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OSCILLATION CRITERIA FOR A HIGHER ORDER FUNCTIONAL DIFFERENCE EQUATION WITH OSCILLATING COEFFICIENT

Abstract. In this paper we are concerned with the oscillatory behaviour of solutions of a certain higher order nonlinear neutral type functional difference equation with oscillating coefficient. We obtain two sufficient criteria for oscillatory behaviour.

1. Introduction

We consider the higher order nonlinear difference equation of the form

$$\Delta^n[y_k + h_k g(y_k, y_{k-\tau})] + q_k f(y_k, y_{k-\sigma_1}, y_{k-\sigma_2}, \dots, y_{k-\sigma_n}) = 0 \tag{1.1}$$

where $n, k \in N$(natural numbers), $N(a) = \{a, a+1, \dots\}$, $N(a,b) = \{a, a+1, \dots, b\}$, $y(k) = y_k$ and the following conditions are always assumed to hold:

i) $n \geq 2$

ii) τ is a positive integer and σ_j are nonnegative integers for $j = 1, 2, \dots, n$,

iii) h_k is an oscillating function and q_k is a nonnegative function,

iv) g and f are continuous and monotone functions such that respectively $g(v_0, v_1) : R^2 \to R$, $f(u_0, u_1, u_2, \dots, u_n) : R^{n+1} \to R$. Further $v_i g(v_0, v_1) > 0$ for every $v_i \neq 0$, $i = 0, 1$ and $u_j f(u_0, u_1, u_2, \dots, u_n) > 0$ for every $u_j \neq 0$ and $j = 0, 1, 2, \dots, n$,

By a solution of Eq.(1.1), we mean any function y_k which is defined for all $k \geq \min\limits_{\gamma \geq 0}\{\gamma - \tau, \gamma - \sigma_j\}$ and satisfies Eq.(1.1) for sufficiently large k. We consider only such solutions which are nontrivial for all large k. As it is customary, a solution $\{y_k\}$ is said to be oscillatory if the terms y_k of the sequence are not eventually positive or not eventually negative. Otherwise,

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the solution is called nonoscillatory. A difference equation is called oscillatory if all of its solutions oscillate. Otherwise, it is nonoscillatory. In this paper, we restrict our attention to real valued solutions y_k.

The neutral delay difference equations arise in a number of important applications including problems in population dynamics when maturation and gestation are included, in "cobweb" models in economics where demand depends on current price but supply depends on the price at an earlier time, and in electrical transmission and in loss transmission lines between circuits in high speed computers.

Recently, much research has been done on the oscillatory and asymptotic behaviour of solutions of higher order delay and neutral type difference equations. In all of the known results, the positive value or negative value case of coefficient h_k is considered; see, for example, [1-9], chapter 7[2] and Section 22[3] and related equations. Firstly only we consider the case of oscillating function of coefficient h_k in our manuscript [7] and in this manuscript.

The purpose of this paper is to study oscillatory behaviour of solutions of Eq. (1.1). For the general theory of difference equations, one can refer to [1-6]. Many references to applications of the difference equations can be found in [4-6].

For the sake of convenience, the function z_k is defined by

$$z_k = y_k + h_k g(y_k, y_{k-\tau}). \tag{1.2}$$

2. Auxiliary lemma

LEMMA 1. [1] *Let y_k be defined for $k \geq k_0 \in N$, and $y_k > 0$ with $\Delta^n y_k$ of constant sign for $k \geq k_0$, $n \in N(1)$ and not identically zero. Then there exists an integer m, $0 \leq m \leq n$ with $(n+m)$ even for $\Delta^n y_k \geq 0$ or $(n+m)$ odd for $\Delta^n y_k \leq 0$ such that*

i) $m \leq n-1$ *implies* $(-1)^{m+l}\Delta^l y_k > 0$ *for all* $k \geq k_0$, $m \leq l \leq n-1$;

ii) $m \geq 1$ *implies* $\Delta^l y_k > 0$ *for all large* $k \geq k_0$, $1 \leq l \leq m-1$.

3. Main results

THEOREM 1. *Assume that n is odd and*

(C_1) $\lim_{k\to\infty} h_k = 0$;

(C_2) $|g(v_0, v_1)| \leq p$, *where p is a positive constant;*

(C_3) $\sum_{s=k_{10}}^{+\infty} s^{n-1} q_s = +\infty$.

