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ADJOINT FUNCTIONS TO BOUNDARY SOLUTIONS OF DIFFERENTIAL INCLUSIONS AND SMOOTHNESS OF BARRIER SOLUTIONS ON SEMIPERMEABLE SURFACES

Abstract. The problem of existence of adjoint functions to boundary solutions is considered – it depends on the geometry of the attained set at the end point. This is applied to prove the smoothness of boundary solutions in the case of strictly convex right-hand side of differential inclusion which in turn permitts to show the smoothness of barrier solutions on semipermeable surfaces.

1. Introduction

This work is a continuation of investigations included in [10] concerning the regularity of solutions of differential inclusions which do not exit so called semipermeable surfaces. Such problems were considered in [2] and [3] where one of main assumptions was the smoothness of $\partial F(x)$, where $F(\cdot)$ is the right-hand side of corresponding differential inclusion. We are interested in the regularity of barrier solutions under weaker assumption of strict convexity of $F(x)$.

Let us fix first the basic notions – we refer to [10] for more details. The solutions of a differential inclusion

$$\dot{x} \in F(x) \tag{1}$$

where $x \in \Omega \subset \mathbb{R}^d$, Ω open, $F(x) \subset \mathbb{R}^d$, are understood in the sense of Carathéodory (absolutely continuous functions $x(\cdot)$ defined on an interval with $\dot{x}(t) \in F(x(t))$ almost everywhere in the domain).

DEFINITION 1.1. We call a multifunction $F : \Omega \rightsquigarrow \mathbb{R}^d$ of Lipschitz type if the sets $F(x) \subset \mathbb{R}^d$ are nonempty, compact, convex and

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(2) $$\exists\, L \geq 0,\, \forall\, x, y \in \Omega:\; F(y) \subset F(x) + L\|x - y\| B_1,$$

where B_1 denotes the closed unit ball in $\mathbb{R}^d$.

Remark that (2) is a usual Lipschitz condition for multifunctions with respect to the Hausdorff metric. We add the compactness and convexity of $F(x)$ as we always require it.

$\mathrm{Sol}_F(x_0, T)$, where $x_0 \in \Omega$ and $T > 0$, is the set of all solutions of (1) defined on $[0, T]$ and satisfying

(3) $$x(0) = x_0.$$

$\mathcal{A}(x_0, t) = \{x(t) : x(\cdot) \in \mathrm{Sol}_F(x_0, t)\}$ is the set of points which can be reached starting from x_0 and using solutions of (1),(3) on $[0, t]$.

A solution $x(\cdot) \in \mathrm{Sol}_F(x_0, T)$ is called boundary if $x(t) \in \partial\mathcal{A}(x_0, t)$ for all $t \in [0, T]$.

Let $K \subset \mathbb{R}^d$. We refer to [1] for the notions of contingent cone $\mathrm{T}_K(x)$, Dubovitski-Miliutin tangent cone $\mathrm{D}_K(x)$, the Clarke's tangent cone and the hypertangent cone $\mathrm{C}_K^\circ(x)$.

DEFINITION 1.2. The boundary ∂M of a closed set $M \subset \mathbb{R}^d$ is semipermeable in an open set U with respect to the differential inclusion (1) if the following conditions hold:

(i) $\forall\, \xi \in M \cap U,\, \exists\, T > 0,\, \exists\, x(\cdot) \in \mathrm{Sol}_F(\xi, T),\, \forall\, t \in [0, T]:\; x(t) \in M,$

(ii) $\forall\, \xi \in M \cap U,\, \exists\, T > 0,\, \forall\, x(\cdot) \in \mathrm{Sol}_{-F}(\xi, T),\, \forall\, t \in [0, T]:\; x(t) \in M.$

For Lipschitz type multifunctions $F(\cdot)$ the third important condition implied by (i) and (ii) is valid for semipermeable surfaces:

(iii) $\forall\, \xi \in \partial M \cap U,\, \exists\, T > 0,\, \forall\, x(\cdot) \in \mathrm{Sol}_F(\xi, T),\, \forall\, t \in [0, T]:\; x(t) \in \widehat{M},$

where $\widehat{M} = \overline{\mathbb{R}^d \setminus M}$.

