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IMPLICIT DIFFERENCE METHODS FOR HAMILTON–JACOBI DIFFERENTIAL FUNCTIONAL EQUATIONS

Abstract. Classical solutions of the local Cauchy problem on the Haar pyramid are approximated in the paper by solutions of suitable quasilinear systems of difference functional equations. The numerical methods are difference schemes which are implicit with respect to time variable. A complete convergence analysis for the methods is given and it is shown that the new methods are considerable better than the explicit schemes. The proof of the stability is based on a comparison technique with nonlinear estimates of the Perron type. Numerical examples are given.

1. Introduction

For any metric spaces X and Y we denote by $C(X,Y)$ the class of all continuous functions from X into Y. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. Let E be the Haar pyramid

$$E = \{(t,x) \in \mathbf{R}^{1+n} : t \in [0,a], -b + Mt \leq x \leq b - Mt\},$$

where $x = (x_1, \ldots, x_n)$, $a > 0$, $M = (M_1, \ldots, M_n) \in \mathbf{R}^n_+$, $\mathbf{R}_+ = [0, +\infty)$, $b = (b_1, \ldots, b_n) \in \mathbf{R}^n$ and $b \geq Ma$. Write $E_0 = [-b_0, 0] \times [-b, b] \subset \mathbf{R}^{1+n}$, where $b_0 \in \mathbf{R}_+$ and $\Omega = E \times \mathbf{R} \times \mathbf{R}^n$. Suppose that the functions

$$f : \Omega \to \mathbf{R}, \quad \varphi : E_0 \to R, \quad V : C(E_0 \cup E, \mathbf{R}) \to C(E, \mathbf{R})$$

are given. We consider the differential functional equation

$$\partial_t z(t,x) = f(t, x, V[z](t,x), \partial_x z(t,x)) \tag{1}$$

with the initial condition

$$z(t,x) = \varphi(t,x) \quad \text{for} \quad (t,x) \in E_0, \tag{2}$$

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where $\partial_x z = (\partial_{x_1} z, \ldots, \partial_{x_n} z)$. A function $z : E_0 \cup E \to \mathbf{R}$ is called a classical solution of the above problem if

(i) $z \in C(E_0 \cup E, \mathbf{R})$ and z is of class C^1 on E,
(ii) z satisfies (1) on E and initial condition (2) holds.

We consider classical solutions of (1), (2) and assuming that V satisfies the Volterra condition defined below. We are interested in establishing a method of numerical approximation of solutions of problem (1), (2) by means of solutions of associated systems of difference functional equations and in estimating of the difference between the exact and approximate solutions.

In recent years, a number of papers concerning numerical methods for functional partial differential equations have been published. The main question in these investigations is to find a difference functional equation which satisfies the consistency conditions on all classical solutions of the original problem and it is stable. The method of difference inequalities or theorems on linear recurrent inequalities are used in the investigations of the stability. The proofs of the convergence are also based on a general theorem on the error estimates of approximate solutions to functional difference equations of the Volterra type with initial boundary conditions and with unknown function of several variables.

Difference schemes for (1), (2) in the case when differential equation does not contain a functional variable were considered in [1], [5], [7], [8]. Finite difference approximations relative to initial or initial boundary value problems for functional differential equations were investigated in [2], [3], [5], [6], [10]. The monograph [4] contains an exposition of recent developments of numerical methods for hyperbolic functional differential problems.

In the paper we present a new class of difference schemes for (1), (2). The numerical methods are difference schemes which are implicit with respect to time variable.

Two type of assumptions are needed in theorems on the convergence of difference schemes corresponding to (1), (2). The first type conditions deal with the regularity of given functions. The assumptions of the second type are connected with relations between the steps of the mesh. We show in the paper that the assumptions of the second type can be omitted for implicit difference schemes.

In Section 2 we present relations between classical difference methods and implicit difference schemes.

Our considerations are based on the following idea. In the first step we transform the nonlinear equation (1) into a quasilinear system of functional differential equations, where unknown functionals are z and the partial derivatives of z with respect to spatial variables. In the second step we construct an implicit Euler method for z and for their spatial derivatives. It is important in our considerations that the method of discretizations of quasilinear systems corresponding to (1), (2) depend on local properties of given functions. The stability of the methods is investigated by using a comparison technique.

The paper is organized as follows. In section 2 we construct an implicit difference functional problem corresponding to (1), (2) and we prove that there exists exactly one solution of a difference scheme. In Section 3 we prove a convergence result and we give an error estimate for implicit schemes. Examples of interpolating operators are given in Section 4. Numerical experiments are presented in the last part of the paper.

Differential equations with deviated variables and differential integral problems can be derived from (1), (2) by specializing the operator V. Existence and uniqueness results for functional differential problems on the Haar pyramid can be found in [4] (Th. 2.4, p. 49).

First order partial functional differential equations find applications in different fields of knowledge.

For additional bibliography on partial functional differential equations and their applications see the monographs [4], [12].

Let us denote by $F(X,Y)$ the class of all functions defined on X and taking values in Y, where X and Y are arbitrary sets. Let $\mathbf{N}$ and $\mathbf{Z}$ be the sets of natural numbers and integers, respectively.

Denote by $\mathbf{R}^n$ the Euclidean real space of vectors $x = (x_1, \ldots, x_n)$ and by $\mathbf{R}^{n\times n}$ the space all $n \times n$ matrices $U = [u_{ij}]_{i,j=1,\ldots,n}$ with real elements. In $\mathbf{R}^n$ and $\mathbf{R}^{n\times n}$ we introduce the norms

$$\|x\| = \sum_{j=1}^{n} |x_j| \quad \text{and} \quad \|U\| = \max\Big\{\sum_{j=1}^{n} |u_{ij}| : 1 \le i \le n\Big\}.$$

If $U \in \mathbf{R}^{n\times n}$ then U^T is the transpose matrix. Write

$$E_t = (E_0 \cup E) \cap ([-b_0, t] \times \mathbf{R}^n), \quad 0 \le t \le a.$$

We will say that the operator $V : C(E_0 \cup E, \mathbf{R}) \to C(E, \mathbf{R})$ satisfies the Volterra conditions if for each $(t,x) \in E$ and for $z, \bar{z} \in C(E_0 \cup E, \mathbf{R})$ such that $z|_{E_t} = \bar{z}|_{E_t}$ we have $V[z](t,x) = V[\bar{z}](t,x)$.

For functions $z \in C(E_0 \cup E, \mathbf{R})$, $u \in C(E_0 \cup E, \mathbf{R}^n)$ and for a point $t \in [0,a]$ we put

$$\|z\|_t = \max\{|z(\tau,y)| : (\tau,y) \in E_t\} \quad \text{and}$$
$$\|u\|_t = \max\{\|u(\tau,y)\| : (\tau,y) \in E_t\}.$$

We formulate a difference problem corresponding to (1), (2). We define a mesh on the set $E_0 \cup E$ in the following way. Suppose that $(h_0, \hat{h})$, $\hat{h} =$

$(h_1, \ldots, h_n)$, stand for steps of the mesh. Denote by $\widetilde{H}$ the set of all $h = (h_0, \hat{h})$ such that there are $N_0 \in \mathbf{N}$, $N = (N_1, \ldots, N_n) \in \mathbf{N}^n$ with the properties $N_i h_i = b_i$ for $0 \le i \le n$ and $\hat{h} \le M h_0$. Let $K \in \mathbf{N}$ be defined by $K h_0 \le a < (K+1) h_0$. For $h \in \widetilde{H}$ and $(r, m) \in \mathbf{Z}^{1+n}$, where $m = (m_1, \ldots, m_n)$, we define nodal points as follows

$$t^{(r)} = r h_0, \ \ x^{(m)} = (x_1^{(m_1)}, \ldots, x_n^{(m_n)}) = (m_1 h_1, \ldots, m_n h_n).$$

Write

$$\mathbf{R}_h^{1+n} = \{(t^{(r)}, x^{(m)}) : (r, m) \in \mathbf{Z}^{1+n}\}$$

and

$$E_h = E \cap \mathbf{R}_h^{1+n}, \ \ E_{h.0} = E_0 \cap \mathbf{R}_h^{1+n},$$
$$I_h = \{t^{(r)} : 0 \le r \le K\},$$
$$E_{h.r} = (E_{h.0} \cup E_h) \cap ([-b_0, t^{(r)}] \times \mathbf{R}^n), \quad 0 \le r \le K.$$

For functions $\eta : I_h \to \mathbf{R}$, $z : E_{h.0} \cup E_h \to \mathbf{R}$, $u : E_{h.0} \cup E_h \to \mathbf{R}^{n \times n}$ we write $\eta^{(r)} = \eta(t^{(r)})$, $z^{(r,m)} = z(t^{(r)}, x^{(m)})$, $u^{(r,m)} = u(t^{(r)}, x^{(m)})$ and

$$\|z\|_{h.i} = \max\{|z^{(r,m)}| : (t^{(r)}, x^{(m)}) \in E_{h.i}\},$$
$$\|u\|_{h.i} = \max\{\|u^{(r,m)}\| : (t^{(r)}, x^{(m)}) \in E_{h.i}\},$$

where $0 \le i \le K$. Let $e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbf{R}^n$, 1 standing on the j-th place and $\theta = (0, \ldots, 0) \in \mathbf{R}^n$. Write

(3) $$E'_h = \{(t^{(r)}, x^{(m)}) \in E_h : (t^{(r+1)}, x^{(m)}) \in E_h\}.$$

Classical difference methods for (1), (2) consist in replacing partial derivatives ∂_t and $(\partial_{x_1}, \ldots, \partial_{x_n}) = \partial_x$ with difference operators δ_0 and $(\delta_1, \ldots, \delta_n) = \delta$, respectively. Approximate solutions of (1), (2) are functions z_h defined on the mesh $E_{h.0} \cup E_h$. On the other hand, equation (1) contains the functional variable $V[z]$ which is an element of the space $C(E, \mathbf{R})$. Therefore we need an interpolating operator $V_h : F(E_{h.0} \cup E_h, \mathbf{R}) \to C(E, \mathbf{R})$. This leads to the difference equation

(4) $$\delta_0 z^{(r,m)} = f(t^{(r)}, x^{(m)}, V_h[z]^{(r,m)}, \delta z^{(r,m)})$$

with the initial condition

(5) $$z^{(r,m)} = \varphi_h^{(r,m)} \quad \text{on} \quad E_{h.0},$$

where $\varphi_h : E_{h.0} \to \mathbf{R}$ is a given function.

