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ON THE UNBOUNDED SOLUTIONS FOR PARABOLIC DIFFERENTIAL–FUNCTIONAL CAUCHY PROBLEM

Abstract. We consider the initial value problem for second order differential–functional equation. Functional dependence on an unknown function is of the Hale type. We prove the existence theorem for unbounded classical solution. Our formulation admits a large group of nonlocal problems. We put particular stress on "retarded and deviated" argument as it seems to be the most difficult.

1. Introduction

In this paper we consider the Cauchy problem for nonlinear differential - functional heat equation. We extend the result obtained in [5] for bounded solution and apply it to the case of unbounded solution.

Let $E = \Theta_0 \cup \Theta$ where $\Theta_0 = [-a_0, 0] \times \mathbb{R}^m$, $\Theta = (0, T) \times \mathbb{R}^m$ and $T > 0$, $a_0 \geq 0$. Set $D = [-a_0, 0] \times B(r)$, where $B(r) = \{x \in \mathbb{R}^m : \ |x| \leq r\}$, $r \geq 0$ and $|\cdot|$ denotes the norm in $\mathbb{R}^m$. For every $z : E \to \mathbb{R}$ and $(t,x) \in \Theta$ we define a function $z_{(t,x)} : D \to \mathbb{R}$ by $z_{(t,x)}(s,y) = z(t+s, x+y)$. We call the operator $z \to z_{(t,x)}$ Hale's operator and functional dependence in the equation "of the Hale type" (see [1] for an ordinary differential equations).

Let $Y \subset \mathbb{R}^{1+m}$. Throughout the paper $C(Y)$ stands for the space of all continuous functions $w : Y \to \mathbb{R}$ with the finite supremum norm $\|w\|_Y = \sup_{(t,x)\in Y} |w(t,x)|$.

Assume that $f : \Theta \times C(D) \times \mathbb{R}^m \to \mathbb{R}$ and $\Psi : \Theta_0 \to \mathbb{R}$ are continuous functions. We investigate the problem

$$D_t u - \varepsilon \Delta u = f(t, x, u_{(t,x)}, Du) \quad \text{in} \quad \Theta, \tag{1}$$

$$u = \Psi \quad \text{in} \quad \Theta_0. \tag{2}$$

where $\varepsilon > 0$. In the above $\Delta u = \sum_{i=1}^m D^2_{x_i x_i} u$ and Du is a gradient of u,

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both with respect to space variable x. In (1) we write $D_t u$, Δu, Du, u for the values at point (t,x) and $u_{(t,x)}$ for the Hale operator.

The above problem contains as a particular case a large group of differential - functional equation. The most important are : equations with a retarded and deviated argument, differential-integral equations and of course equation without functional dependence (i.e. with component u). This can be derived from (1), (2) by specializing the function f (see [5]). The main problem that arise here is, how to formulate assumptions on f in order to obtain theorems for well known types of differential—functional equations. We will focus on "retarded and deviated argument" as it is more difficult.

We based on the result obtained in [5] for bounded solution. The result for unbounded solutions is obtained after transformation of our problem. To do this, we first need to weaken assumption in [5]. All the differences between the theory of bounded and unbounded solutions, significant only for functional dependence, are contain in our transformation. Our result can be extended to weakly coupled systems without assuming quasimonotone conditions. With the method presented in the paper we can treat any strictly parabolic equation of constant coefficients.

For a deep discussion of the related literature we refer the reader to [5].

We write $CLS(f,\Psi)$ for the set of all classical solution of (1), (2) i.e. $u \in CLS(f,\Psi)$ if $u : E \mapsto \mathbb{R}$ and $D_t u, D^2 u, Du$ exist, are continuous in Θ and (1), (2) are satisfied.

Let $K(R) = \{z \in C(D) : \|z\|_D \leq R\}$. For $G \subseteq \mathbb{R}^{1+m}$ we write $G_t = \{(s,x) \in G : -a_0 \leq s \leq t\}$.