The properties (i) and (iii) ensure the existence of barrier solutions defined below.

DEFINITION 1.3. We call barrier solution on a semipermeable boundary ∂M every solution of (1) which starts on ∂M and does not quit it.

Barrier solutions start from any point of ∂M but not necessarily every point of ∂M is crossed by some barrier solutions. This is connected with some regularity properties of ∂M.

The C^1 regularity of barrier solutions was already discussed in [2] and [10]. A natural condition which is required is the strict convexity of $F(x)$, examples given in [10] show why this assumption should not be omitted. The general idea is based on the use of an adjoint function to the barrier solutions $x(\cdot)$ i.e. such absolutely continuous map $p : [0, T] \to \mathbb{R}^d$ for which

(4) $$H(x(t), p(t)) = \langle \dot{x}(t), p(t) \rangle \quad \text{a.e. in } [0, T]$$

where $H(x,p) = \max\{\langle v,p\rangle : v \in F(x)\}$. If the sets $F(x)$ are strictly convex then

(5) $$\dot{x}(t) = \operatorname{Arg\,max}_{v\in F(x(t))}\langle v, p(t)\rangle \text{ a.e. in } [0,T]$$

and due to the continuity of $(x,p) \to \operatorname{Arg\,max}_{v\in F(x)}\langle v,p\rangle$ the function $\dot{x}(\cdot)$ is almost everywhere equal to a continuous function. This implies that $\dot{x}(\cdot)$ itself must be continuous as it is a derivative of the integral of continuous function.

In view of the above remarks crucial for getting the desired regularity of barrier solutions is the existence of adjoint functions. In [10] the existence of adjoint functions was obtained for barrier solutions which are also time optimal solutions. In this paper we present two other approaches. One based on parametrization of multivalued maps, the other on application of differential Hamiltonian inclusions.

2. Application of smooth parametrization

The content of this section is based on a classical result – a version of the Maximum Principle of Pontryagin ([8], chapter 4, Theorem 3) which we recall at the beginning.

Consider the differential equation below

(6) $$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0$$

where $f : \Omega \times U \to \mathbb{R}^d$, $\Omega \subset \mathbb{R}^d$, $U \subset \mathbb{R}^m$. Measurable functions $u(\cdot)$ occuring in (6) are usually interpreted as controls. Solutions are again understood in the sense of Carathéodory and $\mathcal{A}^f(T, x_0)$ is the set of points attained by solutions of (6) at time T.

THEOREM 2.1. *Suppose that f is continuous, $\frac{\partial f}{\partial x}$ exists and is also continuous in (x,u). We fix a measurable control function $u(\cdot)$ and let $x(\cdot)$ be the corresponding solution of (6). If $x(T) \in \partial\mathcal{A}^f(T,x_0)$ then there exists a nonzero, absolutely continuous function $p : [0,T] \to \mathbb{R}^d$ which satisfies the equation*

$$-\dot{p} = \frac{\partial f}{\partial x}(x(t), u(t))^* \cdot p$$

(the asterisk above denotes the transposition of a matrix) and for almost all t the condition

$$\langle p(t), f(x(t), u(t))\rangle = \max_{u\in U}\langle p(t), f(x(t), u)\rangle.$$

In order to apply the above theorem to our goals we would need to represent the initial value problem (1), (3) under the form of some control system (6). This can be achieved using so called parametrization of the multifunction $F(\cdot)$ by which we mean a function $f : \Omega \times B_1 \to \mathbb{R}^d$ with

(7) $$f(x, B_1) = F(x) \text{ for all } x \in \Omega$$

and f satisfying the assumptions of Theorem 2.1.

In view of remarks given in the previous section we may formulate now the following theorem which is implied by Theorem 2.1.

