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OSCILLATION THEOREMS FOR A CLASS OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATION WITH PERTURBATION

Abstract. In this paper we discuss the oscillatory and asymptotic behavior of a second order nonlinear differential equation with perturbation and establish two theorems which develop and generalize some known results.

1. Introduction

In the past few years, the oscillation problem for the following second order nonlinear differential equation with damping

$$(E_1)\qquad (a(t)\psi(x(t))x'(t))' + p(t)x'(t) + q(t)f(x(t)) = 0, \quad ' = \frac{d}{dt}$$

has been studied [1,2], and the oscillation of the following second order nonlinear differential equations

$$(E_2)\qquad (a(t)\psi(x(t))x'(t))' + q(t)f(x(g(t))) = 0$$

and

$$(E_3)\qquad (a(t)\psi(x(t))x'(t))' + q(t)f(x(t)) = 0$$

have been investigated in [3,4]. And Jurang Yan[5] has given the oscillation theorems for a second order linear differential equations with damping

$$(E_4)\qquad (r(t)x'(t))' + p(t)x'(t) + q(t)x(t) = 0.$$

In this paper we discuss the oscillatory behavior of the solutions of the second order nonlinear differential equation with perturbation of the form

$$(1)\qquad (a(t)\psi(x(t))x'(t))' + Q(t, x(t)) = P(t, x(t), x'(t)), \quad ' = \frac{d}{dt}$$

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where the condition

(2) $$\lim_{t\to+\infty}\int_{t_0}^{t}[q(s)-\frac{p^2(s)}{4ca(s)}]ds=+\infty$$

is not assumed and

(A_1) $a:[t_0,+\infty)\to R$ $(R=(-\infty,+\infty))$ is positive continuously differentiable;

(A_2) $\psi:R\to R$ is continuously differentiable and $\psi(u)>0$ for $u\neq 0$;

(A_3) $Q\in([t_0,+\infty)\times R\to R)$, and there exists a continuous function $q(t)$ and continuously differentiable function $f(x)$ such that $\frac{Q(t,x)}{f(x)}\geq q(t)$ for $x\neq 0$, where $q:[t_0,+\infty)\to R, q(t)$ not identically zero, i.e. there exists $\{t_k\}$, $t_k\to+\infty$ such that $q(t_k)\neq 0$, $f:R\to R$, $uf(u)>0$ and $f'(u)>0$ for $u\neq 0$;

(A_4) $P\in([t_0,+\infty)\times R^2\to R)$ and there exists $p(t)\in([t_0,+\infty)\to R)$ such that $x(t)P(t,x(t),x'(t))\leq x(t)p(t)x'(t)$ for $x\neq 0$.

Throughout by a *solution* of Eq.(1) we shall mean a function which exists on $[t_0,+\infty)$ satisfies Eq.(1) and $x(t)\not\equiv 0$, $t\in[T,+\infty)$. As usual, a solution of Eq.(1) is said to be oscillatory if it has arbitrarily large zeros; otherwise it is said to be *nonoscillatory*. A nonoscillatory solution of Eq.(1) is said to be *weakly oscillatory* if $x'(t)$ changes sign for arbitrarily large values of t (see [3, 6]). Eq.(1) is called *oscillatory* if all its solutions are oscillatory.

With respect to their asymptotic behavior, all the solutions of Eq.(1) can be divided into the following four types:

$S^+=\{x=x(t)$ solution of Eq.(1): there exists $t_x\geq t_0$, such that $x(t)x'(t)\geq 0$ for $t\geq t_x\}$;

$S^-=\{x=x(t)$ solution of Eq.(1): there exists $t_x\geq t_0$, such that $x(t)x'(t)<0$ for $t\geq t_x\}$;

$S^O=\{x=x(t)$ solution of Eq.(1): there exists $\{t_n\},t_n\to+\infty$, such that $x(t_n)=0\}$;

$S^{WO}=\{x=x(t)$ solution of Eq.(1): $x(t)\neq 0$ for t sufficiently large and for all $t_\alpha>t_0$ there exist $t_{\alpha_1}>t_\alpha$, $t_{\alpha_2}>t_\alpha$ such that $x'(t_{\alpha_1})x'(t_{\alpha_2})<0\}$.