Suppose that the interpolating operator V_h is fixed. The following examples of difference schemes are considered in literature. Write

(6) $$\delta_0 z^{(r,m)} = \frac{1}{h_0}[z^{(r+1,m)} - z^{(r,m)}]$$

and

$$\delta_i z^{(r,m)} = \frac{1}{h_i}[z^{(r,m+e_i)} - z^{(r,m)}] \quad \text{for} \quad 1 \le i \le \kappa, \tag{7}$$

$$\delta_i z^{(r,m)} = \frac{1}{h_i}[z^{(r,m)} - z^{(r,m-e_i)}] \quad \text{for} \quad \kappa+1 \le i \le n, \tag{8}$$

where $0 \le \kappa \le n$ is fixed. Numerical method (4), (5) with the above given δ_0 and δ is known as the Euler method.

The Lax difference scheme is the second important example. It is obtained by putting

$$\delta_0 z^{(r,m)} = \frac{1}{h_0}[z^{(r+1,m)} - \frac{1}{2n}\sum_{j=1}^{n}(z^{(r,m+e_j)} + z^{(r,m-e_j)})] \tag{9}$$

and

$$\delta_i z^{(r,m)} = \frac{1}{2h_i}[z^{(r,m+e_i)} - z^{(r,m-e_i)}], \quad 1 \le i \le n. \tag{10}$$

Assumptions on the regularity of f in convergence theorems are the same for both methods. It is required that the function f of the variables (t, x, p, q) satisfies the Lipschitz condition with respect to p and it is of class C^1 with respect to $(q_1, \ldots, q_n) = q$ and that the function $\partial_q f$ is bounded. The second type of assumptions are the Courant-Friedrichs-Levy conditions. In the case of the Lax method they have the form

$$\frac{1}{n} - h_0 \frac{1}{h_j}|\partial_{q_j} f(t, x, p, q)| \ge 0. \tag{11}$$

For the analysis of the stability of the Euler method we need the assumption that

$$1 - h_0 \sum_{j=1}^{n} \frac{1}{h_j}|\partial_{q_j} f(t, x, p, q)| \ge 0 \tag{12}$$

and that the functions $\partial_{q_i} f$, $i = 1, \ldots, n$, have constant signs on Ω.

Condition (11) and (12) are similar and they require some relations between h_0 and $(h_1, \ldots, h_n)$. Then the strong assumption that the functions sign $\partial_{q_i} f$, $i = 1, \ldots, n$, are constant on Ω is the main difference between the above methods.

There are equations (1) for which both the methods can be used. We give comments on the relations between the Euler method and the Lax scheme in this case. Suppose that

(i) $\partial_q f \in C(\Omega, \mathbf{R}^n)$ and the function $\partial_q f$ satisfies the Lipschitz condition with respect to (x, p, q),

(ii) for $P=(t,x,p,q)\in\Omega$ we have

(13) $\partial_{q_i}f(P)\ge 0$ for $1\le i\le\kappa$ and $\partial_{q_i}f(P)\le 0$ for $\kappa+1\le i\le n$,

where $0\le\kappa\le n$ is fixed, and κ appears in the definitions (7), (8),

(iii) the function $v\in C^1(E_0\cup E,\mathbf{R})$ is a classical solution of (1), (2) and the functions $V[v]$ and $\partial_x v$ satisfy the Lipschitz condition with respect to x.

For given $z\in C(E_0\cup E,\mathbf{R})$, $u\in C(E,\mathbf{R}^n)$ denote by $g[z,u]$ the solution of the Cauchy problem

$$\eta'(\tau)=-\partial_q f(\tau,\eta(\tau),V[z](\tau,\eta(\tau)),u(\tau,\eta(\tau))), \tag{14}$$

$$\eta(t)=x. \tag{15}$$

The function $g[z,u](\cdot,t,x)$ is the bicharacteristic of equation (1) corresponding to (z,u). The bicharacteristic $g[v,\partial_x v](\cdot,t,x)$ is defined on some interval $[0,\alpha(t,x)]$ such that $(\alpha(t,x),\ g[v,\partial_x v](\alpha(t,x),t,x))\in\partial E$, where ∂E is the bounder of E.

Let us denote by z_h the solution of (4), (5) with δ_0 and δ defind by (6)–(8). Suppose that z_h is given on $E_{h.r}$ and $(t^{(r+1)},x^{(m)})\in E_h$. Our aim is to calculate the number $z_h^{(r+1,m)}$.

Write

$$A^{(r)}=[x_1^{(m_1)},x_1^{(m_1+1)}]\times\ldots\times[x_\kappa^{(m_\kappa)},x_\kappa^{(m_\kappa+1)}]\times$$
$$\times[x_{\kappa+1}^{(m_{\kappa+1}-1)},x_{\kappa+1}^{(m_{\kappa+1})}]\times\ldots\times[x_n^{(m_n-1)},x_n^{(m_n)}].$$

It follows from (12), (13) that

$$g[v,\partial_x v](t^{(r)},t^{(r+1)},x^{(m)})\in A^{(r)}. \tag{16}$$

The following property of the Euler method is important: $A^{(r)}\subset\mathbf{R}^n$ is the smallest interval of the form $[y,\widetilde{y}]\subset\mathbf{R}^n$ such that $(t^{(r)},y)$, $(t^{(r)},\widetilde{y})\in E_h$ and $g[v,\partial_x v](t^{(r)},t^{(r+1)},x^{(m)})\in[y,\widetilde{y}]$.

Let us denote by $\widetilde{z}_h$ the solution of (4), (5) with δ_0 and δ defined by (9), (10). Suppose that $\widetilde{z}_h$ is given on $E_{h.r}$ and $(t^{(r+1)},x^{(m)})\in E_h$ and we calculate $\widetilde{z}_h^{(r+1,m)}$ by using the Lax scheme. It follows from (9), (10) that the numbers

$$\widetilde{z}_h^{(r,m-e_i)},\quad \widetilde{z}_h^{(r,m+e_i)},\quad i=1,\ldots,n$$

appear in (4). Note that

$$x^{(m-e_i)}\notin A^{(r)}\quad\text{for}\quad i=1,\ldots,\kappa\quad\text{and}\quad x^{(m+e_i)}\notin A^{(r)}$$
$$\text{for}\quad i=\kappa+1,\ldots,n.$$

This is the reason why the Euler method is more suitable than the Lax scheme. Numerical experiments confirm the above theoretical observation.

The monograph [4] contains an exposition of recent developments of numerical methods for hyperbolic functional differential problems.

The aim of the paper is to show that for each equation (1) with sufficiently regular f and V the Euler method can be constructed.

The assumption that the functions sign $\partial_{q_i} f$, $i = 1, \ldots, n$, are constant is omitted in the paper. In other words, we show that the Lax scheme is superfluous for the numerical approximation of classical solutions of (1).

Since we consider implicit difference schemes, then we show that assumption (12) can be also omitted in convergence theorems.

2. Generalized Euler method for initial problems

We formulate implicit difference methods of the Euler type for (1), (2). Write

$$\partial_0 E_i^+ = \{(t,x) \in E : x_i = b_i - M_i t\},$$
$$\partial_0 E_i^- = \{(t,x) \in E : x_i = -b_i + M_i t\},$$

where $1 \le i \le n$. We need the following assumptions on f.

Assumption $H_0[f]$. The function $f \in C(\Omega, \mathbf{R})$ is such that

1) the partial derivatives $\partial_x f = (\partial_{x_1} f, \ldots, \partial_{x_n} f)$, $\partial_p f$, $\partial_q f = (\partial_{q_1} f, \ldots, \partial_{q_n} f)$ exist on Ω and $\partial_x f$, $\partial_q f \in C(\Omega, \mathbf{R}^n)$, $\partial_p f \in C(\Omega, \mathbf{R})$,
2) there is $A \in \mathbf{R}_+$ such that for $P = (t, x, p, q) \in \Omega$ we have
$$\|\partial_x f(P)\|, \quad |\partial_p f(P)|, \quad \|\partial_q f(P)\| \le A,$$
3) there is $\delta > 0$ such that
$$\partial_{q_i} f(t, x, p, q) < -\delta \quad \text{for} \quad (t, x, p, q) \in \partial_0 E_i^+ \times \mathbf{R} \times \mathbf{R}^n$$
and
$$\partial_{q_i} f(t, x, p, q) > \delta \quad \text{for} \quad (t, x, p, q) \in \partial_0 E_i^- \times \mathbf{R} \times \mathbf{R}^n$$
where $1 \le i \le n$.

Remark 2.1. Suppose that Assumption $H_0[f]$ is satisfied. For given $z \in C(E_0 \cup E, \mathbf{R})$, $u \in C(E, \mathbf{R}^n)$ consider the bicharacteristic $g[z,u](\cdot, t, x) = (g_1[z,u](\cdot, t, x), \ldots, g_n[z,u](\cdot, t, x))$ as the solution of the Cauchy problem (14), (15). By $H_0[f]$, 3) there is an $\varepsilon_0 > 0$ such that

(i) functions $g_i[z,u](\cdot, t, x) : (t - \varepsilon_0, t] \to \mathbf{R}$ are strictly increasing for each $(t,x) \in \partial_0 E_i^+$, $1 \le i \le n$,
(ii) functions $g_i[z,u](\cdot, t, x) : (t - \varepsilon_0, t] \to \mathbf{R}$ are strictly decreasing for each $(t,x) \in \partial_0 E_i^-$, $1 \le i \le n$.

