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k-FOLD SYMMETRICAL MAPPINGS AND LOEWNER CHAINS

Abstract. Let B be the unit ball in $\mathbb{C}^n$ with respect to an arbitrary norm on $\mathbb{C}^n$. In this paper, we give a necessary and sufficient condition that a Loewner chain $f(z,t)$, such that $\{e^{-t}f(z,t)\}_{t\geq 0}$ is a normal family on B, is k-fold symmetrical. As a corollary, we give a necessary and sufficient condition that a normalized locally biholomorphic mapping on B is spirallike of type α and k-fold symmetrical. When $\alpha = 0$, this result solves a natural problem that is similar to an open problem posed by Liczberski. We also give two examples of k-fold symmetrical Loewner chains.

1. Introduction

Let $\mathbb{B}^n$ be the Euclidean unit ball in $\mathbb{C}^n$ and let f be a normalized locally biholomorphic mapping on $\mathbb{B}^n$. Suffridge [20] proved that f is starlike if and only if

$$\mathrm{Re}\langle [Df(z)]^{-1}f(z), z\rangle > 0, \quad z \in \mathbb{B}^n \setminus \{0\},$$

where $\langle\cdot,\cdot\rangle$ is the Euclidean inner product in $\mathbb{C}^n$. Let $k \geq 2$ be an arbitrarily fixed integer and let $f_{1,k}$ be the $(1,k)$-symmetrical part of f. Then Liczberski [11] showed that f is starlike and k-fold symmetrical if and only if $f_{1,k}$ is locally biholomorphic and one of the following conditions is satisfied:

(i) $\mathrm{Re}\langle [Df_{1,k}(z)]^{-1}f(z), z\rangle > 0$ for $z \in \mathbb{B}^n \setminus \{0\}$ and $[Df_{1,k}(z)]^{-1}f(z)$ is k-fold symmetrical;

(ii) $\mathrm{Re}\langle [Df(z)]^{-1}f_{1,k}(z), z\rangle > 0$ for $z \in \mathbb{B}^n \setminus \{0\}$ and $[Df(z)]^{-1}f_{1,k}(z)$ is k-fold symmetrical.

It is natural to pose the following open problem (cf. [11, Open problem 4.6]).

OPEN PROBLEM: Assume that $\mathrm{Re}\langle [Df(z)]^{-1}f(z), z\rangle > 0$ for $z \in \mathbb{B}^n \setminus \{0\}$ and $[Df(z)]^{-1}f(z)$ is k-fold symmetrical. Is $f(z)$ k-fold symmetrical?

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Recently, k-fold symmetrical mappings are studied in [7], [9] and [13].

Let B be the unit ball in $\mathbb{C}^n$ with respect to an arbitrary norm on $\mathbb{C}^n$. In this paper, we give a necessary and sufficient condition that a Loewner chain $f(z,t)$, such that $\{e^{-t}f(z,t)\}_{t\geq 0}$ is a normal family on B, is k-fold symmetrical. As a corollary, we give a necessary and sufficient condition that a normalized locally biholomorphic mapping on B is spirallike of type α and k-fold symmetrical. When $\alpha = 0$, this result solves the above open problem. It seems that it is very difficult to solve the open problem without using Loewner chain. We also give examples of k-fold symmetrical Loewner chains.

2. Preliminaries

Let $\mathbb{C}^n$ denote the space of n complex variables $z = (z_1, \dots, z_n)$ with respect to an arbitrary norm $\|\cdot\|$. Let $B = \{z \in \mathbb{C}^n : \|z\| < 1\}$. The Euclidean unit ball in $\mathbb{C}^n$ is denoted by $\mathbb{B}^n$. Let U be the unit disc in $\mathbb{C}$. Let $H(G)$ denote the set of holomorphic mappings from an open set $G \subset \mathbb{C}^n$ into $\mathbb{C}^n$. Further, let $L(\mathbb{C}^n, \mathbb{C}^m)$ be the space of all continuous linear operators from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the standard operator norm. Let I be the identity in $L(\mathbb{C}^n, \mathbb{C}^n)$. A mapping $f \in H(B)$ is called normalized if $f(0) = 0$ and $Df(0) = I$.

For each $z \in \mathbb{C}^n \setminus \{0\}$, we set $T(z) = \{l_z \in L(\mathbb{C}^n, \mathbb{C}) : l_z(z) = \|z\|, \|l_z\| = 1\}$. Then this set is nonempty by the Hahn-Banach theorem.