With very simple argument we can prove that S^+,S^-,S^O,S^{WO} are mutually disjoint. By the above definitions, it turns out that solutions in the class S^+ are eventually either positive nondecreasing or negative nonincreasing, solutions in the class S^- are eventually either positive nonincreasing or negative nondecreasing, solutions in the class S^O are oscillatory, and finally, solutions in the class S^{WO} are weakly oscillatory.

2. Main results

In this section, we establish two oscillatory theorems of Eq. (1).

LEMMA 1. *Assume that $\psi(x)f'(x) \geq c > 0$ for $x \neq 0$. If for sufficiently large T*

$$\liminf_{t\to+\infty} \int_T^t \left[q(s) - \frac{p^2(s)}{4ca(s)}\right] ds \geq 0, \tag{3}$$

then $S^{WO} = \emptyset$ for Eq. (1).

Proof. Suppose that Eq.(1) has a solution $x(t) \in S^{WO}$. There is no loss of generality in assuming that there exists $t_1 \geq t_0$ such that $x(t) > 0$ for all $t \geq t_1$. (For $x(t) < 0$, the proof is similar.) Thus for all $t_\alpha > t_1$ there exist $t_{\alpha_1}, t_{\alpha_2} > t_\alpha$, such that $x'(t_{\alpha_1})x'(t_{\alpha_2}) < 0$. Therefore there exists the sequence $\{C_n\} \to +\infty$ such that $x'(C_n) < 0$. Let sufficiently large N be such that C_N satisfies the condition (3). i.e.,

$$\liminf_{t\to+\infty} \int_{C_N}^t \left[q(s) - \frac{p^2(s)}{4ca(s)}\right] ds \geq 0.$$

Consider the function

$$W(t) = \frac{a(t)\psi(x(t))x'(t)}{f(x(t))}, \quad t \geq t_1.$$

Then it follows from Eq.(1) when $t \geq t_1$

$$\begin{aligned} W'(t) &= -\frac{Q(t,x(t))}{f(x(t))} + \frac{x(t)P(t,x(t),x'(t))}{x(t)f(x(t))} - a(t)\psi(x(t))f'(x(t))\frac{x'^2(t)}{f^2(x(t))} \\ &\leq -q(t) + \frac{p(t)x'(t)}{f(x(t))} - a(t)\psi(x(t))f'(x(t))\frac{x'^2(t)}{f^2(x(t))} \\ &= -q(t) + \frac{p^2(t)}{4a(t)\psi(x(t))f'(x(t))} \\ &\quad -\left[\sqrt{a(t)\psi(x(t))f'(x(t))}\frac{x'(t)}{f(x(t))} - \frac{p(t)}{2\sqrt{a(t)\psi(x(t))f'(x(t))}}\right]^2 \\ &\leq -q(t) + \frac{p^2(t)}{4ca(t)}. \end{aligned}$$

For all $b \geq t_1$, integrating the above inequality from b to t, we have

$$\frac{a(t)\psi(x(t))x'(t)}{f(x(t))} \leq \frac{a(b)\psi(x(b))x'(b)}{f(x(b))} - \int_b^t \left[q(s) - \frac{p^2(s)}{4ca(s)}\right] ds. \tag{4}$$

Then for the above C_N when $t \geq C_N$ we have

$$\frac{a(t)\psi(x(t))x'(t)}{f(x(t))} \leq \frac{a(C_N)\psi(x(C_N))x'(C_N)}{f(x(C_N))} - \int_{C_N}^t \left[q(s) - \frac{p^2(s)}{4ca(s)}\right] ds.$$

Therefore

$$\limsup_{t\to+\infty}\frac{a(t)\psi(x(t))x'(t)}{f(x(t))}\le\frac{a(C_N)\psi(x(C_N))x'(C_N)}{f(x(C_N))}+\limsup_{t\to+\infty}\{-\int_{C_N}^{t}\left[q(s)-\frac{p^2(s)}{4ca(s)}\right]ds\}<0.$$

Then for all $t\ge C_N$ we obtain $x'(t)<0$, which gives a contradiction since $x'(t_{\alpha_1})x'(t_{\alpha_2})<0$. The proof is now complete.