This property of bicharacteristics is important in the construction of implicit difference methods for (1), (2).

Now we formulate assumptions on V and on interpolating operators.

ASSUMPTION $H[V, T_h, L_h]$. Suppose that the operator $V : C(E_0 \cup E, \mathbf{R}) \to C(E, \mathbf{R})$ satisfies the Volterra condition and

1) if $z \in C^1(E_0 \cup E, \mathbf{R})$ then there exist partial derivatives $\partial_x V[z] = (\partial_{x_1} V[z], \ldots, \partial_{x_n} V[z])$, $\partial_x V[z] \in C(E, \mathbf{R}^n)$,
2) there is an operator $T_h : F(E_{h.0} \cup E_h, \mathbf{R}) \to C(E, \mathbf{R})$ such that

   (i) there is $L \in \mathbf{R}_+$ such that for $z, \bar{z} \in F(E_{h.0} \cup E_h, \mathbf{R})$ we have

$$\|T_h[z] - T_h[\bar{z}]\|_{t^{(r)}} \leq L\|z - \bar{z}\|_{h.r}, \quad 0 \leq r \leq K, \tag{17}$$

   (ii) there is $\mu > 0$ such that for each function $v \in C^2(E_0 \cup E, \mathbf{R})$ there is $c_0 \in \mathbf{R}_+$ such that

$$\|V[v] - T_h[v_h]\|_{t^{(r)}} \leq c_0 h_0^{\mu}, \quad 0 \leq r \leq K, \tag{18}$$

   where v_h is the restriction of v to the set $E_{h.0} \cup E_h$.

3) there is an interpolating operator $L_h : F(E_{h.0} \cup E_h, \mathbf{R}^{1+n}) \to C(E, \mathbf{R}^n)$ with the properties

   (i) there is $L_0 \in \mathbf{R}_+$ such that for (z, u), $(\bar{z}, \bar{u}) \in F(E_{h.0} \cup E_h, \mathbf{R}^{1+n})$ we have

$$\|L_h[z, u] - L_h[\bar{z}, \bar{u}]\|_{t^{(r)}} \leq L_0[\|z - \bar{z}\|_{h.r} + \|u - \bar{u}\|_{h.r}], \quad 0 \leq r \leq K, \tag{19}$$

   (ii) there is $\nu > 0$ such that for each function $v \in C^2(E_0 \cup E, \mathbf{R})$ there is c_1 with the property

$$\|L_h[v_h, (\partial_x v)_h] - \partial_x V[v]\|_{t^{(r)}} \leq c_1 h_0^{\nu}, \quad 0 \leq r \leq K, \tag{20}$$

   where $(\partial_x v)_h$ are the restrictions of $\partial_x v$ to the set $E_{h.0} \cup E_h$.

Examples of the operator V, T_h, L_h are given in Section 4.

Note that condition 1) of Assumption $H[V, T_h, L_h]$ implies that T_h satisfies the following Volterra condition: if $(t^{(r)}, x) \in E_{h.r}$ and $z, \bar{z} \in F(E_{h.0} \cup E_h, \mathbf{R})$ and $z(\tau, y) = \bar{z}(\tau, y)$ for $(\tau, y) \in E_{h.r}$ then $T_h[z](t^{(r)}, x) = T_h[\bar{z}](t^{(r)}, x)$. It follows from condition 3) of Assumption $H[V, T_h, L_h]$ that the operator L_h satisfies the Volterra condition.

Suppose that the Assumption $H_0[f]$ is satisfied. Let

$$E_{i.\varepsilon}^+ = \{(t, x) \in E : b_i - M_i t - \varepsilon \leq x_i \leq b_i - M_i t\}$$

and

$$E_{i.\varepsilon}^- = \{(t, x) \in E : -b_i + M_i t \leq x_i \leq -b_i + M_i t + \varepsilon\},$$

where $1 \leq i \leq n$. By condition 3) of Assumption $H_0[f]$ there exists $\varepsilon > 0$ such that

$$\partial_{q_i} f(t, x, p, q) < -\frac{\delta}{2} \quad \text{for} \quad (t, x, p, q) \in E_{i.\varepsilon}^+ \times \mathbf{R} \times \mathbf{R}^n$$

and

$$\partial_{q_i} f(t,x,p,q) > \frac{\delta}{2} \quad \text{for} \quad (t,x,p,q) \in E^-_{i.\varepsilon} \times \mathbf{R} \times \mathbf{R}^n.$$

Let

(21) $$H = \{h = (h_0, \hat{h}) \in \widetilde{H} : h_i < \frac{\delta}{2} \quad \text{for} \quad 1 \le i \le n\}.$$

We write a difference problem corresponding to (1), (2). The unknown functions in a difference system are denoted by (z,u), where $u=(u_1,\dots,u_n)$. Put

(22) $$\begin{aligned} \delta_0 z^{(r,m)} &= \frac{1}{h_0}(z^{(r+1,m)} - z^{(r,m)}), \\ \delta_0 u_i^{(r,m)} &= \frac{1}{h_0}(u_i^{(r+1,m)} - u_i^{(r,m)}), \qquad 1 \le i \le n, \\ \delta_0 u^{(r,m)} &= (\delta_0 u_1^{(r,m)}, \dots, \delta_0 u_n^{(r,m)}), \end{aligned}$$

and

$$P^{(r,m)}[z,u] = (t^{(r)}, x^{(m)}, T_h[z]^{(r,m)}, u^{(r,m)}).$$

We consider the system of difference equations

(23) $$\delta_0 z^{(r,m)} = f(P^{(r,m)}[z,u]) + \partial_q f(P^{(r,m)}[z,u]) \, [\, \delta z^{(r+1,m)} - u^{(r,m)} \,]^T,$$

(24) $$\begin{aligned} \delta_0 u^{(r,m)} = {} & \partial_x f(P^{(r,m)}[z,u]) + \partial_p f(P^{(r,m)}[z,u]) \, L_h[z,u]^{(r,m)} \\ & + \partial_q f(P^{(r,m)}[z,u]) \, [\, \delta u^{(r+1,m)} \,]^T \end{aligned}$$

with the initial condition

(25) $$z^{(r,m)} = \varphi_h^{(r,m)}, \quad u^{(r,m)} = \psi_h^{(r,m)} \quad \text{for} \quad (t^{(r)}, x^{(m)}) \in E_{h.0},$$

where $\varphi_h : E_{h.0} \to \mathbf{R}$, $\psi_h : E_{h.0} \to \mathbf{R}^n$ are given functions. The difference operators $(\delta_1, \dots, \delta_n)$ for spatial variables are given in the following way. Suppose that $(t^{(r)}, x^{(m)}) \in E'_h$ and that the functions (z,u) are known on the set $E_{h.r}$.

(26) If $\partial_{q_i} f(P^{(r,m)}[z,u]) \ge 0$ then $\delta_i z^{(r+1,m)} = \frac{1}{h_i}(z^{(r+1,m+e_i)} - z^{(r+1,m)})$,

and

(27) $$\delta_i u_j^{(r+1,m)} = \frac{1}{h_i}(u_j^{(r+1,m+e_i)} - u_j^{(r+1,m)}), \quad 1 \le j \le n.$$

(28) If $\partial_{q_i} f(P^{(r,m)}[z,u]) < 0$ then $\delta_i z^{(r+1,m)} = \frac{1}{h_i}(z^{(r+1,m)} - z^{(r+1,m-e_i)})$

and

(29) $$\delta_i u_j^{(r+1,m)} = \frac{1}{h_i}(u_j^{(r+1,m)} - u_j^{(r+1,m-e_i)}), \quad 1 \le j \le n.$$

We put $i = 1, \dots, n$ in (26)–(29), and

$$\delta z^{(r+1,m)} = (\delta_1 z^{(r+1,m)}, \dots, \delta_n z^{(r+1,m)}), \quad \delta u^{(r+1,m)} = [\delta_j u_i^{(r+1,m)}]_{i,j=1,\dots,n}.$$

The above difference functional problem is called a generalized implicit Euler method for (1), (2). It is important in our considerations that the difference expressions δz and δu appear in (23), (24) at the point $(t^{(r+1)}, x^{(m)})$.

REMARK 2.2. It follows from (26)–(29) that the generalized implicit Euler method has the following property: the definition of the operator δz at the point $(t^{(r+1)}, x^{(m)})$ depends on the local properties of the function $\partial_q f$. More precisely, it depend on the numbers

$$(\text{sign}\, \partial_{q_1} f(P^{(r,m)}[z,u]), \dots, \text{sign}\, \partial_{q_n} f(P^{(r,m)}[z,u])).$$

We prove that under natural assumptions on given functions and on the mesh there exists exactly one solution $(z_h, u_h) : E_{h.0} \cup E_h \to \mathbf{R}^{1+n}$ of implicit difference problem (23)–(25).

REMARK 2.3. If we apply method (4), (5) to solve problem (1), (2) numerically then we approximate derivatives with respect to spatial variables with difference expressions which are calculated by using the previous values of the approximate solution. If we use method (23)–(25) then we approximate the spatial derivatives of the unknown function by using adequate difference equations which are generated by the original problem. Therefore numerical results obtained by (23)–(25) are better than those obtained by (4), (5). We give suitable examples in Section 5. Notice that the assumptions on the right-hand sides of equation (1) are more restrictive for the methods (23)–(25) than for the classical schemes.

REMARK 2.4. Note that the implicit difference method generated by (4) has the form

$$\delta_0 z^{(r,m)} = f(t^{(r)}, x^{(m)}, V_h[z]^{(r,m)}, \delta z^{(r+1,m)}. \tag{30}$$

Then the solution $\widetilde{z}_h : E_{h.0} \cup E_h \to \mathbf{R}$ of problem (5), (30) is obtained by solving of nonlinear systems of algebraic equations. The solution $(z_h, u_h) : E_{h.0} \cup E_h \to \mathbf{R}^{1+n}$ of implicit difference problem (23)–(25) is obtained by solving of linear systems.