We call $\omega : \mathbb{R}_+ \mapsto \mathbb{R}_+$ a *modulus* if ω is nondecreasing and $\lim_{t \mapsto 0^+} \omega(t) = 0$.

Let as above $Y \subseteq \mathbb{R}^{1+m}$. We will write $u \in BUC_t(Y)$ if $u : Y \mapsto \mathbb{R}$ is bounded, continuous and $|u(t,x) - u(\bar{t},x)| \leq \omega(|t-\bar{t}|)$ in Y for some *modulus* ω.

From now we will always assume that $\Psi \in BUC_t(\Theta_0)$. Note, if $a_0 = 0$, $\Theta_0 = \{0\} \times \mathbb{R}^m$, it means that Ψ is bounded and continuous. In this case we have no delay in the equation but it can be still interesting functional dependence.

Set $CLS^*(f,\Psi) = CLS(f,\Psi) \cap BUC_t(E)$.

Let $\alpha \in (0,1)$, $l = \alpha,\ 2+\alpha$. We will denote by $C^{l/2,l}(Y)$ the space of all function $u : Y \to \mathbb{R}$ of the variable (t,x) such that $D_x^r u$ (for $r = 0, \dots [l]$) and $D_t^k u$ (for $k = 0, 1 \dots [l/2]$) exist and are continuous and bounded in Y, $D_x^{[l]} u$ satisfies Hölder condition with exponent $l - [l]$ with respect to x and $D_t^{[l/2]} u$ satisfies Hölder condition with exponent $l/2 - [l/2]$ with respect to t. It is well known that $(C^{l/2,l}(Y),\ \|\cdot\|_l)$ is a Banach space. For the definition of the norm $\|\cdot\|_l$ and for the properties of $C^{l/2,l}(Y)$ we refer the reader to [2]. We will only define here the symbols that we use in the paper.

Let $\alpha \in (0,1]$ and $1 > \rho > 0$. Define

$$H_t^\alpha[u] = \sup\left\{\frac{|u(t,x) - u(\bar{t},x)|}{|t-\bar{t}|^\alpha} : (t,x), (\bar{t},x) \in Y, t \neq \bar{t}\right\},$$

$$H_x^\alpha[u] = \sup\left\{\frac{|u(t,x) - u(t,\bar{x})|}{|x-\bar{x}|^\alpha} : (t,x), (t,\bar{x}) \in Y, x \neq \bar{x},\ |x-\bar{x}| \leq \rho\right\}.$$

We write $L_x[u] = H_x^1[u]$, $L_t[u] = H_t^1[u]$ for the Lipschitz constant in x and t for u. We write $z \in C^{2+\alpha}(\mathbb{R}^m)$ if $\tilde{z}$ defined by $\tilde{z}(t,x) = z(x)$ belongs to $C^{1+\alpha/2,2+\alpha}(\Theta)$.

We will also use the following spaces:

$$C^{L,\alpha}(Y) = \{u \in C(Y) : L_t[z] < \infty,\ H_x^\alpha[z] < \infty\},$$
$$C^{L,0}(Y) = \{u \in C(Y) : L_t[z] < \infty\},$$
$$C^{0,L}(Y) = \{u \in C(Y) : L_x[z] < \infty,\},$$
$$C^{0,\alpha}(Y) = \{u \in C(Y) : H_x^\alpha[z] < \infty\}.$$
$$C^{L,L}(Y) = \{u \in C(Y) : L_t[z] < \infty,\ L_x[z] < \infty,\}.$$

2. The existence theorem

In the paper [5] we proved theorem on the existence of unique bounded solutions for (1), (2) under the following assumption on f.