If $f, g \in H(B)$, we say that f is subordinate to g, and write $f \prec g$, if there exists a Schwarz mapping v (i.e. $v \in H(B)$, $v(0) = 0$, and $\|v(z)\| < 1$, $z \in B$) such that $f = g \circ v$ on B. If g is biholomorphic on B, this condition is equivalent to $f(0) = g(0)$ and $f(B) \subset g(B)$.

We recall that a mapping $f : B \times [0, \infty) \to \mathbb{C}^n$ is called a Loewner chain if $f(\cdot, t)$ is biholomorphic on B, $f(0,t) = 0$, $Df(0,t) = e^t I$ for $t \geq 0$, and

$$f(z,s) \prec f(z,t), \quad z \in B, \quad 0 \leq s \leq t < \infty.$$

The above condition is equivalent to the fact that there exists a unique biholomorphic Schwarz mapping $v = v(z,s,t)$, called the transition mapping of $f(z,t)$, such that $f(z,s) = f(v(z,s,t),t)$, $z \in B$, $t \geq s \geq 0$. The normalization of $f(z,t)$ implies the normalization $Dv(0,s,t) = e^{s-t}I$ for $t \geq s \geq 0$.

A fundamental role in the study of Loewner chains and the Loewner differential equation in several complex variables is played by the following sets

$$\mathcal{N} = \{p \in H(B) : p(0) = 0, \quad \operatorname{Re} l_z(p(z)) > 0,\ z \in B \setminus \{0\},\ l_z \in T(z)\}$$

and $\mathcal{M} = \{p \in \mathcal{N} : Dp(0) = I\}$.

The set $\mathcal{M}$ is the generalization to higher dimensions of the Carathéodory set in one complex variable.

The basic existence theorem for the Loewner differential equation on B, originally due to Pfaltzgraff (see [14, Theorem 2.1]), can be improved by omitting the boundedness assumption on $h(z,t)$. The following proposition is due to [3, Theorem 1.4, Lemma 1.6] (cf. [5], [14], [19], [10]).

PROPOSITION 1. *Let* $h = h(z,t) : B \times [0,\infty) \to \mathbb{C}^n$ *satisfy the following conditions:*

(i) *for each* $t \geq 0$, $h(\cdot,t) \in \mathcal{M}$;
(ii) *for each* $z \in B$, $h(z,t)$ *is a measurable function of* $t \in [0,\infty)$.

Let $f(z,t)$ *be a Loewner chain which satisfies the differential equation*

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad a.e. \quad t \geq 0, \quad \forall z \in B. \tag{2.1}$$

Suppose that $\{e^{-t}f(z,t)\}$ *be a normal family. Then for each* $s \geq 0$ *and* $z \in B$, *the initial value problem*

$$\frac{\partial v}{\partial t} = -h(v,t), \quad a.e. \quad t \geq s, \quad v(z,s,s) = z, \tag{2.2}$$

has a unique solution $v = v(z,s,t)$. *The mapping* $v(z,s,t) = e^{s-t}z + \cdots$ *is a univalent Schwarz mapping on* B *and is a locally Lipschitz function of* $t \geq s$ *locally uniformly with respect to* $z \in B$. *Moreover,*

$$\lim_{t\to\infty} e^t v(z,s,t) = f(z,s)$$

locally uniformly on B *for each* $s \geq 0$.

REMARK 2. Let $f(z,t)$ be a Loewner chain. Then $f(z,\cdot)$ is locally absolutely continuous on $[0,\infty)$ locally uniformly with respect to $z \in B$ (see e.g. [5]). In [3], it is shown that there exists a mapping $h = h(z,t)$ such that $h(\cdot,t) \in \mathcal{M}$, $t \geq 0$, $h(z,\cdot)$ is measurable on $[0,\infty)$ for $z \in B$, and

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad a.e. \quad t \geq 0, \quad \forall z \in B.$$

Moreover, if $\{e^{-t}f(z,t)\}_{t\geq 0}$ is a normal family, then $f(z,t)$ satisfies the condition (2.2) in Proposition 1.