The difference functional problem (23)–(25) is obtained in the following way. Suppose that Assumption $H_0[f]$ is satisfied and that the derivatives $\partial_x \varphi = (\partial_{x_1} \varphi, \dots, \partial_{x_n} \varphi)$ exist on E_0. The method of quasilinearization for nonlinear equations consists in replacing problem (1), (2) with the following one. We first introduce an additional unknown function $u = \partial_x z$ in equation (1). Then we consider the linearization of (1) with respect to the last variable:

(31) $\partial_t z(t,x) = f(U[z,u;t,x]) + \partial_q f(U[z,u;t,x]) \, [\, \partial_x z(t,x) - u(t,x) \,]^T,$

where $U[z,u;t,x] = (t,x,V[z](t,x),u(t,x))$. Differential equations for u we get by differentiation of equation (1) with respect to x. The result is

(32)
$$\partial_t u(t,x) = \partial_x f(U[z,u;t,x])$$
$$+\partial_p f(U[z,u;t,x]) \, \partial_x V[z](t,x) + \partial_q f(U[z,u;t,x]) \, [\, \partial_x u(t,x) \,]^T.$$

It is natural to consider the following initial boundary condition for (31), (32):

(33) $$z(t,x) = \varphi(t,x), \quad u(t,x) = \partial_x \varphi(t,x) \quad \text{for} \quad (t,x) \in E_0.$$

Difference problem (23)–(25) is a discretization of (31)–(33).

Let

$$\Delta^{(r)} = \{x^{(m)} : x^{(m)} \in [-b + t^{(r)}M, b - t^{(r)}M]\}, \quad 0 \le r \le K.$$

The difference functional equations

(34) $$z^{(r+1,m)} = h_0 \partial_q f(P^{(r,m)}[z,u]) \, [\, \delta z^{(r+1,m)} \,]^T$$

and

(35) $$u^{(r+1,m)} = h_0 \partial_q f(P^{(r,m)}[z,u]) \, [\, \delta u^{(r+1,m)} \,]^T$$

are principal parts of (23) and (24) respectively. We prove a lemma on difference inequalities generated by (34), (35). Put

$$J_+^{(r,m)}[z,u] = \{j \in \{1,\dots,n\} : \partial_{q_j} f(P^{(r,m)}[z,u]) \ge 0\},$$
$$J_-^{(r,m)}[z,u] = \{1,\dots,n\} \setminus J_+^{(r,m)}[z,u].$$

LEMMA 2.1. *Suppose that $h \in H$, and $z_h \in F(E_{h.0} \cup E_h, \mathbf{R})$, $u_h \in F(E_{h.0} \cup E_h, \mathbf{R}^n)$.*

(I) If z_h and u_h satisfy the implicit difference inequalities

$$z_h^{(r+1,m)} \le h_0 \partial_q f(P^{(r,m)}[z_h,u_h]) \, [\, \delta z_h^{(r+1,m)} \,]^T,$$
$$u_h^{(r+1,m)} \le h_0 \partial_q f(P^{(r,m)}[z_h,u_h]) \, [\, \delta u_h^{(r+1,m)} \,]^T,$$

where $(t^{(r)}, x^{(m)}) \in E'_h$ and initial estimates

$$z_h^{(r,m)} \le 0, \quad u_h^{(r,m)} \le \theta$$

are satisfied on $E_{h.0}$ then $z_h^{(r,m)} \le 0$ and $u_h^{(r,m)} \le \theta$ on E_h.

(II) If the difference inequalities

$$z_h^{(r+1,m)} \ge h_0 \partial_q f(P^{(r,m)}[z_h,u_h]) \, [\, \delta z_h^{(r+1,m)} \,]^T,$$
$$u_h^{(r+1,m)} \ge h_0 \partial_q f(P^{(r,m)}[z_h,u_h]) \, [\, \delta u_h^{(r+1,m)} \,]^T,$$

are satisfied on E'_h *and*

$$z_h^{(r,m)} \geq 0, \quad u_h^{(r,m)} \geq \theta \quad on \quad E_{h.0}$$

then $z_h^{(r,m)} \geq 0$ *and* $u_h^{(r,m)} \geq \theta$ *on* E_h.

Proof. Consider the case (I). Suppose that $0 \leq r \leq K-1$ and there exists $x^{(\tilde{m})} \in \Delta^{(r+1)}$ such that $z_h^{(r+1,\tilde{m})} = M$, where

$$M = \max\{z_h^{(r+1,m)} : x^{(m)} \in \Delta^{(r+1)}\},$$

and

$$z_h^{(r+1,\tilde{m})} > 0. \tag{36}$$

It follows from the difference inequality that

$$z_h^{(r+1,\tilde{m})} \leq h_0 \sum_{i \in J_+^{(r,m)}[z_h,u_h]} \frac{1}{h_i} \partial_{q_i} f(P^{(r,m)}[z_h,u_h])(z_h^{(r+1,\tilde{m}+e_i)} - z_h^{(r+1,\tilde{m})}).$$
$$+ h_0 \sum_{i \in J_-^{(r,m)}[z_h,u_h]} \frac{1}{h_i} \partial_{q_i} f(P^{(r,m)}[z_h,u_h])(z_h^{(r+1,\tilde{m})} - z_h^{(r+1,\tilde{m}-e_i)}) \leq 0.$$

We thus get $z_h^{(r+1,\tilde{m})} \leq 0$ which contradicts (36). In a similar way we prove that $u_h^{(r,\tilde{m})} \leq \theta$ on E_h. The case (II) can be treated in the same way. This completes the proof.

LEMMA 2.2. *If Assumptions* $H_0[f]$ *and* $H[V, T_h, L_h]$ *are satisfied then there exists exactly one solution* (z_h, u_h), $z_h : E_{h.0} \cup E_h \to \mathbf{R}$, $u_h : E_{h.0} \cup E_h \to \mathbf{R}^n$, *of difference functional problem (23)–(25).*

Proof. Suppose that $0 \leq r \leq K-1$ is fixed and (z_h, u_h) are known on the set $E_{h.r}$. Consider the linear system

$$\begin{aligned} z^{(r+1,m)} = z_h^{(r,m)} &+ h_0 f(P^{(r,m)}[z_h,u_h]) \\ &+ h_0 \partial_q f(P^{(r,m)}[z_h,u_h]) \, [\, \delta z^{(r+1,m)} - u_h^{(r,m)} \,]^T, \end{aligned} \tag{37}$$

$$\begin{aligned} u^{(r+1,m)} = u_h^{(r,m)} &+ h_0 \partial_x f(P^{(r,m)}[z_h,u_h]) \\ &+ h_0 \partial_p f(P^{(r,m)}[z_h,u_h]) \, L_h[z_h,u_h]^{(r,m)} \\ &+ h_0 \partial_q f(P^{(r,m)}[z_h,u_h]) \, [\, \delta u^{(r+1,m)} \,]^T \end{aligned} \tag{38}$$

with unknown functions $z^{(r+1,m)}$, $u^{(r+1,m)}$ where $x^{(m)} \in \Delta^{(r+1)}$. Suppose that $(t^{(r)}, x^{(m)}) \in E'_h$ and $\partial_{q_i} f(P^{(r,m)}[z_h,u_h]) \geq 0$. Then

$$\delta_i z^{(r+1,m)} = \frac{1}{h_i}[z^{(r+1,m+e_i)} - z^{(r+1,m)}]$$

and

$$\delta_i u_j^{(r+1,m)} = \frac{1}{h_i}[u_j^{(r+1,m+e_i)} - u_j^{(r+1,m)}], \quad 1 \le j \le n.$$

It follows from condition 3) of Assumption $H_0[f]$ that $x^{(m+e_i)} \in \Delta^{(r+1)}$ and the difference expressions $\delta_i z^{(r+1,m)}$, $\delta_i u_j^{(r+1,m)}$, $1 \le j \le n$, are well defined. The same conclusion can be drawn for $\partial_{q_i} f(P^{(r,m)}[z_h, u_h]) < 0$.

The homogeneous problem corresponding to linear system (37), (38) has the form

$$z^{(r+1,m)} = h_0 \partial_q f(P^{(r,m)}[z_h, u_h]) \; [\; \delta z^{(r+1,m)} \;]^T,$$
$$u^{(r+1,m)} = h_0 \partial_q f(P^{(r,m)}[z_h, u_h]) \; [\; \delta u^{(r+1,m)} \;]^T.$$

It follows from Lemma 2.1 that the above system has exactly one zero solution. Then system (37), (38) has exactly one solution $z_h^{(r+1,m)}$, $u_h^{(r+1,m)}$, $x^{(m)} \in \Delta^{(r+1)}$, and consequently the functions (z_h, u_h) are defined and they are unique on $E_{h.r+1}$. Since (z_h, u_h) are given on $E_{h.0}$ then the proof is completed by induction.

3. Convergence of the generalized implicit Euler method

Throughout this section we will need the following assumptions on f.