ASSUMPTION 1. *Suppose that*

1. *There exists* $\gamma \geq 0$ *such that* $\|f(\cdot,\cdot,0,0)\|_{\bar{\Theta}} \leq \gamma$.
2. *There exists* $H \geq 0$ *such that*

$$|f(t,x,w,p) - f(t,x,\bar{w},\bar{p})| \leq H(\|w-\bar{w}\|_D + |p-\bar{p}|) \quad \text{in } \Theta \times C(D) \times \mathbb{R}^m.$$

3. *There exist* $H_1 \geq 0,\ 0 < \rho_0 < 1,\ \alpha \in (0,1)$ *such that for* $|x-\bar{x}| \leq \rho_0$

$$|f(t,x,w,p) - f(t,\bar{x},w,p)| \leq H_1(1 + H_x^\alpha[w])|x-\bar{x}|^\alpha \quad \text{in } \Theta \times C^{0,\alpha}(D) \times \mathbb{R}^m.$$

4. *There exists* $H_2 \geq 0$ *such that*

$$|f(t,x,w,p) - f(\bar{t},x,w,p)| \leq H_2(1 + L_t[w] + H_x^\alpha[w])|t-\bar{t}| \quad \text{in } \bar{\Theta} \times C^{L,\alpha}(D) \times \mathbb{R}^m.$$

(In 1. symbol "0" stands both for the null function and for the null vector.)

A quite general form of 3), 4) allows us to apply the results to equations with a retarded and deviated argument (see [5] for more precise explanation).

The theorem proved in [5] states as follows

THEOREM 1. *Suppose that f satisfies Assumption 1, $\Psi \in C^{L,\alpha}(\Theta_0)$ and $\Psi(0,\cdot) \in C^{2+\alpha}(\mathbb{R}^m)$. Then the initial value problem (1), (2) has exactly one solution $u \in C^{L,\alpha}(E) \cap C^{1+\alpha/2,2+\alpha}(\bar{\Theta})$.*

REMARK 1. Following the proof given in [5] we can put $\alpha = 1$ in Assumption 1, and $\Psi \in C^{L,L}(\Theta_0)$. Together with $\Psi(0,\cdot) \in C^{2+\alpha}(\mathbb{R}^m)$ we get an existence of a solution $u \in C^{L,L}(E) \cap C^{1+\alpha/2,2+\alpha}(\bar{\Theta})$.

In this section we generalize Theorem 1 (for $\Psi \in C^{L,L}(\Theta_0)$ and Assumption 1 with $\alpha = 1$). We will give a sufficient condition to have x-derivative of the solution of (1), (2) uniformly bounded.

In the following we assume that $M \geq 0$.

DEFINITION 1. *We write $\sigma \in O_M$ if $\sigma : [0,T] \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ is continuous, nondecreasing in both variable, and if a maximal solution of the problem*

$$z'(t) = \sigma(t, z(t)), \qquad z(0) = M. \tag{3}$$

exists in $[0,T]$. We will write $\mu_\sigma(\cdot, M)$ for this solution.

DEFINITION 2. *Let $\sigma \in O_M$. We write $f \in X_{\sigma,M}$ if*

(i) $f(t,x,w,0)\mathrm{sgn}w(0,0) \leq \sigma(t, \|w\|_D)$ in $\Theta \times C(D)$;

(ii) For every $R \geq 0$ there exists modulus ω_R such that,

$$|f(t,x,w,p) - f(t,x,w,0)| \leq \omega_R(|p|) \quad \text{in} \quad \Theta \times K(R) \times \mathbb{R}^m.$$

Put $\mu_\sigma(T,M) = R(\sigma,M)$.

Though in [4] we consider the space $BUC(E)$ (bounded uniformly continuous) it is immediate that all the results are valid also in $BUC_t(E)$. Thus in view of Theorem 2 [4] we can write,

PROPOSITION 1. *If $f \in X_{\sigma,M}$, $\|\Psi\|_{\Theta_0} \leq M$ and $u \in CLS^*(f,\Psi)$ then*

$$\|u\|_{E_t} \leq \mu_\sigma(t,M) \leq R(\sigma,M) \quad \text{for} \quad t \in [0,T]. \tag{4}$$

REMARK 2. Let $\sigma(t,z) = \gamma + Cz$ for γ, $C \geq 0$. If $\|\Psi\|_{\Theta_0} = M$, $f \in X_{\sigma,M}$ and $u \in CLS^*(f,\Psi)$. Then

$$\|u\|_{E_t} \leq e^{Ct}(\,\|\Psi\|_{\Theta_0} + \gamma t\,) \quad \text{for} \quad t \in [0,T]. \tag{5}$$