Then every bounded solution of Eq. (1.1) *is either oscillatory or tends to zero as* $k \to +\infty$.

Proof. Assume that Eq. (1.1) has a bounded nonoscillatory solution y_k. Without loss of generality, assume that y_k is eventually positive (the proof is similar when y_k is eventually negative). That is, $y_k > 0$, $y_{k-\tau} > 0$, $y_{k-\sigma_1} > 0$, $y_{k-\sigma_2}, \dots, y_{k-\sigma_n} > 0$ for $k \geq k_1 \geq k_0$. Further, we assume that y_k does not tend to zero as $k \to \infty$. By (1.1), (1.2) we have for $k \geq k_1$

$$\Delta^n z_k = -q_k f(y_k, y_{k-\sigma_1}, y_{k-\sigma_2}, \dots, y_{k-\sigma_n}) \leq 0. \tag{3.1}$$

That is, $\Delta^n z_k \leq 0$. It follows that $\Delta^a z_k$ $(a = 0, 1, 2, \dots, n-1)$ is strictly monoton and eventually of constant sign. Since y_k does not tend to zero as $k \to \infty$ and $h_k \to 0$ as $k \to \infty$ by (C_1) and $0 < g(v_0, v_1) \leq p$ by (iv) and (C_2), there exists a $k_2 \geq k_1$ such that for $k \geq k_2$ we have $z_k > 0$. Since y_k is bounded function and $h_k g(y_k, y_{k-\tau}) \to 0$ as $k \to \infty$, there is a $k_3 \geq k_2$ such that z_k is bounded for $k \geq k_3$. Because n is odd and z_k is bounded, by Lemma 1, since $m = 0$ (otherwise, z_k is not bounded), there exists $k_4 \geq k_3$ such that for $k \geq k_4$ we have $(-1)^l \Delta^l z_k > 0$ $(l = 0, 1, 2, \dots, n-1)$. In particular, since $\Delta z_k < 0$ for $k \geq k_4$, z_k is decreasing. Since z_k is bounded, we may write $\lim_{k\to\infty} z_k = L$ $(-\infty < L < +\infty)$. Assume that $0 \leq L < +\infty$. Let $L > 0$. Then there exists a constant $c > 0$ and a k_5 with $k_5 \geq k_4$ such that $z_k > c > 0$ for $k \geq k_5$. Since $\lim_{k\to\infty} h_k g(y_k, y_{k-\tau}) = 0$ by (C_1) and (C_2), there exist a constant $c_1 > 0$ and a k_6 with $k_6 \geq k_5$ such that $y_k = z_k - h_k g(y_k, y_{k-\tau}) > c_1 > 0$ for $k \geq k_6$. So, we may find k_7 with $k_7 \geq k_6$ such that $y_{k-\sigma_1} > c_1 > 0$, $y_{k-\sigma_2} > c_1 > 0, \dots$, $y_{k-\sigma_n} > c_1 > 0$ for $k \geq k_7$. From (3.1) we have

$$\Delta^n z_k \leq -q_k f(c_1, c_1, \dots, c_1) \quad (k \geq k_7). \tag{3.2}$$

If we multiply (3.2) by k^{n-1} and summing it from k_7 to $k-1$, we obtain

$$F_k - F_{k_7} \leq -f(c_1, c_1, \dots, c_1) \sum_{s=k_7}^{k-1} s^{n-1} q_s, \tag{3.3}$$

where

$$F_k = \sum_{\gamma=2}^{n-1} (-1)^\gamma \Delta^\gamma k^{(n-1)} \Delta^{n-\gamma-1} z_{k+\gamma}.$$

Since $(-1)^l \Delta^l z_k > 0$ for $l = 0, 1, 2, \dots, n-1$ and $k \geq k_4$, we have $F_k > 0$ for $k \geq k_7$. From (3.3) we have

$$-F_{k_7} \leq -f(c_1, c_1, \dots, c_1) \sum_{s=k_7}^{k-1} s^{n-1} q_s.$$

By (C_3), we obtain

$$-F_{k_7} \leq -f(c_1, c_1, \dots, c_1) \sum_{s=k_7}^{\infty} s^{n-1} q_s = -\infty$$

as $k \to \infty$. This is a contradiction. So, $L > 0$ is impossible. Therefore, $L = 0$ is the only possible case. That is, $\lim_{k\to\infty} z_k = 0$. Hence, by virtue of (C_1) and (C_2), we obtain from (1.2)

$$\lim_{k\to\infty} y_k = \lim_{k\to\infty} z_k - \lim_{k\to\infty} h_k g(y_k, y_{k-\tau}) = 0.$$