Let $\alpha \in \mathbb{R}$ with $|\alpha| < \pi/2$. A normalized locally biholomorphic mapping $f \in H(B)$ is said to be spirallike of type α if f is biholomorphic and the spiral $\exp(-e^{-i\alpha}t)f(z)$ $(t \geq 0)$ is contained in $f(B)$ for any $z \in B$. When $\alpha = 0$, we obtain the usual notion of starlikeness.

Then according to [8, Theorem 2.1], a normalized locally biholomorphic mapping $f \in H(B)$ is spirallike of type α if and only if

$$\operatorname{Re} l_z\left(e^{-i\alpha}[Df(z)]^{-1}f(z)\right) > 0, \quad z \in B\setminus\{0\},\ l_z \in T(z).$$

Especially, a normalized locally biholomorphic mapping $f \in H(B)$ is starlike if and only if (see [20])

$$\operatorname{Re} l_z\left([Df(z)]^{-1}f(z)\right) > 0, \quad z \in B \setminus \{0\},\, l_z \in T(z).$$

In [8, Theorem 3.1], the following alternative characterization of spirallikeness of type α is proved: f is spirallike of type α if and only if $f(z,t) = e^{(1-ia)t}f(e^{iat}z)$ is a Loewner chain, where $a = \tan\alpha$. Especially, a normalized locally biholomorphic mapping $f \in H(B)$ is starlike if and only if $f(z,t) = e^t f(z)$ is a Loewner chain [16].

Let $f \in H(B)$ be a normalized locally biholomorphic mapping on B. Then f is called close-to-starlike with respect to a normalized starlike mapping g if

$$\operatorname{Re} l_z\Big([Df(z)]^{-1}g(z)\Big) > 0,\ z \in B \setminus \{0\},\ l_z \in T(z).$$

This definition was introduced by Pfaltzgraff and Suffridge [16]. They proved that f is close-to-starlike with respect to g if and only if

$$f(z,t) = f(z) + (e^t - 1)g(z)$$

is a Loewner chain (see [16]).

Definition 3 ([12]). Let $j = 0, 1, \ldots, k-1$, where $k \geq 2$ is a natural number. A mapping $f : B \to \mathbb{C}^n$ will be called (j,k)-symmetrical, if

$$f(\varepsilon_k z) = \varepsilon_k^j f(z), \quad z \in B,$$

where $\varepsilon_k = \exp(2\pi i/k)$.

Liczberski and Połubiński [12] proved the following decomposition theorem.

Theorem 4. *For every mapping $f : B \to \mathbb{C}^n$, there exists exactly one sequence of (j,k)-symmetrical mappings $f_{j,k}$, $j = 0, 1, \ldots, k-1$ such that*

$$f = \sum_{j=0}^{k-1} f_{j,k}.$$

Moreover,

$$f_{j,k}(z) = \frac{1}{k}\sum_{l=0}^{k-1} \varepsilon_k^{-jl} f(\varepsilon_k^l z), \quad z \in B.$$

By the uniqueness of the above decomposition, the mappings $f_{j,k}$ are called (j,k)-symmetrical parts of the mapping f.

$(1,k)$-symmetrical mappings are also called k-fold symmetrical mappings. Then f is k-fold symmetrical if and only if $f = f_{1,k}$. A Loewner chain $f(z,t)$ is said to be k-fold symmetrical if $f(z,t)$ is k-fold symmetrical for each $t \geq 0$.

3. Main results

First, we give a necessary and sufficient condition that a Loewner chain which satisfies the assumptions of Proposition 1 is k-fold symmetrical.

THEOREM 5. *Let $f(z,t)$ be a Loewner chain which satisfies the assumptions of Proposition* 1. *Also let $h(z,t)$ be the mapping which satisfies the assumptions of Proposition* 1. *Then $f(z,t)$ is k-fold symmetrical if and only if $h(z,t)$ is k-fold symmetrical for almost all $t \geq 0$.*

This result has the following equivalent formulation by Remark 2.