THEOREM 2.2. *Let a multifunction* $F : \Omega \rightsquigarrow \mathbb{R}^d$ *has strictly convex values and admit a continuous parametrization* $f : \Omega \times B_1 \to \mathbb{R}^d$ *satisfying (7). If* $\frac{\partial f}{\partial x}$ *exists and is continuous in* (x, u) *then every boundary solution of (1) is of class* C^1.

So, if only we know that such regular parametrization exists then the regularity of boundary solutions, and in consequence of barrier solutions on semipermeable surfaces, is assured.

There exist theorems on parametrization of multifunctions in the literature. One based on projections on convex sets published by I. Ekeland and M. Valadier in [7] and other due to S. Łojasiewicz Jr. using the notion of Steiner points of convex sets (see [1]) with a variant given by A. Ornelas [11].

These mentioned authors prove the regularity of their parametrizations with respect to x – continuity in the case of Ekeland and Valadier, Lipschitz condition in the case of Łojasiewicz and Ornelas. But we need more to apply in Theorem 2.2, which is our goal. The existence (and continuity) of $(\partial f/\partial x)(x, u)$ would be necessary. The problem is whether under some reasonable assumptions on our multifunction F the parametrization described in [7]), [1] or [11] has this property. Unfortunately, this is not the case.

To see why the answer to the above question is negative one has to analyse the way how the parametrizations are constructed. We think that showing their deeper nature in this discussion presents an interest in itself although does not provide a tool to resolve our main problem – that will be done farther.

All this does not mean that Theorem 2.2 is useless. In some situations the right-hand side of differential inclusion (1) may be given directly through a parametrization. In such case this theorem provides the regularity of boundary solutions and barrier solutions on semipermeable surfaces.

2.1. Parametrizations of Ekeland-Valadier and Łojasiewicz

We assume now that F is bounded and let $M > 0$ be such that $F(x) \subset M \cdot B_1$ for $x \in \Omega$.

We start with the description of parametrization defined in [7]. It can be given the following form where the projection on convex sets $F(x)$ is used.

$$f(x, u) = \mathrm{proj}_{F(x)}(Mu) , \quad \text{for } x \in \Omega , \quad u \in B_1 .$$

The condition (7) is satisfied, f is Lipschitzean in u and continuous in x but in general the derivative $\partial f/\partial x$ does not exist apart particular cases like when F does not depend on x or $F(x) = \{\phi(x)\}$ and $\phi(\cdot)$ is a usual, single-valued, differentiable map. This is illustrated by Example 2.1 below which will serve also to discuss the parametrization of Łojasiewicz.

The definition of parametrization of Łojasiewicz requires the notion of Steiner point of a convex set which we recall now. By ω we mean the Lebesgue measure on the sphere S^{d-1} in $\mathbb{R}^d$ and $\sigma(p,K) = \max\{\langle p,u\rangle : u \in K\}$ is the support function of a convex set $K \subset \mathbb{R}^d$. Then the Steiner point of K is defined by the integral

$$s(K) = \frac{1}{\operatorname{vol}(B_1)} \int_{S^{d-1}} p\,\sigma(p,K)\,d\omega(p).$$

An auxiliary multifunction $\Phi : \Omega \times B_1 \to \operatorname{Conv}(\mathbb{R}^d)$ is defined (a slightly simplified version in comparison to [1])

$$\Phi(x,u) = B(Mu, 2\operatorname{dist}(Mu, F(x)) \cap F(x).$$

Remark that when $Mu \in F(x)$ then $\Phi(x,u)$ is reduced to $\{Mu\}$ and in the opposite case it is a piece of $F(x)$ contained in the ball centered at Mu.

The parametrization is now defined as the Steiner point

$$f(x,u) = s(\Phi(x,u)).$$

Φ is Lipschitz in x (Lemma 9.4.2 in [1]). The Steiner point $s(K)$ is also Lipschitzean in K with respect to the Hausdorff metric so f is Lipschitzean in x. It is easy to see that f is also Lipschitzean in u.

Unfortunately, when the interiors of $F(x)$ are not empty and F is not constant then for none of those parametrizations $\partial f/\partial x$ exists – the reason for this can be seen in the following example.