In the following we will use

DEFINITION 3. *Let $(X, \|\cdot\|)$ be any normed space, $R \geq 0$ any constant. Define $I_R : X \mapsto X$ by*

$$I_R(x) = \begin{cases} x, & \text{if} \;\; \|x\| \leq R; \\ \frac{x}{\|x\|}R & \text{if} \;\; \|x\| > R. \end{cases} \tag{6}$$

Of course

$$\|I_R(x)\| = \min(\|x\|, R), \quad \|I_R(x) - I_R(y)\| \leq 2\|x - y\| \quad \text{in} \quad X. \tag{7}$$

For a given $f : \Theta \times C(D) \times \mathbb{R}^m \to \mathbb{R}$ we define $f_R : \Theta \times C(D) \times \mathbb{R}^m \to \mathbb{R}$ by

$$f_R(t,x,w,p) = f(t,x,I_R(w),p). \tag{8}$$

Let $R \geq R(\sigma, M)$. It is easy to verify that

REMARK 3. *Let $\|\Psi\|_{\Theta_0} \leq M$, $\sigma \in O_M$. If $f \in X_{\sigma,M}$ then, $f_R \in X_{\sigma,M}$ and $CLS^*(f,\Psi) = CLS^*(f_R,\Psi)$.*

Since conditions 1), 2) of the Assumption 1 gives (see Remark 2) estimation by $R = e^{HT}(M + \gamma T)$ for any solution of (1), (2) we can assume that $2), 3), 4)$ are satisfied for $w \in K(R)$.

ASSUMPTION 2. *Suppose that*

1. *$f \in X_{\sigma,M}$ for some $\sigma \in O_M$;*
2. *there exists $C \geq 0$ such that*
$$|f(t,x,w,p) - f(t,x,\bar{w},p)| \leq C\|w - \bar{w}\|_D \quad \text{in } \Theta \times K(R) \times \mathbb{R}^m;$$
3. *there exists $\tilde{C} \geq 0$ such that*
$$|f(t,x,w,p) - f(t,\bar{x},w,p)| \leq \tilde{C}(1 + |p| + L_x[w])|x - \bar{x}| \quad \text{in } \Theta \times K(R) \cap C^{0,L}(D) \times \mathbb{R}^m;$$
4. *for every $\tilde{L} \geq 0$ there exists modulus $\omega_{\tilde{L}}$ such that*
$$|f(t,x,w,p) - f(t,x,w,\bar{p})| \leq \omega_{\tilde{L}}(|p - \bar{p}|) \quad \text{in } \Theta \times K(R) \times B(\tilde{L}),$$

where $R = R(\sigma, M)$.

REMARK 4. Let $\|\Psi\|_0 \leq M$, $\sigma \in 0_M$, $R = R(\sigma, M)$. If f satisfies Assumption 2 with $\sigma, M, C, \tilde{C}$ then f_R satisfies it with $\sigma, M, 2C, \tilde{C}$.

Define $CLS_b^*(f,\Psi) = \{u \in CLS^*(f,\Psi) : Du \text{ is bounded in } \Theta\}$ and $BUC_t(\Theta_0, L_0) = \{\Psi \in BUC_t(\Theta_0) : |\Psi(t,x) - \Psi(t,y)| \leq L_0|x - y| \text{ in } \Theta_0\}$.