THEOREM 6. *Let $f(z,t)$ be a Loewner chain such that $\{e^{-t}f(z,t)\}_{t\geq 0}$ is a normal family on B. Also let $h(z,t)$ be given by* (2.1). *Then $f(z,t)$ is k-fold symmetrical if and only if $h(z,t)$ is k-fold symmetrical for almost all $t \geq 0$.*

Proof of Theorem 5. First, assume that $f(z,t)$ is k-fold symmetrical, that is, $f(\varepsilon_k z,t) = \varepsilon_k f(z,t)$ for all $z \in B$, $t \geq 0$. Therefore, we obtain that $Df(\varepsilon_k z,t) = Df(z,t)$, $z \in B$, $t \geq 0$, and since $f(z,\cdot)$ is differentiable a.e. on $[0,\infty)$, we have that

$$\frac{\partial f}{\partial t}(\varepsilon_k z,t) = \varepsilon_k \frac{\partial f}{\partial t}(z,t), \quad a.e.\ t \geq 0, \forall z \in B.$$

Thus, in view of (2.1) we obtain that $h(\varepsilon_k z,t) = \varepsilon_k h(z,t)$ for almost all $t \geq 0$ and for all $z \in B$.

Next, we assume that $h(z,t)$ is k-fold symmetrical for almost all $t \geq 0$. That is, there exists a subset E of $[0,\infty)$ of measure zero such that $h(z,t)$ is k-fold symmetrical in B, for all $t \in [0,\infty)\setminus E$. The solution of the initial value problem (2.2) is constructed by the method of successive approximation as follows (see the proof of [14, Theorem 2.1])

$$v_0(z,s,t) = z$$

$$v_m(z,s,t) = z - \int_s^t h(v_{m-1}(z,s,\tau),\tau)d\tau, \quad m \geq 1$$

and the solution of (2.2) is defined by

$$v(z,s,t) = \lim_{m\to\infty} v_m(z,s,t),$$

where the above limit holds locally uniformly on B. Since every $v_m(z,s,t)$ is k-fold symmetrical by induction on m, $v(z,s,t)$ is k-fold symmetrical. By Proposition 1, $f(z,s) = \lim_{t\to\infty} e^t v(z,s,t)$ locally uniformly on B. So $f(z,s)$ is k-fold symmetrical too. This completes the proof.

REMARK 7. Let $f(z,t)$ be a Loewner chain such that $\{e^{-t}f(z,t)\}_{t\geq 0}$ is a normal family on B. Assume that $f(z,t)$ is k-fold symmetrical. Then Hamada, Honda and Kohr [7] showed that

$$\frac{\|z\|}{(1+\|z\|^k)^{2/k}} \le \|e^{-t}f(z,t)\| \le \frac{\|z\|}{(1-\|z\|^k)^{2/k}},\ z\in B,\ t\ge 0.$$

In Liu and Liu [13], it is shown that this result is sharp (cf. [2, Corollary 3.2.1]).

As corollaries to the above theorem, we obtain the following results.

COROLLARY 8. *Let $f \in H(B)$ be a normalized locally biholomorphic mapping on B. Then f is spirallike of type α and k-fold symmetrical if and only if $e^{-i\alpha}[Df(z)]^{-1}f(z) \in \mathcal{N}$ and $[Df(z)]^{-1}f(z)$ is k-fold symmetrical.*

Proof. First, assume that f is spirallike of type α and k-fold symmetrical. Then $e^{-i\alpha}[Df(z)]^{-1}f(z) \in \mathcal{N}$ and $f(\varepsilon_k z) = \varepsilon_k f(z)$. Therefore, $Df(\varepsilon_k z) = Df(z)$ and this implies that $[Df(z)]^{-1}f(z)$ is k-fold symmetrical.

Next, assume that $e^{-i\alpha}[Df(z)]^{-1}f(z) \in \mathcal{N}$ and $[Df(z)]^{-1}f(z)$ is k-fold symmetrical. Also let $a = \tan\alpha$. Then f is spirallike of type α by [8, Theorem 2.1], and $f(z,t) = e^{(1-ia)t}f(e^{iat}z)$ is a Loewner chain by [8, Theorem 3.1], which satisfies the assumptions of Proposition 1. Since $f(z,t)$ is of class C^∞ on $B\times[0,\infty)$, it follows in view of the relation (2.1) that

$$h(z,t) = [Df(z,t)]^{-1}\frac{\partial f}{\partial t}(z,t) = iaz + (1-ia)e^{-iat}[Df(e^{iat}z)]^{-1}f(e^{iat}z)$$

is k-fold symmetrical. Hence f is k-fold symmetrical by Theorem 5. This completes the proof.

If we put $\alpha = 0$ in the above corollary, we obtain the following corollary. This corollary solves the open problem which is described in the introduction (cf. [11, Open problem 4.6]).