ASSUMPTION $H[\sigma, f]$. Suppose that Assumption $H_0[f]$ is satisfied and there is a function $\sigma : [0,a] \times \mathbf{R}_+ \to \mathbf{R}_+$ such that

1) σ is continuous and it is nondecreasing with respect to both variables,
2) $\sigma(t, 0) = 0$ for $t \in [0, a]$ and for each $c \ge 1$ the maximal solution of the Cauchy problem

$$\eta'(t) = c[\eta(t) + \sigma(t, c\eta(t))], \quad \eta(0) = 0,$$

is $\bar{\eta}(t) = 0$ for $t \in [0, a]$,

3) the terms

$$\|\partial_x f(t,x,p,q) - \partial_x f(t,x,\bar{p},\bar{q})\|, \qquad |\partial_p f(t,x,p,q) - \partial_p f(t,x,\bar{p},\bar{q})|,$$
$$\|\partial_q f(t,x,p,q) - \partial_q f(t,x,\bar{p},\bar{q})\|$$

are bounded from above by $\sigma(t, |p - \bar{p}| + \|q - \bar{q}\|)$,

THEOREM 3.1. *Suppose that Assumptions $H[f, \sigma]$ and $H[V, T_h, L_h]$ are satisfied and*

1) the function $\varphi : E_0 \to \mathbf{R}$ is of class C^2 and $v : E_0 \cup E \to \mathbf{R}$ is the solution of problem (1), (2) and v is of class C^2 on $E_0 \cup E$,

2) $(z_h, u_h) : E_{h.0} \cup E_h \to \mathbf{R}^{1+n}$ is the solution of difference problem (23)–(25) with operators δ_0, δ defined by (26)–(29) and there is $\alpha_0 : H \to \mathbf{R}_+$ such that

(39) $$|\varphi^{(r,m)} - \varphi_h^{(r,m)}| + \|\partial_x\varphi^{(r,m)} - \psi_h^{(r,m)}\| \le \alpha_0(h) \quad on \quad E_{h.0}$$

and $\lim_{h\to 0}\alpha_0(h) = 0$, *where* H *is defined by (21).*

Then there is $\alpha : H \to \mathbf{R}_+$ *such that*

(40) $$\|v_h - z_h\|_{h.r} + \|(\partial_x v)_h - u_h\|_{h.r} \le \alpha(h), \quad 0 \le r \le K,$$

and

$$\lim_{h\to 0}\alpha(h) = 0,$$

where $v_h = v|_{E_{h.0}\cup E_h}$ *and* $(\partial_x v)_h = \partial_x v|_{E_{h.0}\cup E_h}$.

Proof. Write $w = \partial_x v$. Then the functions $(v, w) : E_0 \cup E \to \mathbf{R}^{1+n}$ satisfy quasilinear system (31), (32) and initial condition (33). Let us denote by $\xi_h : E_{h.0} \cup E_h \to \mathbf{R}$ and $\lambda_h : E_{h.0} \cup E_h \to \mathbf{R}^n$ the functions

(41) $$\xi_h^{(r,m)} = v_h^{(r,m)} - z_h^{(r,m)}, \quad \lambda_h^{(r,m)} = w_h^{(r,m)} - u_h^{(r,m)},$$

where

$$v_h = v|_{E_{h.0}\cup E_h}, \quad w_h = w|_{E_{h.0}\cup E_h}.$$

Put

(42) $$\omega_{h.0}^{(r)} = \|\xi_h\|_{h.r}, \quad \omega_{h.1}^{(r)} = \|\lambda_h\|_{h.r}, \quad 0 \le r \le K,$$

and $\omega_h = \omega_{h.0} + \omega_{h.1}$. We will write a difference inequality for the function ω_h.

We first examine $\omega_{h.0}$. Set

$$U^{(r,m)}[v, w] = (t^{(r)}, x^{(m)}, V[v]^{(r,m)}, w^{(r,m)}).$$

Let the functions $\Gamma_{h.0}$, $\Lambda_{h.0} : E_h' \to \mathbf{R}$ be defined by

(43) $$\Gamma_{h.0}^{(r,m)} = \delta_0 v_h^{(r,m)} - \partial_t v^{(r,m)} + \partial_q f(U^{(r,m)}[v, w])\, [\, \partial_x v^{(r,m)} - \delta v_h^{(r+1,m)}\,]^T$$

and

(44) $$\begin{aligned}\Lambda_{h.0}^{(r,m)} &= f(U^{(r,m)}[v, w]) - f(P^{(r,m)}[z_h, u_h])\\ &- \partial_q f(U^{(r,m)}[v, w])\, [\, w^{(r,m)}\,]^T\\ &+ \partial_q f(P^{(r,m)}[z_h, u_h])\, [\, u_h^{(r,m)}\,]^T\\ &+ [\partial_q f(U^{(r,m)}[v, w]) - \partial_q f(P^{(r,m)}[z_h, u_h])]\, [\, \delta v_h^{(r+1,m)}\,]^T.\end{aligned}$$

It follows from (23) and (31) that

(45) $$\delta_0 \xi_h^{(r,m)} = \partial_q f(P^{(r,m)}[z_h, u_h])\, [\, \delta \xi_h^{(r+1,m)}\,]^T + \Gamma_{h.0}^{(r,m)} + \Lambda_{h.0}^{(r,m)},$$

where $(t^{(r)}, x^{(m)}) \in E_h'$. We conclude that relation (45) is equivalent to

$$(46)\quad \xi_h^{(r+1,m)}[1+h_0\sum_{j=1}^{n}\frac{1}{h_j}|\partial_{q_j}f(P^{(r,m)}[z_h,u_h])|]=\xi_h^{(r,m)}$$
$$+h_0\sum_{j\in J_+^{(r,m)}[z_h,u_h]}\frac{1}{h_j}\partial_{q_j}f(P^{(r,m)}[z_h,u_h])\,\xi_h^{(r+1,m+e_j)}$$
$$-h_0\sum_{j\in J_-^{(r,m)}[z_h,u_h]}\frac{1}{h_j}\partial_{q_j}f(P^{(r,m)}[z_h,u_h])\,\xi_h^{(r+1,m-e_j)}+h_0[\Gamma_{h.0}^{(r,m)}+\Lambda_{h.0}^{(r,m)}].$$

It follows easily that there is $\gamma_0 : H\to\mathbf{R}_+$ such that

$$(47)\qquad |\Gamma_{h.0}^{(r,m)}|\le\gamma_0(h)\quad\text{on}\quad E'_h\quad\text{and}\quad\lim_{h\to 0}\gamma_0(h)=0,$$

where E'_h is given by (3). There is $\widetilde{c}\in\mathbf{R}_+$ such that

$$(48)\qquad \|\partial_x v(t,x)\|,\quad \|\partial_{xx}v(t,x)\|,\quad \|\partial_x V[v](t,x)\|\le\widetilde{c}\quad\text{for}\quad (t,x)\in E.$$

It follows from Assumptions $H[f,\sigma]$ and $H[V,T_h,L_h]$ that

$$(49)\qquad |f(U^{(r,m)}[v,w])-f(P^{(r,m)}[z_h,u_h])|\le A[c_0h_0^{\mu}+\bar{d}\omega_h^{(r)}],$$

where $(t^{(r)},x^{(m)})\in E'_h$ and

$$\bar{d}=\max\{1,L\}.$$

In the same way we can see that

$$\|\partial_q f(U^{(r,m)}[v,w])-\partial_q f(P^{(r,m)}[z_h,u_h])\|\le\sigma(t^{(r)},c_0h_0^{\mu}+\bar{d}\omega_h^{(r)}),$$

where $(t^{(r)},x^{(m)})\in E'_h$ and consequently

$$(50)\qquad |\Lambda_{h.0}^{(r,m)}|\le A[c_0h_0^{\mu}+\bar{d}\omega_h^{(r)}]+2\widetilde{c}\sigma(t^{(r)},c_0h_0^{\mu}+\bar{d}\omega_h^{(r)})+A\omega_{h.1}^{(r)}.$$

We see at once that

$$(51)\quad h_0\sum_{j\in J_+^{(r,m)}[z_h,u_h]}\frac{1}{h_j}\partial_{q_j}f(P^{(r,m)}[z_h,u_h])\,|\xi_h^{(r+1,m+e_j)}|$$
$$-h_0\sum_{j\in J_-^{(r,m)}[z_h,u_h]}\frac{1}{h_j}\partial_{q_j}f(P^{(r,m)}[z_h,u_h])\,|\xi_h^{(r+1,m-e_j)}|$$
$$\le h_0\omega_{h.0}^{(r+1)}\sum_{j=1}^{n}\frac{1}{h_j}|\partial_{q_j}f(P^{(r,m)}[z_h,u_h])|.$$

We conclude from (46), (49), (50), (51) that

$$(52)\qquad \omega_{h.0}^{(r+1)}\le\omega_{h.0}^{(r)}+h_0A\omega_{h.1}^{(r)}+h_0\gamma_0(h)+h_0A[c_0h_0^{\mu}+\bar{d}\omega_h^{(r)}]$$
$$+2h_0\widetilde{c}\sigma(t^{(r)},c_0h_0^{\mu}+\bar{d}\omega_h^{(r)}),\quad 0\le r\le K-1.$$

Now we write a difference inequality for $\omega_{h.1}$. Let the functions $\Lambda_h : E'_h \to \mathbf{R}^n$, $\Gamma_h : E'_h \to \mathbf{R}^n$ be defined by

$$\Gamma_h^{(r,m)} = \delta_0 w_h^{(r,m)} - \partial_t w^{(r,m)} + \partial_q f(U^{(r,m)}[v,w]) \, [\, \partial_x w^{(r,m)} - \delta w_h^{(r+1,m)} \,]^T \tag{53}$$

and

$$\begin{aligned}\Lambda_h^{(r,m)} &= \partial_x f(U^{(r,m)}[v,w]) - \partial_x f(P^{(r,m)}[z_h,u_h]) \\ &\quad + \partial_p f(U^{(r,m)}[v,w]) \partial_x V[v]^{(r,m)} \\ &\quad - \partial_p f(P^{(r,m)}[z_h,u_h]) \, L_h[z_h,u_h]^{(r,m)} \\ &\quad + [\partial_q f(U^{(r,m)}[v,w]) - \partial_q f(P^{(r,m)}[z_h,u_h])] \, [\, \delta w_h^{(r+1,m)} \,]^T.\end{aligned} \tag{54}$$

Then the function λ_h satisfies the difference equation

$$\delta_0 \lambda_h^{(r,m)} = \partial_q f(P^{(r,m)}[z_h,u_h]) \, [\, \delta \lambda_h^{(r+1,m)} \,]^T + \Gamma_h^{(r,m)} + \Lambda_h^{(r,m)}, \quad (t^{(r)}, x^{(m)}) \in E'_h.$$