LEMMA 1. *Suppose that f satisfies Assumption 2 with $\sigma, M, C, \tilde{C}$, $\|\Psi\|_{\Theta_0} \leq M$ and $\Psi \in BUC_t(\Theta_0, L_0)$, for $L_0 \geq 0$. If $u \in CLS_b^*(f,\Psi)$ then there exists $L \geq 0$ depending on $C, \tilde{C}, L_0$ such that $\|Du\|_\Theta \leq L$.*

Proof. By Remarks 3 and 4 we can assume that Assumption 2 is satisfied globally in w. Put $L_t = \max(\|Du\|_{\Theta_t}, L_0)$. Let $\xi \in R^m$ and $u_\xi(t,x) = u(t,x+\xi)$, $\Psi_\xi(t,x) = \Psi(t,x+\xi)$, $f_\xi(t,x,w,p) = f(t,x+\xi,w,p)$. Clearly $u_\xi \in CLS_b^*(f_\xi,\Psi_\xi)$ and Ψ_ξ, f_ξ satisfies assumptions with the same parameters. Define

$$g(t,x,w,p) = f_\xi(t,x,w + u_{(t,x)}, p + Du(t,x)) - f(t,x,u_{(t,x)},Du(t,x)).$$

Notice that, $u_\xi - u \in CLS^*(g, \Psi_\xi - \Psi)$ and $g \in X_{\bar{\sigma},\bar{M}}$, where $\bar{\sigma}(s,z) = 2\tilde{C}(1 + L_t)|\xi| + Cz$, $\bar{M} = L_0|\xi|$ for $s \leq t$. In view of Remark 2 (in the set E_t) we get

$$\|u - u_\xi\|_{E_t} \le e^{Ct}[\|\Psi - \bar{\Psi}\|_{\Theta_0} + 2\tilde{C}t(1+L_t)|\xi|],$$

which yields $L_t \le e^{Ct}[L_0 + 2\tilde{C}t(1+L_t)]$ for $t \in [0,T]$.

Let $k \in \mathbb{N}$ such that $1 - 2\tilde{C}he^{Ch} > 0$ for $h = T/k$. Put $L_i = L_{ih}$ for $i = 0,1,2,\dots k$. Repeating argument leads to

$$L_i \le e^{Ch}[L_{i-1} + 2h\tilde{C}(1+L_i)] \qquad \text{for} \quad i = 1,2,\dots k.$$

and by a standard procedure to

$$(9) \quad L = L_k \le [\alpha(h)]^k(L_0 + 2\tilde{C}T) \quad \text{where} \quad \alpha(h) = \frac{e^{Ch}}{1 - 2\tilde{C}he^{Ch}} \ge 1.$$

Since $[\alpha(h)]^k \to e^{(C+2\tilde{C})T}$ while $h \to 0$ we get

$$(10) \qquad \|Du\|_\Theta = L \le e^{(C+2\tilde{C})T}(L_0 + 2\tilde{C}T) = L = L(C,\tilde{C},L_0),$$

which proves the lemma.

Let $R \ge 0$ and $L \ge 0$. For every function $f : \Theta \times C(D) \times \mathbb{R}^m \to \mathbb{R}$ we define

$$f_{R,L}(t,x,w,p) = f(t,x,I_R(w),I_L(p)).$$

In view of the above lemma and earlier consideration we have

PROPOSITION 2. *If $\|\Psi\|_{\Theta_0} \le M$ and f satisfies Assumption 2 with $\sigma, C, \tilde{C}, M$ then $f_{R,L}$ satisfies it with $\sigma, 2C, \tilde{C}, M$ globally in w and p. Moreover, if $\Psi \in BUC_t(\Theta_0, L_0)$ for some $L_0 \ge 0$ and $L \ge L(2C,\tilde{C},L_0)$, $R \ge R(\sigma,M)$ then $CLS_b^*(f,\Psi) = CLS_b^*(f_{R,L},\Psi)$.*

ASSUMPTION 3. *Suppose that*

1. *Assumption 2 1), 2), 3) are satisfied, $R = R(\sigma,M)$,*
2. *for every $\tilde{L} \ge 0$ there exists $C_{\tilde{L}} \ge 0$ such that*

$$|f(t,x,w,p) - f(t,x,w,\bar{p})| \le C_{\tilde{L}}|p - \bar{p}| \quad \text{in } \Theta \times K(R) \times B(\tilde{L}),$$

3. *for every $\tilde{L} \ge 0$ there exists $H_{\tilde{L}} \ge 0$ such that*

$$|f(t,x,w,p) - f(\bar{t},x,w,p)| \le H_{\tilde{L}}(1 + L_t[w] + L_x[w])|t - \bar{t}|$$

in $\bar{\Theta} \times K(R) \cap C^{L,L}(D) \times B(\tilde{L})$.