LEMMA 2. *Assume that $\psi(x)f'(x)\ge c>0$ for $x\ne 0$. If*

$$\left|\int_{t_0}^{+\infty}\left[q(s)-\frac{p^2(s)}{4ca(s)}\right]ds\right|<+\infty \tag{5}$$

and

$$\lim_{t\to+\infty}\int_{t_0}^{t}\frac{1}{a(s)}\int_{s}^{+\infty}\left[q(\tau)-\frac{p^2(\tau)}{4ca(\tau)}\right]d\tau ds=+\infty, \tag{6}$$

furthermore $\frac{f(u)}{\psi(u)}$ is strongly superlinear, that is

$$\int_{\varepsilon}^{+\infty}\frac{\psi(u)}{f(u)}du<+\infty,\quad \int_{-\infty}^{-\varepsilon}\frac{\psi(u)}{f(u)}du>-\infty \tag{7}$$

for all $\varepsilon>0$, then for Eq.(1), we have $S^+=\emptyset$.

Proof. Suppose that Eq.(1) has a solution $x(t)\in S^+$. There is no loss of generality in assuming that there exists $t_1\ge t_0$ such that $x(t)>0, x'(t)\ge 0$ for all $t\ge t_1$. (For $x(t)<0$, $x'(t)\le 0$, the proof is similar.) As in the the proof of lemma 1 we can acquire (4). From (5) we obtain ($t\ge b$, $x'(t)\ge 0$),

$$0\le\frac{a(b)\psi(x(b))x'(b)}{f(x(b))}-\int_{b}^{+\infty}[q(s)-\frac{p^2(s)}{4ca(s)}]ds.$$

Thus for all $t\ge b$ we have

$$\int_{t}^{+\infty}[q(s)-\frac{p^2(s)}{4ca(s)}]ds\le\frac{a(t)\psi(x(t))x'(t)}{f(x(t))}.$$

So we can obtain

$$\int_{b}^{t}\frac{1}{a(s)}\int_{s}^{+\infty}[q(\tau)-\frac{p^2(\tau)}{4ca(\tau)}]d\tau ds\le\int_{b}^{t}\frac{\psi(x(s))x'(s)}{f(x(s))}ds.$$

Letting $t\to+\infty$, which contradicts condition (6) and (7). The proof is complete.

THEOREM 1. *Assume that* $p(t) \geq 0, t \geq t_0$ *and* $\psi(x)f'(x) \geq c > 0$ *for* $x \neq 0$. *If the conditions* (3), (5), (6) *and* (7) *hold and*

$$\lim_{t\to+\infty} \int_{t_0}^{t} \frac{1}{a(s)} ds = +\infty, \tag{8}$$

then Eq. (1) *is oscillatory.*

Proof. It follows from Lemma 1 and Lemma 2 $S^+ = S^{WO} = \emptyset$ for Eq. (1). Therefore, to prove Theorem 1, it suffices to show that $S^- = \emptyset$ for Eq. (1). Let $x(t)$ be a solution of class S^- of Eq. (1). There is no loss of generality in assuming that there exists $t_1 \geq t_0$ such that $x(t) > 0, x'(t) \leq 0$ for all $t \geq t_1$. (For $x(t) < 0, x'(t) \geq 0$, the proof is similar.) It follows from (3) there exists $t_2 \geq t_1$ such that