This contradicts our assumption that y_k does not tend to zero as $k \to \infty$. Now let us consider the case of $y_k < 0$ for $k \geq k_1$. By (1.1) and (1.2),

$$\Delta^n z_k = -q_k f(y_k, y_{k-\sigma_1}, y_{k-\sigma_2}, \dots, y_{k-\sigma_n}) \geq 0 \quad (k \geq k_1).$$

That is, $\Delta^n z_k \geq 0$. It follows that $\Delta^a z_k$ $(a = 0, 1, 2, \dots, n-1)$ is strictly monotone and eventually of constant sign. Since y_k does not tend to zero as $k \to \infty$ and $h_k \to 0$ as $k \to \infty$ by (C_1) and $-p \leq g(v_0, v_1) < 0$ by (iv) and (C_2), there exists a $k_2 \geq k_1$ such that for $k \geq k_2$ we have $z_k < 0$. Since y_k is a bounded function and $h_k g(y_k, y_{k-\tau}) \to 0$ as $k \to \infty$, there is a $k_3 \geq k_2$ such that z_k is a bounded for $k \geq k_3$. Let us set $x_k = -z_k$. Then $\Delta^n x_k = -\Delta^n z_k$. Therefore, $x_k > 0$ and $\Delta^n x_k \leq 0$ for $k \geq k_3$. Since z_k is bounded, we observe that x_k is also bounded. Because n is odd and x_k is bounded, by Lemma 1, since $m = 0$ (otherwise, x_k is not bounded), there exists a $k_4 \geq k_3$ such that $(-1)^l \Delta^l x_k > 0$ for $l = 0, 1, 2, \dots, n-1$ and $k \geq k_4$. That is, $(-1)^l \Delta^l z_k < 0$ for $l = 0, 1, 2, \dots, n-1$ and $k \geq k_4$. In particular, for $k \geq k_4$ we have $\Delta z_k > 0$. Therefore, z_k is increasing. So, we can assume that $\lim_{k\to\infty} z_k = L$ $(-\infty < L \leq 0)$. As in the proof of $y_k > 0$, we may prove that $L = 0$. As for the rest, it is similar to the case of $y_k > 0$. That is, $\lim_{k\to\infty} y_k = 0$. This contradicts our assumption. Hence, the proof is completed. □

THEOREM 2. *Assume that n is even and also (C_1) and (C_2) hold. Further,*

$$(C_4) \qquad \limsup_{k\to\infty} \sum_{s=k_0}^{k-1} q_s = +\infty$$

is satisfied.

Then every bounded solution of Eq. (1.1) is oscillatory.

Proof. Assume that Eq. (1.1) has a bounded nonoscillatory solution y_k. Without loss of generality assume that y_k is eventually positive (the proof is similar when y_k is eventually negative). That is, $y_k > 0$, $y_{k-\tau} > 0$, $y_{k-\sigma_1} > 0$, $y_{k-\sigma_2}, \dots, y_{k-\sigma_n} > 0$ for $k \geq k_1 \geq k_0$. By (1.1), (1.2) we have for $k \geq k_1$

$$(3.4) \qquad \Delta^n z_k = -q_k f(y_k, y_{k-\sigma_1}, y_{k-\sigma_2}, \dots, y_{k-\sigma_n}) \leq 0.$$