THEOREM 2. *Let $\Psi \in C^{L,L}(\Theta_0)$, $\Psi(0,\cdot) \in C^{2+\alpha}(\mathbb{R}^m)$ for some $\alpha \in (0,1)$ and $M = \|\Psi\|_\Theta$. Suppose that Assumption 3 is satisfied. Then problem (1), (2) has a unique solution $u \in C^{1+\alpha/2,2+\alpha}(\bar{\Theta}) \cap C^{L,L}(E)$.*

Proof. Notice that, Assumption 3 implies Assumption 2. In view of Proposition 2 $f_{R,L}$ satisfies Assumption 1. Thus we can apply Theorem 1 (with a Remark 1) to $f_{R,L}$.

3. Unbounded solutions

In this section we extend our result to some class of unbounded solutions.

Let $\phi(x) = e^{b\sqrt{1+|x|^2}}$ where $b > 0, x = (x_1, \dots, x_m)$. Suppose that Ψ is continuous function such that $\frac{\Psi}{\phi}$ is bounded. First we look for the solution in the class of function u such that $\frac{u}{\phi}$ is bounded. In comparison to the case of bounded solutions we need

ASSUMPTION 4. *Suppose that*

1. *there exists* $C \geq 0$ *such that*
$$|f(t,x,w,p) - f(t,x,\bar{w},\bar{p})| \leq C(\|w-\bar{w}\|_D + |p-\bar{p}|) \quad \text{in } \Theta \times C(D) \times \mathbb{R}^m;$$
2. *there exists* $\tilde{C} \geq 0$ *such that*
$$|f(t,x,w,p) - f(t,\bar{x},w,p)| \leq \tilde{C}(1 + |w|_D + |p| + L_x[w])|x - \bar{x}| \quad \text{in } \Theta \times C^{0,L}(D) \times \mathbb{R}^m;$$
3. *there exists nondecreasing* $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*
$$|f(t,x,w,p) - f(\bar{t},x,w,p)| \leq \Gamma(|p|)(1 + L_t[w] + L_x[w])|t - \bar{t}|$$
in $\bar{\Theta} \times C^{L,L}(D) \times \mathbb{R}^m$.

REMARK 5. It follows from the Assumption 4 2) and from the continuity of f that there exists $\gamma \geq 0$ such that $\|\frac{1}{\phi} f(\cdot,\cdot,0,0)\|_{\bar{\Theta}} \leq \gamma$.

THEOREM 3. *Suppose that* $\frac{\Psi}{\phi} \in C^{L,L}(\Theta_0)$, $\frac{\Psi(0,\cdot)}{\phi} \in C^{2+\alpha}(\mathbb{R}^m)$ *and Assumption 4 is satisfied. Then there exists a unique solution of (1), (2) such that* $\frac{u}{\phi} \in C^{1+\alpha/2,2+\alpha}(\bar{\Theta}) \cap C^{L,L}(E)$.

Proof. It is not difficult to verify that u satisfies (1),(2) if and only if $v = u/\phi$ satisfies

$$D_t v - \varepsilon \Delta v = g(t,x,v_{(t,x)}, Dv) \quad \text{in} \quad \Theta, \tag{11}$$

$$v = \frac{\Psi}{\phi} \quad \text{in} \quad \Theta_0, \tag{12}$$

with $g : \Theta \times C(D) \times \mathbb{R}^m \mapsto \mathbb{R}$ given by

$$g(t,x,w,p) = 2\varepsilon \left\langle p, \frac{1}{\phi} D\phi \right\rangle + \frac{\varepsilon w(0)}{\phi} \Delta\phi + \frac{1}{\phi} f(t,x,w\phi_{(x)}, p\phi + w(0)D\phi), \tag{13}$$

where $\phi_{(x)}(y) = \phi(x+y)$ for $y \in B(r)$ and $<,>$ denotes the standard scalar product (we omit writing x with $\phi, D\phi, \Delta\phi$ since there is no functional dependence). Here and later w write $w(0) = w(0,0)$, where $(0,0) \in D \subset \mathbb{R}^{1+m}$. We claim that g satisfies assumptions of Theorem 2. First we show that g satisfies Assumption 3 1) which means that g satisfies Assumption 2 1), 2), 3).