$$\int_{t_2}^{t} [q(s) - \frac{p^2(s)}{4ca(s)}] ds \geq 0$$

for $t \geq t_2$. From Eq. (1), $x'(t) \not\equiv 0$ for $t \geq t_2$. Suppose for $t \geq t_2$ then $x'(t) \equiv 0$. So it follows from Eq.(1) $Q(t, x(t)) = P(t, x(t), x'(t))$. Because of (A_3) and (A_4) we have $q(t)f(x) \leq p(t)x'(t)$, then $q(t) \leq 0$, which contradicts condition (3). So $x'(t) \not\equiv 0$ for $t \geq t_2$. There exists $t_3 \geq t_2$ such that $x'(t_3) < 0$. Integrating Eq.(1) from t_3 to t, we have

$$\begin{aligned} a(t)\psi(x(t))x'(t) &= a(t_3)\psi(x(t_3))x'(t_3) + \int_{t_3}^{t} P(s, x(s), x'(s)) ds \\ &\quad - \int_{t_3}^{t} Q(s, x(s)) ds \\ &\leq a(t_3)\psi(x(t_3))x'(t_3) + \int_{t_3}^{t} p(s)x'(s) ds - \int_{t_3}^{t} q(s)f(x(s)) ds. \end{aligned}$$

Because $p(t) \geq 0$ and $x'(t) \leq 0$, we get

$$\begin{aligned} a(t)\psi(x(t))x'(t) &\leq a(t_3)\psi(x(t_3))x'(t_3) - \int_{t_3}^{t} q(s)f(x(s)) ds \\ &= a(t_3)\psi(x(t_3))x'(t_3) - \int_{t_3}^{t} \left[q(s) - \frac{p^2(s)}{4ca(s)}\right] f(x(s)) ds \\ &\quad - \int_{t_3}^{t} \frac{p^2(s)}{4ca(s)} f(x(s)) ds \end{aligned}$$

$$\leq a(t_3)\psi(x(t_3))x'(t_3) - \int_{t_3}^{t}\left[q(s) - \frac{p^2(s)}{4ca(s)}\right]f(x(s))ds$$

$$= a(t_3)\psi(x(t_3))x'(t_3) - f(x(t))\int_{t_3}^{t}\left[q(s) - \frac{p^2(s)}{4ca(s)}\right]ds$$

$$+\int_{t_3}^{t} f'(x(s))x'(s)\int_{t_3}^{s}\left[q(\tau) - \frac{p^2(\tau)}{4ca(\tau)}\right]d\tau ds$$

$$\leq a(t_3)\psi(x(t_3))x'(t_3) = k \quad (k < 0).$$

Consequently, for all $t \geq t_3$ we have

$$\int_{x(t_3)}^{x(t)} \psi(u)du \leq k\int_{t_3}^{t}\frac{1}{a(s)}ds.$$

Observe that the condition (8) and the fact $0 < x(t) \leq x(t_3)$ imply that

$$\lim_{t\to+\infty}\int_{x(t)}^{x(t_3)} \psi(u)du = +\infty,$$

and so a contradiction since $\lim\limits_{t\to+\infty} x(t)$ exists finite and ψ is continuous. The proof is now complete.

LEMMA 3. *Assume that $\psi(x)f'(x) \geq c > 0$ for $x \neq 0$. If the condition (7) hold and*

$$\lim_{t\to+\infty}\int_{T}^{t}\left[q(s) - \frac{p^2(s)}{4ca(s)}\right]\int_{T}^{s}\frac{1}{a(\tau)}d\tau ds = +\infty \tag{9}$$

is satisfied, then $S^+ = \emptyset$ and $S^{WO} = \emptyset$ for Eq. (1).

Proof. (I) Suppose that Eq.(1) has a solution $x(t) \in S^+$. There is no loss of generality in assuming that there exists $t_1 \geq t_0$ such that $x(t) > 0, x'(t) \geq 0$ for all $t \geq t_1$.(For $x(t) < 0, x'(t) \leq 0$, the proof is similar.) Consider the function

$$W(t) = \frac{-a(t)\psi(x(t))x'(t)}{f(x(t))}\int_{t_1}^{t}\frac{1}{a(s)}ds, \quad t \geq t_1.$$