EXAMPLE 2.1. Let $\Omega = (0,2)$, $F(x) = [x,3]$ and denote by f_E, f_L respectively the parametrizations of Ekeland-Valadier and Łojasiewicz. Then

$$f_E(x,1) = \begin{cases} 1 & \text{for } x \in (0,1] \\ x & \text{for } x \in (1,2), \end{cases}$$

$$f_L(x,1) = \begin{cases} 1 & \text{for } x \in (0,1] \\ \frac{3}{2}x - \frac{1}{2} & \text{for } x \in (1,2). \end{cases}$$

We see that although the behaviour of $F(x)$ with respect to x is 'smooth' both parametrizations are not smooth.

3. Existence of adjoint functions in the case of nonsmooth differential inclusions

In this section we prove some results on existence of adjoint functions to boundary solutions – and thus also on regularity of barrier solutions – in the case when a smooth parametrization of the right-hand side of the inclusion (1) may not exist so in a sense this right-hand side itself is not smooth. It is natural to expect such kind of result having in mind that for an ordinary differential equation $\dot{x} = g(x)$ if only g is continuous (and need not be smooth) then every Carathéodory solution is of class C^1.

We shall apply a theorem which makes use of Clarke's generalized gradient of locally Lipschitzean functions – we recall now its definition (see [5] for details). Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be such function, $x_0 \in \mathbb{R}^d$ and take any $v \in \mathbb{R}^d$. The generalized directional derivative is defined by

$$\varphi^\circ(x_0; v) = \limsup_{x \to x_0,\, h \to 0+} \frac{\varphi(x + hv) - \varphi(x)}{h}$$

and the Clarke's gradient of φ at x_0

$$\partial^C \varphi(x_0) = \{p \in \mathbb{R}^d : \langle p, v \rangle \leq \varphi^\circ(x_0; v) \text{ for all } v \in \mathbb{R}^d\}.$$

THEOREM 3.1 ([5], Thm. 9.1). *Let $x(\cdot)$ be a solution of the minimization problem*

$$\min\{l(x(T)) : \dot{x}(t) \in F(x(t)) \text{ a.e. in } [0,T],\ x(0) = x_0\} \tag{8}$$

where $l : \Omega \to \mathbb{R}$ is a Lipschitzean function. Then there exists an absolutely continuous function $p : [0,T] \to \mathbb{R}^d$ such that $-p(T) \in \partial^C l(x(T))$ and the pair $(x(\cdot), p(\cdot))$ satisfies the differential inclusion

$$(-\dot{p}(t), \dot{x}(t)) \in \partial^C H(x(t), p(t)). \tag{9}$$

We shall use the notion of a proximal normal to a set K at a point $y \in \partial K$. This is every vector v for which there exists $\alpha > 0$ such that

$$K \cap \bar{B}(y + \alpha v, \alpha\|v\|) = \{y\}.$$

The set of all proximal normals at a given point is a cone – it may reduce to the origin 0. It is known that the set of points at which nonzero proximal normals exist is dense in the boundary ∂K (see, for example, [5], Thm.3.1).

We consider the differential inclusion (1) and corresponding attained set $\mathcal{A}(x_0, T)$ for some fixed $x_0 \in \mathbb{R}^d$ and $T > 0$.

THEOREM 3.2. *Let $F : \Omega \rightsquigarrow \mathbb{R}^d$ be a Lipschitz type multifunction and $y \in \partial\mathcal{A}(x_0, T)$. For at least one $x(\cdot) \in \mathrm{Sol}(x_0, T)$, with $x(T) = y$, exists an adjoint function $p : [0,T] \to \mathbb{R}^d$ such that $p(t) \neq 0$ for all $t \in [0,T]$. If in addition at y there is a nonzero proximal normal to $\mathcal{A}(x_0, T)$ then to every such solution $x(\cdot)$ an adjoint function exists.*

Proof. We consider first the case when a nonzero proximal normal v to $\mathcal{A}(x_0, T)$ at y exists and we fix an $\alpha > 0$ for which $\mathcal{A}(x_0, T) \cap \bar{B}(y + \alpha v, \alpha\|v\|) = \{y\}$. Every solution $x(\cdot)$ of the initial value problem (1), (3) for which $x(T) = y$ is a solution of the optimisation problem (8) for $l(u) = \|u - (y + \alpha v)\|$.