It follows from the definition of difference operators $(\delta_1, \ldots, \delta_n)$ that the above relation is equivalent to

$$\begin{aligned}\lambda_h^{(r+1,m)}[1 + h_0 \sum_{j=1}^{n} \frac{1}{h_j} |\partial_{q_j} f(P^{(r,m)}[z_h,u_h])|] &= \lambda_h^{(r,m)} \\ + h_0 \sum_{j \in J_+^{(r,m)}[z_h,u_h]} &\frac{1}{h_j} \partial_{q_j} f(P^{(r,m)}[z_h,u_h]) \, \lambda_h^{(r+1,m+e_j)} \\ - h_0 \sum_{j \in J_-^{(r,m)}[z_h,u_h]} \frac{1}{h_j} \partial_{q_j} f(P^{(r,m)}[z_h,u_h]) \, &\lambda_h^{(r+1,m-e_j)} + h_0[\Gamma_h^{(r,m)} + \Lambda_h^{(r,m)}],\end{aligned} \tag{55}$$

where $(t^{(r)}, x^{(m)}) \in E'_h$. There is $\gamma : H \to \mathbf{R}_+$ such that

$$\|\Gamma_h^{(r,m)}\| \le \gamma(h) \quad \text{on} \quad E'_h \quad \text{and} \quad \lim_{h \to 0} \gamma(h) = 0. \tag{56}$$

It follows form Assumptions $H[f]$ and $H[V, T_h, L_h]$ that

$$\|\partial_x f(U^{(r,m)}[v,w]) - \partial_x f(P^{(r,m)}[z_h,u_h])\| \le \sigma(t^{(r)}, c_0 h_0^{\mu} + \bar{d}\omega_h^{(r)})$$

and the same estimate we obtain for the derivatives $\partial_p f$ and $\partial_q f$. It follows from Assumption $H[V, T_h, L_h]$ that

$$\|\partial_x V[v]^{(r,m)} - L_h[z_h,u_h]^{(r,m)}\| \le c_1 h_0^{\nu} + L_0 \omega_h^{(r)},$$

where $(t^{(r)}, x^{(m)}) \in E'_h$. We thus get

$$\|\Lambda_h^{(r,m)}\| \le (1 + 2\tilde{c})\sigma(t^{(r)}, c_0 h_0^{\mu} + \bar{d}\omega_h^{(r)}) + A[c_1 h_0^{\nu} + L_0 \omega_h^{(r)}], \tag{57}$$

where $(t^{(r)}, x^{(m)}) \in E'_h$. It is easily seen that

$$(58)\quad h_0 \sum_{j\in J_+[r,m]} \frac{1}{h_j}\partial_{q_j} f(P^{(r,m)}[z_h,u_h])\|\lambda_h^{(r+1,m+e_j)}\| - h_0 \sum_{j\in J_-[r,m]} \frac{1}{h_j}\partial_{q_j} f(P^{(r,m)}[z_h,u_h])\|\lambda_h^{(r+1,m-e_j)}\| \le h_0\omega_{h.1}^{(r+1)} \sum_{j=1}^{n} \frac{1}{h_j}|\partial_{q_j} f(P^{(r,m)}[z_h,u_h])|,$$

where $(t^{(r)}, x^{(m)}) \in E'_h$. We conclude from (55), (57), (58) that

$$(59)\quad \omega_{h.1}^{(r+1,m)} \le \omega_{h.1}^{(r)} + h_0(1+2\tilde{c})\sigma(c_0h_0^\mu + \bar{d}\omega_{h.1}^{(r)}) + h_0A[c_1h_0^\nu + L_0\omega_h^{(r)}] + h_0\gamma(h), \quad 0\le r\le K-1,$$

where $(t^{(r)}, x^{(m)}) \in E'_h$. Adding inequalities (52), (59) we get

$$(60)\quad \omega_h^{(r+1)} \le \omega_h^{(r)} + h_0\tilde{a}\sigma(t^{(r)}, c_0h_0^\mu + \bar{d}\omega_h^{(r)}) + h_0\tilde{d}\omega_h^{(r)} + h_0\tilde{\gamma}(h), \quad 0\le r\le K-1,$$

where

$$\tilde{a} = 1 + 4\tilde{c}, \quad \tilde{d} = A(1+\bar{d}+L_0),$$
$$\tilde{\gamma}(h) = \gamma_0(h) + \gamma(h) + A(c_0h_0^\mu + c_1h_0^\nu).$$

Consider the Cauchy problem

$$(61)\quad \omega'(t) = \tilde{d}\omega(t) + \tilde{a}\sigma(t, c_0h_0^\mu + \bar{d}\omega(t)) + \tilde{\gamma}(h),$$

$$(62)\quad \omega(0) = \alpha_0(h).$$

It follows from Assumption $H[f,\sigma]$ that there is $\varepsilon_0 > 0$ such that for $\|h\| < \varepsilon$ there exists the maximal solution $\eta_h : [0,a] \to \mathbf{R}_+$ of (61), (62) and

$$\lim_{h\to 0} \eta_h(t) = 0 \quad \text{uniformly on} \quad [0,a].$$

The function η_h satisfies the recurrent inequality

$$\eta_h^{(r+1)} \ge \eta_h^{(r)} + h_0\tilde{d}\eta_h^{(r)} + h_0\tilde{a}\sigma(t^{(r)}, c_0h_0^\mu + \bar{d}\eta_h^{(r)}) + h_0\tilde{\gamma},$$

where $0 \le r \le K-1$. Since $\omega_h^{(r)} \le \eta_h^{(r)}$, by the above inequality and (60) we have

$$\omega_h^{(r)} \le \eta_h^{(r)} \quad \text{for} \quad 0\le r\le K.$$

Then we obtain estimate (40) for $\alpha(h) = \eta_h(a)$. This proves the theorem.

REMARK 3.1. Suppose that all the assumptions of Theorem 3.1 are satisfied with

$$\sigma(t,p) = \bar{M}p, \quad (t,p) \in [0,a]\times\mathbf{R}_+,$$

where $M \in \mathbf{R}_+$. Then we have assumed that the functions $\partial_x f$, $\partial_p f$, $\partial_q f$ satisfy the Lipschitz condition with respect to (p, q) and we have the estimates

$$\|v_h - z_h\|_{h.r} + \|\partial_x v_h - u_h\|_{h.r} \le \tilde{\alpha}(h), \quad 0 \le r \le K,$$

where

$$\tilde{\alpha}(h) = \alpha_0(h) e^{\bar{L}a} + \bar{\gamma}(h) \frac{e^{\bar{L}a} - 1}{\bar{L}} \quad \text{if} \quad \bar{L} > 0,$$
$$\tilde{\alpha}(h) = \alpha_0(h) + a\bar{\gamma}(h)$$

and

$$\bar{L} = \tilde{d} + \tilde{a}\bar{d}\bar{M}, \quad \bar{\gamma}(h) = \bar{a}h_0^{\mu} + c_1 h_0^{\nu} + \tilde{b},$$
$$\bar{a} = \tilde{a}c_0\bar{M} + Ac_0, \quad \tilde{b} = \tilde{C}(1 + \|M\|).$$

The above estimates are obtained by solving problem (61), (62).

REMARK 3.2. In our considerations we need estimates for the partial derivatives of the solution v of problem (1), (2). One may obtain them by the method of differential inequalities, see [9] vol. II (Th. 9.2.1 p.120, Th. 9.2.2 p. 123) and [11] (Th. 37.1 p.113).

4. Examples of interpolating operators

In this section we assume that $\hat{h} = Mh_0$. Then we can write the definitions of $E_{0.h}$ and E_h in the following way:

$$E_{0.h} = \{ (t^{(r)}, x^{(m)}) : -N_0 \le r \le 0, \ -N \le m \le N \},$$
$$E_h = \{ (t^{(r)}, x^{(m)}) : 0 \le r \le K, \ |m_i| \le N_i - i, \ i = 1, \dots, n \}.$$

Put $B = [-b, b]$ and $B_h = \{ x^{(m)} : -N \le m \le N \}$. For a function $w : B_h \to \mathbf{R}$ and for a point $x^{(m)} \in B_{h'}$ we write $w^{(m)} = w(x^{(m)})$. Set

$$S_+ = \{ s = (s_1, \dots, s_n) : s_i \in \{0, 1\} \text{ for } 1 \le i \le n \}.$$

We first consider the operator $Q_h : F(B_{\hat{h}}, \mathbf{R}) \to F(B, \mathbf{R})$ as follows. Let $w \in F(B_{\hat{h}}, \mathbf{R})$ and $x \in B$. There exists $m \in \mathbf{Z}^n$ such that $x^{(m)} \le x \le x^{(m+1)}$ and $x^{(m)}, x^{(m+1)} \in B_{\hat{h}}$, where $m + 1 = (m_1 + 1, \dots, m_n + 1)$. We define

$$Q_h[w](x) = \sum_{s \in S_+} w^{(m+s)} \left(\frac{x - x^{(m)}}{\hat{h}} \right)^s \left(1 - \frac{x - x^{m)}}{\hat{h}} \right)^{1-s},$$

where

$$\left(\frac{x - x^{(m)}}{\hat{h}} \right)^s = \prod_{i=1}^{n} \left(\frac{x_i - x_i^{(m_i)}}{h_i} \right)^{s_i},$$

$$\left(1-\frac{x-x^{(m)}}{\hat{h}}\right)^{1-s}=\prod_{i=1}^{n}\left(1-\frac{x_i-x_i^{(m_i)}}{h_i}\right)^{1-s_i},$$

and we take $0^0 = 1$ in the above formulas. If $[a_0, b_0] \subset \mathbf{R}$ and $w : [a_0, b_0] \times B_{\hat{h}} \to \mathbf{R}$ then we write

$$Q_h[w](t,x)=\sum_{s\in S_+} w(t,x^{(m+s)})\left(\frac{x-x^{(m)}}{\hat{h}}\right)^{s}\left(1-\frac{x-x^{m)}}{\hat{h}}\right)^{1-s},$$

where $x \in B$, $x^{(m)} \le x \le x^{(m+1)}$ and $x^{(m)}, x^{(m+1)} \in B_h$.