Then it follows from Eq.(1) that

$$W'(t) = \frac{-(a(t)\psi(x(t))x'(t))'}{f(x(t))}\int_{t_1}^{t}\frac{1}{a(s)}ds + \frac{a(t)\psi(x(t))x'^2(t)f'(x(t))}{f^2(x(t))}\int_{t_1}^{t}\frac{1}{a(s)}ds$$

$$-\frac{\psi(x(t))x'(t)}{f(x(t))}$$

$$= \left[\frac{Q(t,x(t))}{f(x(t))} - \frac{x(t)P(t,x(t),x'(t))}{x(t)f(x(t))} + \frac{a(t)\psi(x(t))x'^2(t)f'(x(t))}{f^2(x(t))}\right]\int_{t_1}^{t}\frac{1}{a(s)}ds - \frac{\psi(x(t))x'(t)}{f(x(t))}$$

$$\geq \left[q(t) - \frac{p(t)x'(t)}{f(x(t))} + a(t)\psi(x(t))f'(x(t))\frac{x'^2(t)}{f^2(x(t))}\right]\int_{t_1}^{t}\frac{1}{a(s)}ds - \frac{\psi(x(t))x'(t)}{f(x(t))}$$

$$= \left[q(t) - \frac{p^2(t)}{4a(t)\psi(x(t))f'(x(t))}\right]\int_{t_1}^{t}\frac{1}{a(s)}ds - \frac{\psi(x(t))x'(t)}{f(x(t))}$$

$$+ \left[\sqrt{a(t)\psi(x(t))f'(x(t))}\frac{x'(t)}{f(x(t))} - \frac{p(t)}{2\sqrt{a(t)\psi(x(t))f'(x(t))}}\right]^2\int_{t_1}^{t}\frac{1}{a(s)}ds$$

$$\geq \left[q(t) - \frac{p^2(t)}{4ca(t)}\right]\int_{t_1}^{t}\frac{1}{a(s)}ds - \frac{\psi(x(t))x'(t)}{f(x(t))}.$$

So

$$W(t) \geq \int_{t_1}^{t}\left[q(s) - \frac{p^2(s)}{4ca(s)}\right]\int_{t_1}^{s}\frac{1}{a(\tau)}d\tau ds - \int_{t_1}^{t}\frac{\psi(x(s))x'(s)}{f(x(s))}ds. \tag{10}$$

Noting the condition (7) and (9) we obtain

$$\lim_{t\to+\infty} W(t) = +\infty,$$

which contradicts with the assumption $W(t) < 0$.

(II) Suppose that Eq. (1) has a solution $x(t) \in S^{WO}$. There is no loss of generality in assuming that there exists $t_1 \geq t_0$ such that $x(t) > 0$ for all $t \geq t_1$ (For $x(t) < 0$, the proof is similar). For all $t_\alpha > t_1$ there exist $t_{\alpha_1}, t_{\alpha_2} > t_\alpha$, such that $x'(t_{\alpha_1})x'(t_{\alpha_2}) < 0$. Proceeding as in the proof of the above (I), we obtain (10), i.e.,

$$\liminf_{t\to+\infty} W(t) \geq \liminf_{t\to+\infty}\int_{t_1}^{t}\left[q(s) - \frac{p^2(s)}{4ca(s)}\right]\int_{t_1}^{s}\frac{1}{a(\tau)}d\tau ds + \liminf_{t\to+\infty}\left\{-\int_{t_1}^{t}\frac{\psi(x(s))x'(s)}{f(x(s))}ds\right\}.$$

Noting the condition (7),

$$\limsup_{t\to+\infty}\int_{t_1}^{t}\frac{\psi(x(s))x'(s)}{f(x(s))}ds$$

has upper bound. In fact, from the condition (7) we know it has upper bound for $x'(s) > 0$. And for $x'(s) < 0$ we know 0 is upper bound. Then

$$\liminf_{t \to +\infty} \{ - \int_{t_1}^{t} \frac{\psi(x(s))x'(s)}{f(x(s))} ds \}$$

has lower bound. Noting the condition (9) we have $x'(t) < 0$ for all large t, which gives a contradiction since $x'(t_{\alpha_1})x'(t_{\alpha_2}) < 0$. The proof is now complete.