We apply now Theorem 3.1. Remark first that $\partial^C l(y) = -v/\|v\|$ so there exists an absolutely continuous function $p : [0,T] \to \mathbb{R}^d$ such that $p(T) = v/\|v\|$ and the pair $(x(\cdot), p(\cdot))$ satisfies the differential inclusion

$$(10) \qquad (-\dot{p}(t), \dot{x}(t)) \in \partial^C H(x(t), p(t)).$$

Due to the Lipschitz condition imposed on F it is true that for every $(q, v) \in \partial^C H(x, p)$ the inequality $\|q\| \le L\|p\|$ holds, where L is the Lipschitz constant. Comparing with (10) we see that

$$\|\dot{p}(t)\| \le L\|p(t)\| \text{ a.e. in } [0,T].$$

As $p(T) \neq 0$ so $p(t) \neq 0$ for all $t \in [0,T]$ – if there was t_0 with $p(t_0) = 0$ then the Gronwall's lemma would imply $p(t) \equiv 0$.

We apply Proposition 3.2.4 in [4] to deduce from (10) that $\dot{x}(t) \in \partial_p^C H(x(t), p(t))$ a.e. in $[0,T]$ – the generalized gradient is here taken with respect to the variable p. In view of convexity of function $H(x,p)$ with respect to p the Clarke's gradient $\partial_p^C H$ coincides with the usual generalized gradient $\partial H/\partial p$ ([4], Proposition 2.2.7). As $(\partial H/\partial p)(x,p) = \{v \in F(x) : \langle p, v\rangle = H(x,p)\}$ so we finally get (4).

We consider now the situation when at a point $y \in \partial\mathcal{A}(x_0, T)$ there is no nonzero proximal normal to $\mathcal{A}(x_0, T)$. As the set of points at which a proximal normal exists is dense in $\partial\mathcal{A}(x_0, T)$ so there is a sequence y_n of such points converging to y. We may also fix a sequence $x_n(\cdot)$ of solutions of (1), (3) with $x_n(T) = y_n$. According to the first part of the proof we can also fix a sequence of their adjoint functions $p_n(\cdot)$ such that

$$(-\dot{p}_n(t), \dot{x}_n(t)) \in \partial^C H(x_n(t), p_n(t)).$$

The sets $\partial^C H(x,p)$ are convex and bounded and the map $(x,p) \to \partial^C H(x,p)$ is upper semicontinuous ([5], Chapter 2, Prop. 1.5). Using a standard procedure (see, for example, [1], Chapter 7.2) we get a subsequence $(x_{n_k}(\cdot), p_{n_k}(\cdot))$ uniformy convergent to a solution $(x_0(\cdot), p_0(\cdot))$ of (10), $x_0(0) = x_0$, $x_0(T) = y$ and $\|p_0(T)\| = 1$.

In the same way as in the first part of the proof one may show that $p_0(t) \neq 0$ for all $t \in [0,T]$ and $(x_0(\cdot), p_0(\cdot))$ satisfies the condition (4).

According to the discussion at the end of Section 1 we get the following property of smoothness of boundary solutions.

COROLLARY 3.1. *Let $F : \Omega \rightsquigarrow \mathbb{R}^d$ be a Lipschitz type multifunction and the sets $F(x)$ strictly convex for all x. For every point $y \in \partial\mathcal{A}(x_0, T)$ exists at least one solution $x(\cdot)$ of (1), (3) satisfying $x(T) = y$ which is of class C^1. If at $y \in \partial\mathcal{A}(x_0, T)$ exists a nonzero proximal normal to $\mathcal{A}(x_0, T)$ then every such solution is of class C^1.*

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