We will base on the following estimation:

$$\frac{\|\phi_{(x)}\|_D}{\phi(x+\xi)} \le e^{2b(r+\delta)} \quad \text{for} \quad |\xi| \le \delta. \tag{14}$$

In order to demonstrate it , notice that if $|y| \le r$ and $|\xi| \le \delta$ we have

$$\frac{\phi(x+y)}{\phi(x+\xi)} = e^{b(\sqrt{1+|x+y|^2}-\sqrt{1+|x+\xi|^2})} \le e^{b(2|y|+2|\xi|)} \le e^{2b(r+\delta)}.$$

We begin with Assumption 2 2). Of course $\frac{1}{\phi}D\phi$ and $\frac{1}{\phi}\Delta\phi$ are bounded by some $N \ge 0$. We can write

$$\begin{aligned} |g(t,x,w,p) - g(t,x,\bar{w},p)| &\le \varepsilon N|w(0) - \bar{w}(0)| + C\frac{1}{\phi(x)}(\|w\phi_{(x)} - \bar{w}\phi_{(x)}\|_D \\ &\qquad + |w(0)D\phi(x) - \bar{w}(0)D\phi(x)|) \\ &\le (\varepsilon + C)N|w(0) - \bar{w}(0)| + C\frac{\|\phi_{(x)}\|_D}{\phi(x)}\|w - \bar{w}\|_D \\ &\le (\varepsilon + C)N\|w - \bar{w}\|_D + Ce^{2br}\|w - \bar{w}\|_D. \end{aligned}$$

In the last inequality we have used (14) with $\xi = 0$, $\delta = 0$. Thus point 2) of Assumption 2 follows. This gives also Assumption 2 1). Indeed putting $\bar{w} = 0$, $p = \bar{p} = 0$ in the Assumption 4 2) we get $g \in X_{\sigma,M}$, where σ is linear and $M \ge 0$ arbitrary.

Consider now point 3) of Assumption 2. Since $D(\frac{1}{\phi}D\phi)$, $D(\frac{1}{\phi}\Delta\phi)$ are bounded, we will consider only the term with f. Let $\|w\|_D \le R = R(\sigma, M)$, $M = \|\frac{\Psi}{\phi}\|_{\Theta_0}$, $|x - \bar{x}| \le \delta$, $\delta > 0$ and $w \in C^{0,L}(D)$. Observe that

$$\begin{aligned} &\left|\frac{1}{\phi(x)}f(t,x,w\phi_{(x)},p\phi(x) + w(0)D\phi(x)) - \frac{1}{\phi(\bar{x})}f(t,\bar{x},w\phi_{(\bar{x})},p\phi(\bar{x}) + w(0)D\phi(\bar{x}))\right| \\ &\le \left|\frac{1}{\phi(x)} - \frac{1}{\phi(\bar{x})}\right| |f(t,x,w\phi_{(x)},p\phi(x) + w(0)D\phi(x))| \\ &+ \frac{1}{\phi(\bar{x})}|f(t,x,w\phi_{(x)},p\phi(x) + w(0)D\phi(x)) - f(t,\bar{x},w\phi_{(\bar{x})},p\phi(\bar{x}) + w(0)D\phi(\bar{x}))| \\ &\le \left|\frac{1}{\phi(x)} - \frac{1}{\phi(\bar{x})}\right| (|f(t,x,w\phi_{(x)},p\phi(x) + w(0)D\phi(x)) - f(t,x,0,0)| + |f(t,x,0,0)|) \\ &+ \frac{1}{\phi(\bar{x})}(|f(t,x,w\phi_{(x)},p\phi(x) + w(0)D\phi(x)) - f(t,\bar{x},w\phi_{(x)},p\phi(x) + w(0)D\phi(x))| \\ &+ |f(t,\bar{x},w\phi_{(x)},p\phi(x) + w(0)D\phi(x)) - f(t,\bar{x},w\phi_{(\bar{x})},p\phi(\bar{x}) + w(0)D\phi(\bar{x}))|). \end{aligned}$$