THEOREM 2. *Assume that $p(t) \geq 0$ for $t \geq t_0$ and $\psi(x)f'(x) \geq c > 0, x \neq 0$. If the assumptions* (7), (8) *and* (9) *are satisfied, then Eq.* (1) *is oscillatory.*

Proof. It follows from Lemma 3 that $S^+ = S^{WO} = \emptyset$. Therefore, to prove Theorem 2, it suffices to show that $S^- = \emptyset$ for Eq.(1). Let $x(t)$ be a solution of type S^- of Eq.(1). Without loss of generality, we may assume that there exists $t_1 \geq t_0$ such that $x(t) > 0, x'(t) \leq 0$ for all $t \geq t_1$. (For $x(t) < 0, x'(t) \geq 0$, the proof is similar.) Consider the function

$$W(t) = \frac{-a(t)\psi(x(t))x'(t)}{f(x(t))} \int_{t_1}^{t} \frac{1}{a(s)} ds, \quad t \geq t_1.$$

As in the proof of Lemma 3 we obtain (10), i.e.,

$$W(t) \geq \int_{t_1}^{t} [q(s) - \frac{p^2(s)}{4ca(s)}] \int_{t_1}^{s} \frac{1}{a(\tau)} d\tau ds - \int_{t_1}^{t} \frac{\psi(x(s))x'(s)}{f(x(s))} ds.$$

In view of condition (9), $W(t) \to +\infty$ for $t \to +\infty$. Then there exists $t_2 \geq t_1$ such that $W(t) \geq 1$ for $t \geq t_2$. Therefore

$$\frac{\psi(x(t))x'(t)}{f(x(t))} \leq -\frac{1}{a(t)\int_{t_1}^{t} \frac{1}{a(s)} ds}, \quad t \geq t_2.$$

Let $A(t, t_1) = \int_{t_1}^{t} \frac{1}{a(s)} ds$. From the above we can acquire $x'(t) < 0$ for $t \geq t_2$ and

$$\int_{x(t_2)}^{x(t)} \frac{\psi(u)}{f(u)} du \leq - \int_{t_2}^{t} \frac{1}{a(s)A(s, t_1)} ds = -\ln \frac{A(t, t_1)}{A(t_2, t_1)} \to -\infty \quad (t \to +\infty).$$

Then $x(t) \to 0$ $(t \to +\infty)$. It also follows from (9) that there exists $t_3 \geq t_2$ such that

$$\int_{t_3}^{t} \left[q(s) - \frac{p^2(s)}{4ca(s)} \right] \int_{t_3}^{s} \frac{1}{a(\tau)} d\tau ds \geq 0, \quad t \geq t_3.$$

Integrating Eq. (1) from t_3 to t, we have

$$\int_{t_3}^{t}(a(s)\psi(x(s))x'(s))'\int_{t_3}^{s}\frac{1}{a(\tau)}d\tau ds=\int_{t_3}^{t}P(s,x(s),x'(s))\int_{t_3}^{s}\frac{1}{a(\tau)}d\tau ds$$
$$-\int_{t_3}^{t}Q(s,x(s))\int_{t_3}^{s}\frac{1}{a(\tau)}d\tau ds.$$

Then integrating the left side we obtain

$$a(t)\psi(x(t))x'(t)\int_{t_3}^{t}\frac{1}{a(\tau)}d\tau\le\int_{x(t_3)}^{x(t)}\psi(u)du+\int_{t_3}^{t}p(s)x'(s)\int_{t_3}^{s}\frac{1}{a(\tau)}d\tau ds$$
$$-\int_{t_3}^{t}q(s)f(x(s))\int_{t_3}^{s}\frac{1}{a(\tau)}d\tau ds.$$