We define now the interpolating operator $U_h : F(E_{0.h} \cup E_h, \mathbf{R}) \to F(E_0 \cup E, \mathbf{R})$ in the following way. Suppose that $w \in F(E_{0.h} \cup E_h, \mathbf{R})$ and $(t,x) \in E_0 \cup E$, $-b_0 \le t \le Kh_0$. Two cases will be distinguished.

I. Suppose that $(t,x) \in E_0 \cup E$ and there is $(r,m) \in \mathbf{Z}^{1+n}$ such that $[t^{(r)}, t^{(r+1)}] \times [x^{(m)}, x^{(m+1)}] \subset E_{0.h} \cup E_h$. We define

$$U_h[w](t,x)=\frac{t-t^{(r)}}{h_0}Q_h[w](t^{(r+1)},x)+\left(1-\frac{t-t^{(r)}}{h_0}\right)Q_h[w](t^{(r)},x). \tag{63}$$

II. Suppose that $(t,x) \in E$ and there is $(r,m) \in \mathbf{Z}^{1+n}$ such that

(i) $t^{(r)} \le t < t^{(r+1)}$ and $x^{(m)} \le x \le x^{(m+1)}$,

(ii) $(t^{(r)}, x^{(m)}), (t^{(r)}, x^{(m+1)}) \in E$ and $(t^{(r)}, x^{(m)}) \in \partial_0 E$ or $(t^{(r)}, x^{(m+1)}) \in \partial_0 E$.

Define the sets of integers $I_+[r,m]$, $I_-[r,m]$, $I_0[r,m]$ (possibly empty) as follows

$$I_+[r,m]=\{i : 1\le i\le n,\ x_i^{(m_i+1)}=b_i-M_i t^{(r)}\},$$
$$I_-[r,m]=\{i : 1\le i\le n,\ x_i^{(m_i)}=-b_i+M_i t^{(r)}\},$$
$$I_0[r,m]=\{1,\dots,n\}\setminus(I_+[r,m]\cup I_-[r,m]).$$

Write $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)$ and $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n)$, where

$$\bar{x}_i=x_i^{(m_i)}+\frac{h_0}{t^{(r+1)}-t}(x_i-x_i^{(m_i)}) \text{ and } \tilde{x}_i=x_i^{(m_i)} \text{ for } i\in I_+[r,m],$$

$$\bar{x}_i=x_i^{(m_i+1)}+\frac{h_0}{t^{(r+1)}-t}(x_i-x_i^{(m_i+1)}) \text{ and } \tilde{x}_i=x_i^{(m_i+1)} \text{ for } i\in I_-[r,m]$$

and $\bar{x}_i = \tilde{x}_i = x_i$ for $i \in I_0[r,m]$. Therefore we define $U_h[w](t,x)$ by the formula

$$U_h[w](t,x)=\frac{t-t^{(r)}}{h_0}Q_h[w](t^{(r+1)},\tilde{x})+\left(1-\frac{t-t^{(r)}}{h_0}\right)Q_h[w](t^{(r)},\bar{x}). \tag{64}$$

If $(t,x) \in E_0 \cup E$ and $Kh_0 < t \le a$ then we put $U_h[w](t,x) = U_h[w](Kh_0, x)$. Then we have defined $U_h[w] : E_0 \cup E \to \mathbf{R}$. It it easy to see that $U_h[w] \in \mathbf{C}(E_0 \cup E, \mathbf{R})$.

The above interpolating operator was first introduced in [4], Chapter 3. The following properties of U_h are important in our considerations.

LEMMA 4.1. *Suppose that the function* $w \in C^2(E_0 \cup E, \mathbf{R})$ *and*

$$|\partial_{tt} w(t,x)|,\ |\partial_{tx_i} w(t,x)|,\ |\partial_{x_i x_j} w(t,x)| \le \tilde{C},\ i,j = 1,\dots,n,$$

where $(t,x) \in E_0 \cup E$. *Let us denote by* w_h *the restriction of* w *to the set* $E_{0.h} \cup E_h$.

Then

$$|U_h[w_h](t,x) - w(t,x)| \le C h_0^2,$$

where $(t,x) \in E_0 \cup E$, $t \le K h_0$, *and* $C = \frac{1}{2}\tilde{C}(1 + \|M\|)^2$.

The above lemma is a consequence of Theorem 3.18 in [4], Chapter 3.

LEMMA 4.2. *Suppose that the function* $w \in C^1(E_0 \cup E, \mathbf{R})$ *and*

$$|\partial_t w(t,x)|,\ |\partial_{x_i} w(t,x)| \le \bar{C},\ i = 1,\dots,n,$$

where $(t,x) \in E_0 \cup E$. *Let us denote by* w_h *the restriction of* w *to the set* $E_{0.h} \cup E_h$. *Then*

$$|U_h[w_h](t,x) - w(t,x)| \le C^\star h_0, \tag{65}$$

where $(t,x) \in E_0 \cup E$, $t \le K h_0$, *and* $C^\star = \bar{C}(1 + \|M\|)$.

Proof. It is easy to show by induction with respect to n that for $x^{(m)} \le x \le x^{(m+1)}$ we have

$$\sum_{s \in S_+} \left(\frac{x - x^{(m)}}{\hat{h}} \right)^s \left(1 - \frac{x - x^{m)}}{\hat{h}} \right)^{1-s} = 1. \tag{66}$$

Write

$$A[s,P] = \sum_{i=1}^{n} \partial_{x_i} w(P)(x_i^{(m_i + s_i)} - x_i),$$

where $s \in S_+$, $P \in E_0 \cup E$.

Suppose that $(t,x) \in E_0 \cup E$ and $t^{(r)} \le t \le t^{(r+1)}$, $x^{(m)} \le x \le x^{(m+1)}$ and $[t^{(r)}, t^{(r+1)}] \times [x^{(m)}, x^{(m+1)}] \subset E_0 \cup E$. Then $U_h[w_h]$ is defined by (63) and there are $P, Q \in E_0 \cup E$ such that

$$U_h[w_h](t,x) - w(t,x) = \frac{t - t^{(r)}}{h_0} \Big\{ \sum_{s \in S_+} [w(t,x) + \partial_t w(P)(t^{(r+1)} - t)$$

$$+ A[s,P]]\Big(\frac{x-x^{(m)}}{\hat{h}}\Big)^{s}\Big(1-\frac{x-x^{m)}}{\hat{h}}\Big)^{1-s} - w(t,x)\Big\}$$
$$+ \Big(1-\frac{t-t^{(r)}}{h_0}\Big)\Big\{\sum_{s\in S_+}[w(t,x)+\partial_t w(Q)(t^{(r)}-t)$$
$$+ A[s,Q]]\Big(\frac{x-x^{(m)}}{\hat{h}}\Big)^{s}\Big(1-\frac{x-x^{m)}}{\hat{h}}\Big)^{1-s} - w(t,x)\Big\}.$$

The above relation and (66) imply (65).

In a similar way we prove (65) in the case when $U_h[w_h](t,x)$ is defined by (64). This proves the lemma.

Now we give examples of the operators V and the corresponding interpolating operators T_h and L_h.

EXAMPLE 4.1. Suppose that the functions

$$\psi_0 : [0,a] \to \mathbf{R}, \quad \psi = (\psi_1,\dots,\psi_n) : E \to \mathbf{R}^n$$

are given. We assume that

1) $\psi_0 \in C([0,a],\mathbf{R})$ and $-b_0 \le \psi_0(t) \le t$ for $t \in [0,a]$,
2) $\psi \in C(E,\mathbf{R}^n)$ and the partial derivatives

$$[\partial_{x_j}\psi_i(t,x)]_{i,j=1,\dots,n} = \partial_x\psi(t,x)$$

exist on E and $\partial_x\psi \in C(E,M_{n\times n})$,
3) $(\psi_0(t),\psi(t,x)) \in E_0 \cup E$ for $(t,x)\in E$.

Let the operator $V : C(E_0\cup E,\mathbf{R}) \to C(E,\mathbf{R})$ be given by $V[z](t,x) = z(\psi_0(t),\psi(t,x))$. Then V satisfies the Volterra condition and equation (1) is equivalent to the equation with deviated variables

$$\partial_t z(t,x) f(t,x,z(\psi_0(t),\psi(t,x)),\partial_x z(t,x)).$$

Let the operators $T_h : F(E_{h.0}\cup E_h,\mathbf{R}) \to C(E,\mathbf{R})$, $L_h : F(E_{h.0}\cup E_h,\mathbf{R}^n) \to C(E,\mathbf{R}^n)$ be defined by

$$T_h[z](t,x) = U_h[z](\psi_0(t),\psi(t,x)), \quad (t,x)\in E,$$
$$L_h[z,u] = (L_{h.1}[z,u],\dots,L_{h.n}[z,u]),$$
$$L_{h.j}[z,u](t,x) = \sum_{i=1}^{n} U_h[u_i](\psi_0(t),\varphi(t,x))\partial_{x_j}\psi_j(t,x), \quad (t,x)\in E.$$

Note that L_h does not depend on the function z in our example.

It follows from Lemma 4.1 and 4.2 that Assumption $H[V,T_h,L_h]$ is satisfied with $\mu = 2$, $\nu = 1$, and

$$L_0 = \max\{\|\partial_x\psi(t,x)\| : (t,x)\in E\},$$
$$c_0 = \frac{1}{2}\widetilde{C}(1+\|M\|), \quad c_1 = \tilde{c}L_0(1+\|M\|),$$

where $\widetilde{C} \in \mathbf{R}_+$ is a constant such that estimates

$$|\partial_{tt}v(t,x)|, \quad |\partial_{tx_i}v(t,x)|, \quad |\partial_{x_ix_j}v(t,x)| \leq \widetilde{C} \tag{67}$$

are satisfied on $E_0 \cup E$.

