1/2\le C<1-A,\quad\text{or}\quad 0<C<1/2\le A<1-C,\ \text{or}\\ &0<A,C<1/2\quad\text{and}\quad A\ne C+3/4-\sqrt{3C+1/16}.\end{aligned}\tag{24}$$

Taking for example $A=1/4, C=1/2$ by Theorem 2 the homogeneous M_{f,g,μ_1} means are of the form

$$\exp\left(\frac{1}{a}\arctan\frac{2x^c\sin(a\ln x)+\left(\frac{x+y}{2}\right)^c\sin\left(a\ln\frac{x+y}{2}\right)+y^c\sin(a\ln y)}{2x^c\cos(a\ln x)+\left(\frac{x+y}{2}\right)^c\cos\left(a\ln\frac{x+y}{2}\right)+y^c\cos(a\ln y)}\right),$$

$$\left(\frac{2x^{c+a}+\left(\frac{x+y}{2}\right)^{c+a}+y^{c+a}}{2x^{c-a}+\left(\frac{x+y}{2}\right)^{c-a}+y^{c-a}}\right)^{\frac{1}{2a}},$$

$$\exp\left(\frac{2x^c\ln x+\left(\frac{x+y}{2}\right)^c\ln\frac{x+y}{2}+y^c\ln y}{2x^c+\left(\frac{x+y}{2}\right)^c+y^c}\right),$$

where $x,y\in I$, and $a\neq 0, c\in\mathbb{R}$ are constants.

As *second example* take the mean M_{f,g,μ_3} where $\mu_3(t)=t^2$. It is easy to check that the moments of μ_3 satisfy (6). *Assume that f,g and the interval I satisfy the conditions of Theorem 2.* By Theorem 2 (transforming the integrals to Riemann integrals) we get that the *homogeneous* M_{f,g,μ_3} means are the following

$$\exp\left(\frac{1}{a}\arctan\frac{\int_0^1 t(tx+(1-t)y)^c\sin(a\ln(tx+(1-t)y))dt}{\int_0^1 t(tx+(1-t)y)^c\cos(a\ln(tx+(1-t)y))dt}\right),$$

$$\left(\frac{\int_0^1 t(tx+(1-t)y)^{c+a}dt}{\int_0^1 t(tx+(1-t)y)^{c-a}dt}\right)^{\frac{1}{2a}},$$

$$\exp\left(\frac{\int_0^1 t(tx+(1-t)y)^c\ln(tx+(1-t)y)dt}{\int_0^1 t(tx+(1-t)y)^c dt}\right),$$

where $x,y\in I$, and $a\neq 0, c\in\mathbb{R}$ are constants.

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