Since $p(t)\ge 0$ and $x'(t)\le 0$, we have

$$a(t)\psi(x(t))x'(t)\int_{t_3}^{t}\frac{1}{a(\tau)}d\tau\le\int_{x(t_3)}^{x(t)}\psi(u)du-\int_{t_3}^{t}q(s)f(x(s))\int_{t_3}^{s}\frac{1}{a(\tau)}d\tau ds.$$

Then

$$\begin{aligned}
(11)\qquad & a(t)\psi(x(t))x'(t)\int_{t_3}^{t}\frac{1}{a(\tau)}d\tau\\
&\le\int_{x(t_3)}^{x(t)}\psi(u)du-\int_{t_3}^{t}\left[q(s)-\frac{p^2(s)}{4ca(s)}\right]f(x(s))\int_{t_3}^{s}\frac{1}{a(\tau)}d\tau ds\\
&\quad-\int_{t_3}^{t}\frac{p^2(s)}{4ca(s)}f(x(s))\int_{t_3}^{s}\frac{1}{a(\tau)}d\tau ds\\
&\le\int_{x(t_3)}^{x(t)}\psi(u)du-\int_{t_3}^{t}\left[q(s)-\frac{p^2(s)}{4ca(s)}\right]f(x(s))\int_{t_3}^{s}\frac{1}{a(\tau)}d\tau ds\\
&=\int_{x(t_3)}^{x(t)}\psi(u)du-f(x(t))\int_{t_3}^{t}\left[q(s)-\frac{p^2(s)}{4ca(s)}\right]\int_{t_3}^{s}\frac{1}{a(\tau)}d\tau ds\\
&\quad+\int_{t_3}^{t}f'(x(s))x'(s)\int_{t_3}^{s}\left[q(u)-\frac{p^2(u)}{4ca(u)}\right]\int_{t_3}^{u}\frac{1}{a(\tau)}d\tau du ds\\
&\le\int_{x(t_3)}^{x(t)}\psi(u)du.
\end{aligned}$$

Consequently, for $t \to +\infty$ we have $x(t) \to 0$. Then there exists $t_4 \geq t_3$ such that $x(t) < \dfrac{x(t_3)}{2}$ for all $t \geq t_4$. And there exists the constant L such that

$$\int_{x(t_3)}^{x(t)} \psi(u)du < -L.$$

Noting Eq.(11) we can acquire

$$\psi(x(t))x'(t) \leq -L\frac{1}{a(t)\int_{t_3}^{t} \frac{1}{a(\tau)}d\tau}, \quad t \geq t_4.$$

Then it follows from the above inequality that

$$\int_{x(t_4)}^{x(t)} \psi(u)du \leq -L \ln \frac{A(t,t_3)}{A(t_4,t_3)} \to -\infty \quad (t \to +\infty),$$

which contradicts with the facts that the left of inequality has lower bound. The proof is now complete.

3. Examples

In this section, we give two illustrative examples.

EXAMPLE 1. Consider the equation

$$\left(\frac{1}{x^2(t)}x'(t)\right)' - \frac{1}{t}x'(t) + \frac{1}{t^{\frac{3}{2}}}x^3(t) = 0, \quad (t > 0), \tag{12}$$

where $a(t) = 1, \psi(u) = u^{-2}$. Let $q(t) = t^{-\frac{3}{2}}, p(t) = t^{-1}, f(u) = u^3$. It is easy to verify that Eq. (12) satisfies the conditions of Theorem 1. Therefore, Eq. (12) is oscillatory. However, using any known results, we can not obtain the conclusion.

EXAMPLE 2. Consider the equation

$$\left(\frac{1}{x^2(t)}x'(t)\right)' - \frac{1}{1+t}x'(t) + \frac{1}{t^2}x^3(t) = 0, \quad (t > 0), \tag{13}$$

where $a(t) = 1, \psi(u) = u^{-2}$. Let $q(t) = t^{-2}$, $p(t) = (1+t)^{-1}$, $f(u) = u^3$. It is easy to check that Eq. (13) satisfies all the conditions of Theorem 2. Therefore, Eq. (13) is oscillatory. However, using any known results, we can not obtain the conclusion.

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