That is, $\Delta^n z(k) \leq 0$. It follows that $\Delta^a z_k$ $(a = 0, 1, 2, \dots, n-1)$ is strictly monotone and eventually of constant sign. Since $y_k \neq 0$ is positive and bounded and $0 < g(v_0, v_1) \leq p$ by (iv) and (C_1), there exists a $k_2 \geq k_1$

such that for $k \geq k_2$ we have $z_k > 0$. Since y_k is a bounded function and $h_k g(y_k, y_{k-\tau}) \to 0$ as $k \to \infty$, there is a $k_3 \geq k_2$ such that z_k is a bounded function for $k \geq k_3$. Because n is even, by Lemma 1, since $m = 1$ (otherwise, z_k is not bounded), there exists $k_4 \geq k_3$ such that for $k \geq k_4$

$$(3.5) \qquad (-1)^{l+1}\Delta^l z_k > 0 \quad (l = 1, 2, \dots, n-1).$$

In particular, since $\Delta z_k > 0$ for $k \geq k_4$, z_k is increasing. Since y_k is bounded and $\lim_{k\to\infty} h_k g(y_k, y_{k-\tau}) = 0$ by (C_1) and (C_2), there exists a $k_5 \geq k_4$ by (1.2)

$$y_k = z_k - h_k g(y_k, y_{k-\tau}) \geq \frac{1}{2} z_k > 0$$

for $k \geq k_5$. We may find a $k_6 \geq k_5$ such that for $k \geq k_6$ we have

$$(3.6) \quad y_{k-\sigma_1} \geq \frac{1}{2} z_{k-\sigma_1} > 0,\ y_{k-\sigma_2} \geq \frac{1}{2} z_{k-\sigma_2} > 0, \dots,\ y_{k-\sigma_n} \geq \frac{1}{2} z_{k-\sigma_n} > 0.$$

From (3.4), (3.6) and the properties of f we have

$$(3.7) \quad \Delta^n z_k \leq -q_k f\left(\frac{1}{2} z_k, \frac{1}{2} z_{k-\sigma_1}, \frac{1}{2} z_{k-\sigma_2}, \dots, \frac{1}{2} z_{k-\sigma_n}\right)$$
$$= -q_k \frac{f(\frac{1}{2} z_k, \frac{1}{2} z_{k-\sigma_1}, \frac{1}{2} z_{k-\sigma_2}, \dots, \frac{1}{2} z_{k-\sigma_n})}{z_{k-\sigma}} z_{k-\sigma} \quad (k \geq k_6),$$

where $\sigma = \min_{1\leq j\leq n}\{\sigma_j\}$. Since z_k is bounded and increasing, $\lim_{k\to\infty} z_k = L \ (0 < L < +\infty)$. By the continuity of f, we have

$$\lim_{k\to\infty} \frac{f(\frac{1}{2} z_k, \frac{1}{2} z_{k-\sigma_1}, \frac{1}{2} z_{k-\sigma_2}, \dots, \frac{1}{2} z_{k-\sigma_n})}{z_{k-\sigma}} = \frac{f(\frac{1}{2}L, \frac{1}{2}L, \frac{1}{2}L, \dots, \frac{1}{2}L)}{L} > 0.$$

Then there is a $k_7 \geq k_6$ such that for $k \geq k_7$ we have

(3.8)
$$\lim_{k\to\infty} \frac{f(\frac{1}{2} z_k, \frac{1}{2} z_{k-\sigma_1}, \frac{1}{2} z_{k-\sigma_2}, \dots, \frac{1}{2} z_{k-\sigma_n})}{z_{k-\sigma}} \geq \frac{f(\frac{1}{2}L, \frac{1}{2}L, \frac{1}{2}L, \dots, \frac{1}{2}L)}{2L} = \alpha > 0.$$

By (3.7) and (3.8),

$$(3.9) \qquad \Delta^n z_k \leq -\alpha q_k z_{k-\sigma} \text{ for } k \geq k_7.$$

Let us set

$$(3.10) \qquad G_k = \frac{\Delta^{n-1} z_k}{z_{k-\sigma}}.$$

Since $\Delta^{n-1} z_k > 0$ and $z_{k-\sigma} > 0$ by (3.5), we can find a k_8 with $k_8 \geq k_7$ such that for $k \geq k_8$ we have $G_k > 0$. If we apply the forward difference operator Δ to (3.10), since $\Delta^{n-1}z_k$ and Δz_k are decreasing and z_k is increasing by (3.5), we obtain

$$\begin{aligned} \Delta G_k &= \frac{z_{k-\sigma}\Delta^n z_k - \Delta z_{k-\sigma}\Delta^{n-1}z_k}{z_{k-\sigma}Ez_{k-\sigma}} \\ &\le \frac{z_{k-\sigma}\Delta^n z_k}{z_{k-\sigma}^2} - \frac{\Delta z_{k-\sigma}\Delta^{n-1}z_k}{z_{k-\sigma}^2} \\ &\le \frac{\Delta^n z_k}{z_{k-\sigma}} - G_k\frac{\Delta z_k}{z_{k-\sigma}}. \end{aligned} \tag{3.11}$$