Now we consider differential integral equations. Suppose that $\kappa \in N$, $1 \leq \kappa \leq n$, is fixed for each $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ we write $x = (x', x'')$ where $x' = (x_1, \ldots, x_\kappa)$, $x'' = (x_{\kappa+1}, \ldots, n)$. We have $x' = x$ if $\kappa = n$.

EXAMPLE 4.2. Consider the operator $V : C(E_0 \cup E, \mathbf{R}) \to C(E, \mathbf{R})$ given by

$$V[z](t,x) = \int_{-b_0}^{t} \int_{-x'}^{x'} z(\tau, s', x'')ds'd\tau,$$

where $s' = (s_1, \ldots, s_\kappa)$. Then (1) is equivalent to the differential integral equation

$$\partial_t z(t,x) = f(t, x, \int_{-b_0}^{t} \int_{-x'}^{x'} z(\tau, s', x'')ds'd\tau, \partial_x z(t,x)).$$

Put

$$T_h[z](t,x) = \int_{-b_0}^{t} \int_{-x'}^{x'} U_h[z](\tau, s', x'')ds'd\tau, \quad (t,x) \in E,$$

where $z \in F(E_{h.0} \cup E, \mathbf{R})$. Then the Lipschitz condition (17) is satisfied with

$$L = (a + b_0)2^\kappa \prod_{i=1}^{\kappa} b_i.$$

Note that the numbers $T_h[z]^{(r,m)}$ may be calculated by using the results presented in [4], Chapter 5. If $v \in C^2(E_0 \cup E, \mathbf{R})$ then estimate (18) holds with $\mu = 2$ and

$$c_0 = \frac{1}{2}\widetilde{C}(1 + \|M\|)2^\kappa(a + b_0) \prod_{i=1}^{\kappa} b_i,$$

where $\widetilde{C}$ is given by (67). Write

$$x'[j] = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_\kappa), \quad s'[j] = (s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_\kappa),$$
$$s'[j, \tau] = (s_1, \ldots, s_{j-1}, \tau, s_{j+1}, \ldots, s_\kappa),$$

where $1 \leq j \leq \kappa$. Put

$$I_j[z](t,x) = \int_{-b_0}^{t} \int_{-x'[j]}^{x'[j]} z(\tau, s'[j, x_j], x'')ds'[j]d\tau.$$

Consider the operator $L_h[z,u] = (L_{h.1}[z,u],\dots,L_{h.n}[z,u])$ given by

$L_{h.j}[z,u](t,x) = I_j[U_h[z]](t,x) - I_j[U_h[z]](t,x_1,\dots,x_{j-1},-x_j,x_{j+1},\dots,x_n)$

for $1 \le j \le \kappa$ and

$$L_{h.j}[z,u] = \int_{-b_0}^{t}\int_{-x'}^{x'} U_h[u_{h.j}](\tau,s',x'')ds'd\tau \quad \text{for} \quad \kappa+1 \le j \le n.$$

Then we have

$$\|L_h[z,u] - L_h[\bar{z},\bar{u}]\|_{t^{(r)}} \le \widetilde{L}[\|z-\bar{z}\|_{h.r} + \|u-\bar{u}\|_{h.r}]$$

with

$$\widetilde{L} = 2^{\kappa}(a+b_0)\widetilde{D}\prod_{i=1}^{\kappa} b_i, \quad \widetilde{D} = \max\{1, \sum_{i=1}^{\kappa}(b_i)^{-1}\}.$$

If function $v \in C^2(E_0 \cup E, \mathbf{R})$ then

$$\|L_h[v_h,(\partial_x v)_h] - \partial_x V[v]\|_{t^{(r)}} \le c_1 h_0, \quad 0 \le r \le K,$$

where

$$c_1 = [1 + (a+b_0)(n-\kappa+\sum_{i=1}^{n}(b_i)^{-1}]\ (1+\|M\|)\ \widetilde{C}2^{\kappa}\prod_{i=1}^{\kappa} b_i$$

and $\widetilde{C}$ is given by (67). Thus we see that Assumption $H[V,T_h,L_h]$ is satisfied.

5. Numerical example

For $n=1$ we put

$$E = [0,0.5]\times[-1,1], \ \ E_0 = \{0\}\times[-1,1],$$

Consider the differential equation with deviated variable

$$\text{(68)} \quad \partial_t z(t,x) = -x\partial_x z(t,x) + \sin(x\partial_x z(t,x)) + z(t,0.5x) + z(t,-0.5x)$$
$$- \sin(xtz(t,x)) + x(1+t)e^{tx} - e^{0.5tx} - e^{-0.5tx}$$

with the initial condition

$$\text{(69)} \qquad z(0,x) = 1, \qquad x \in [-1,1].$$

The solution of the above problem is given by

$$\widetilde{z}(t,x) = e^{tx}.$$

Let us denote by z_h the solution of the implicit difference problem corresponding to (68), (69). By $\bar{z}_h$ we denote the solution of the explicit problem

$$\begin{aligned} z^{(r+1,m)} = {} & \frac{1}{2}(z^{(r,m+1)} + z^{(r,m-1)}) - h_0 x^{(m)} \frac{1}{2h_1}(z^{(r,m+1)} - z^{(r,m-1)}) \\ & + h_0 \sin[x^{(m)} \frac{1}{2h_1}(z^{(r,m+1)} \\ & - z^{(r,m-1)})] + h_0(z(t^{(r)}, 0.5x^{(m)}) + z(t^{(r)}, -0.5x^{(m)})) \\ & - h_0(\sin(x^{(m)} t^{(r)} z^{(r,m)}) + x^{(m)}(1 + t^{(r)}) \exp(t^{(r)} x^{(m)}) \\ & - \exp(0.5 t^{(r)} x^{(m)}) - \exp(-0.5 t^{(r)} x^{(m)})). \end{aligned}$$

Stability of the above difference method, obtained from (68) by using the Lax difference schemes, requires from steps of time and spatial variable satisfying the Courant-Friedrichs-Levy condition (11).

We give the following information on errors of the methods. Write

$$\varepsilon_h^{(r)} = \frac{1}{2N+1} \sum_{m=-N}^{N} |z_h^{(r,m)} - \widetilde{z}^{(r,m)}|,$$

$$\widetilde{\varepsilon}_h^{(r)} = \frac{1}{2N+1} \sum_{m=-N}^{N} |\bar{z}_h^{(r,m)} - \widetilde{z}^{(r,m)}|.$$

The numbers $\varepsilon_h^{(r)}$ and $\widetilde{\varepsilon}_h^{(r)}$ are the arithmetical mean of the errors with fixed $t^{(r)}$. The values of the functions ε_h and $\widetilde{\varepsilon}_h$ are listed in the table. We write " $\times$ " for $\widetilde{\varepsilon}_h^{(r)} > 100$.

Table of errors (ε_h, $\widetilde{\varepsilon}_h$)

$h_0 = 0.005, \quad h_1 = 0.001$

	ε_h	$\widetilde{\varepsilon}_h$
$t = 0.75$	0.000884	0.000764
$t = 0.80$	0.000979	0.000833
$t = 0.85$	0.001083	0.064984
$t = 0.90$	0.001194	$\times$
$t = 0.95$	0.001315	$\times$
$t = 1.00$	0.001445	$\times$

Table of errors (ε_h, $\widetilde{\varepsilon}_h$)

$h_0 = 0.01$, $h_1 = 0.001$

	ε_h	$\widetilde{\varepsilon}_h$
$t = 0.75$	0.001807	×
$t = 0.80$	0.002005	×
$t = 0.85$	0.002220	×
$t = 0.90$	0.002452	×
$t = 0.95$	0.002701	×
$t = 1.00$	0.002970	×

The results shown in the table are consistent with our mathematical analysis.

Our experiments have the following property. The explicit difference method for steps $h_0 = 0.005$, $h_1 = 0.001$, which are not satisfy the condition (11), is not stable. The implicit Euler method is stable aside from selection of steps.

References

[1] S. Bal, *Convergence of a difference method for a system of first order partial differential equations of hyperbolic type*, Ann. Polon. Math. 30 (1974), 19–36.

[2] P. Brandi, Z. Kamont and A. Salvadori, *Approximate solutions of mixed problems for first order partial differential functional equations*, Atti. Sem Mat. Fis. Univ. Modena 39 (1991), no. 1, 277–302.

[3] D. Jaruszewska-Walczak and Z. Kamont, *Difference methods for quasilinear hyperbolic differential functional systems on the Haar pyramid*, Bull. Belg. Math. Soc. 10 (2003), no. 2, 267–290.

[4] Z. Kamont, *Hyperbolic Functional Differential Inequalities and Applications*, Kluver Acad. Publ., Dordrecht, Boston, London (1999).

[5] Z. Kamont and K. Prządka, *Difference methods for first order partial differential functional equations with initial - boundary conditions*, Zh. Vychisl. Mat. i Mat. Fiz. 31 (1991), no. 10, 1476–1488.

[6] Z. Kamont and K. Prządka, *Difference methods for nonlinear partial differential equations of the first order*, Ann. Polon. Math. 48 (1988), no. 3, 227–246.

[7] Z. Kowalski, *A difference method for certain hyperbolic systems of nonlinear partial differential equations of the first order*, Ann. Polon. Math. 19 (1967), 313–322.

[8] Z. Kowalski, *On the difference method for certain hyperbolic systems on nonlinear partial differential equations of the first order*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 16 (1968), 297–302.

[9] V. Lakshmikantham and S. Leela, *Differential and integral inequalities: Theory and applications*, Academic Press, New York, London, (1969).
[10] H. Leszczyński, *Convergence results for unbounded solutions of first order nonlinear differential—functional equations*, Ann. Polon. Math. 64 (1996), no. 1, 1–16.
[11] J. Szarski, *Differential inequalities*, PWN, Warsaw (1965), 256 pp.
[12] J. Wu, *Theory and Applications of Partial Functional Differential Equations*, Berlin, Springer (1996).

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