First term in the last sum (see Assumption 4 1) and Remark 5) is bounded by

$$\left|\frac{1}{\phi(x)}-\frac{1}{\phi(\bar{x})}\right|(C\|w\phi_{(x)}\|_D+C\phi(x)|p|+C|w(0)||D\phi(x)|+\phi(x)\gamma),$$

while the second (see Assumption 4.1) and 2)) by

$$\frac{1}{\phi(\bar{x})}[\ \tilde{C}(\ 1+\|w\phi_{(x)}\|_D+|p|\phi(x)+|w(0)||D\phi(x)|+L_x[w\phi_{(x)}]\ )|x-\bar{x}|$$
$$+C(\ \|w\phi_{(x)}-w\phi_{(\bar{x})}\|_D+|p||\phi(x)-\phi(\bar{x})|+|w(0)||D\phi(x)-D\phi(\bar{x})|)].$$

Since

$$\left|\frac{1}{\phi(x)}-\frac{1}{\phi(\bar{x})}\right|\|\phi_{(x)}\|_D\le\sup\left\{\left|D\left(\frac{1}{\phi(x+\xi)}\right)\right|:|\xi|\le\delta\right\}\|\phi_{(x)}\|_D|x-\bar{x}|$$
$$\le b\sup\left\{\frac{1}{|\phi(x+\xi)|}:|\xi|\le\delta\right\}\|\phi_{(x)}\|_D|x-\bar{x}|\le be^{2b(r+\delta)}|x-\bar{x}|$$

(in view of (14)) and

$$L_x[w\phi_{(x)}]\le R\|D\phi_{(x)}\|_D+\|\phi_{(x)}\|_DL_x[w],$$

thus

$$(15)\qquad \frac{1}{\phi(x)}L_x[w\phi_{(x)}]\le(Rb+L_x[w])\frac{\|\phi(x)\|_D}{\phi(x)}\le(Rb+L_x[w])e^{2br},$$

(see (14), $\xi=0$). Moreover: $\frac{\phi(x)}{\phi(\bar{x})}\le e^{2b\delta}$ (see (14) $r=0$), $|D\phi(x)|\le b\phi(x)$ and $\frac{1}{\phi(\bar{x})}|D\phi(x)-D\phi(\bar{x})|\le A_{r,\delta}|x-\bar{x}|$ for $A_{r,\delta}>0$. All this gives Assumption 2 3) for g, with a some constant $\tilde{C}_\delta$, while $|x-\bar{x}|\le\delta$. In a standard way we can show that this constant is right for all $x,\bar{x}$. Thus letting $\delta\to0$ we obtain an independent constant $\tilde{C}$.

Since Assumption 3 2) is easy, it is left to the reader.

It remain to consider Assumption 3 3). In this case the conclusion follows easily from (15) and from: $L_t[w\phi_{(x)}]\le\|\phi_{(x)}\|_DL_t[w]$. We skip the details.

By Theorem 2 we have $\tilde{v}$, an unique solution of (11),(12). Putting $\tilde{u}=\tilde{v}\phi$ we get an unique solution of (1),(2) in a class of exponentially bounded function.

It is a good place to underline the difference between bounded and unbounded solutions of (1), (2) in context to unbounded deviation. Notice that all the results of previous sections hold true if we put $r=\infty$. However, in case of unbounded solution, this require from assumption on f to be modified in a natural way.

The above method, after a little modification of Proposition 1, can be applied to a larger class of unbounded solution i.e. to these bounded by $Me^{b|x|^2}$. The main difference is that we can expect only a local existence theorem. The Lipshitz condition on w must be strengthen also in this case (except for r=0) (see [3]).

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