Since $G_k\frac{\Delta z_k}{z_{k-\sigma}} > 0$, from (3.11) and (3.9) we can write

$$\Delta G_k \le -\alpha q_k \tag{3.12}$$

Summing up (3.12) from k_8 to $k-1$ we obtain

$$G_k \le G_{k_8} - \alpha\sum_{s=k_8}^{k-1} q_k. \tag{3.13}$$

By (C_4) we have $G_k \to -\infty$ which is contradiction to the fact that $G_k > 0$.

Now let us consider the case of $y_k < 0$. We do the proof similar to Theorem 1 as in the case of $y_k < 0$. Therefore, there is a $k \ge k_1$ such that $\Delta^n z_k \ge 0$, $z_k < 0$ and z_k is bounded and at the same time there exist an integer $m = 1$ and a $k_4 \ge k_3$ such that $(-1)^{l+1}\Delta^l z_k < 0$ for $k \ge k_4$ and $l = 1, 2, \dots, n-1$. In particular, $\Delta z_k < 0$ for $k \ge k_4$. Let us set $x_k = -z_k$. The rest of proof is similar to the case of $y_k > 0$. Hence, the proof is completed. □

EXAMPLE 1. We consider difference equation of the form

$$\Delta^3\left[y_k + e^{-k}\sin\left(k\frac{\pi}{2}\right)y_k y_{k-1}^2\right] + k y_k^2 y_{k-3} y_{k-2}^3 = 0 \text{ for } k \ge 3, \tag{3.14}$$

where $n = 3$, $q_k = k$, $\sigma_1 = 3$, $\sigma_2 = 2$, $\sigma_3 = 0$, $\tau = 1$, $h_k = e^{-k}\sin(k\frac{\pi}{2})$. Therefore, we have

$$\lim_{k\to\infty} h_k = \lim_{k\to\infty} e^{-k}\sin\left(k\frac{\pi}{2}\right) = 0$$

and

$$\sum_{s=k_0}^{+\infty} s^{n-1}q_s = \sum_{s=k_0}^{+\infty} s^3 = +\infty.$$

Then conditions (C_1) and (C_3) of the Theorem 1 are satisfied. Since y_k is bounded, g is also bounded and condition (C_2) holds. Hence, all conditions of Theorem 1 are satisfied. Then every bounded solution of equation (3.16) is oscillatory. One of such solutions is $\{y_k\} = \{(-1)^k\}$.

EXAMPLE 2. Consider difference equation of the form

$$\Delta^2\left[y_k + \frac{(-1)^k}{k} y_k y_{k-2}\right] + 2\left(2 + \frac{k^2(4k+5)+9k+1}{k^3(k+4)+k(5k+2)}\right) y_k y_{k-2} y_{k-1} = 0, \tag{3.15}$$

where $\tau = 2$, $h_k = \frac{(-1)^k}{k}$, $q_k = 2(2 + \frac{k^2(4k+5)+9k+1}{k^3(k+4)+k(5k+2)})$, $\sigma_1 = 2$, $\sigma_2 = 1$. Therefore, we have

$$\lim_{k\to\infty} h_k = \lim_{k\to\infty} \frac{(-1)^k}{k} = 0,$$

$$\lim_{k\to\infty} \sup \sum_{s=k_0}^{k-1} q_s = \lim_{k\to\infty} \sup \sum_{s=k_0}^{k-1} \left(2 + \frac{s^2(4s+5)+9s+1}{s^3(s+4)+s(5s+2)}\right) = +\infty.$$

Then conditions (C_1) and (C_4) of Theorem 2 are satisfied. Since y_k is bounded, $g(v_0, v_1)$ is bounded and condition (C_2) of Theorem 2 holds. Hence, since all conditions of Theorem 2 are satisfied, every bounded solution of equation (3.17) is oscillatory. In fact, equation (3.17) has an oscillatory solution given by $\{y_k\} = \{(-1)^k\}$.

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