COROLLARY 9. *Let f be a normalized locally biholomorphic mapping on B. Then f is starlike and k-fold symmetrical if and only if $[Df(z)]^{-1}f(z) \in \mathcal{M}$ and $[Df(z)]^{-1}f(z)$ is k-fold symmetrical.*

For the Euclidean unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$, we have the following necessary and sufficient condition for a mapping f which satisfies (3.1) to be k-fold symmetrical (cf. [1], [14]).

THEOREM 10. *Let $f \in H(\mathbb{B}^n)$ be a normalized locally biholomorphic mapping which satisfies the condition*

$$(3.1)\qquad (1-\|z\|^2)\|\,[Df(z)]^{-1}D^2f(z)(z,\cdot)\| \le 1,\ \text{for all } z\in\mathbb{B}^n.$$

Then f is k-fold symmetrical if and only if $M(z) = [Df(z)]^{-1}D^2f(z)(z,\cdot)$ satisfies $M(\varepsilon_k z) = M(z)$ for all $z \in \mathbb{B}^n$.

Proof. First, assume that f is k-fold symmetrical. Then $Df(\varepsilon_k z) = Df(z)$ and $D^2f(\varepsilon_k z)(\varepsilon_k z, \cdot) = D^2f(z)(z, \cdot)$. Therefore, $M(\varepsilon_k z) = M(z)$ for all $z \in \mathbb{B}^n$.

Next, assume that $M(\varepsilon_k z) = M(z)$ for all $z \in \mathbb{B}^n$. Let

$$f(z,t) = f(ze^{-t}) + (e^t - e^{-t})Df(ze^{-t})(z), \quad t \geq 0. \tag{3.2}$$

Since f satisfies the condition (3.1), it follows from [14, Theorem 2.4] that $f(z,t)$ is a Loewner chain which satisfies the assumptions of Proposition 1 and

$$h(z,t) = [Df(z,t)]^{-1}\frac{\partial f}{\partial t}(z,t) = (I - E(z,t))^{-1}(I + E(z,t))(z),$$

for all $t \geq 0$ and $z \in B$, where

$$E(z,t) = -(1 - e^{-2t})[Df(ze^{-t})]^{-1}D^2f(ze^{-t})(ze^{-t}, \cdot).$$

Since $M(\varepsilon_k z) = M(z)$, we obtain that $E(\varepsilon_k z, t) = E(z,t)$ for all $z \in \mathbb{B}^n$, $t \geq 0$. Therefore, we obtain that $h(z,t)$ is k-fold symmetrical for all $t \geq 0$. By Theorem 5, f is k-fold symmetrical. This completes the proof.

In the case of one complex variable, every Loewner chain satisfies the assumptions of Proposition 1, i.e. every Loewner chain $f(z,t)$ has the property that $\{e^{-t}f(z,t)\}_{t\geq 0}$ is a normal family, by [18, Theorem 6.2]. However, in higher dimensions, there exists a Loewner chain which does not satisfy the assumptions of Proposition 1 [3, Example 2.12]. Also, in higher dimensions, there exists a k-fold symmetrical Loewner chain $f(z,t)$ which does not satisfy the assumptions of Proposition 1 [7].

REMARK 11. (i) Let f be a normalized starlike mapping on B. This is equivalent to the fact that $f(z,t) = e^t f(z)$ is a Loewner chain. It is obvious that f is k- fold symmetrical if and only if $f(z,t)$ is k-fold symmetrical.

(ii) If f is a spirallike mapping of type α, then it is clear that f is k-fold symmetrical if and only if $f(z,t) = e^{(1-ia)t}f(e^{iat}z)$ is a k-fold symmetrical Loewner chain, where $a = \tan\alpha$.

(iii) Let $f \in H(\mathbb{B}^n)$ be a normalized locally biholomorphic mapping which satisfies the assumption (3.1). Then f is k-fold symmetrical if and only if

$$f(z,t) = f(ze^{-t}) + (e^t - e^{-t})Df(ze^{-t})(z)$$

is a k-fold symmetrical Loewner chain.

(iv) Let $f : B \to \mathbb{C}^n$ be a mapping which have parametric representation on B, i.e. there exists a Loewner chain $f(z,t)$ such that $\{e^{-t}f(z,t)\}_{t\geq 0}$ is a normal family on B and $f(z) = f(z,0)$, $z \in B$ (see [3]). Clearly if $f(z,t)$ is k-fold symmetrical then f is also k-fold symmetrical. However, if f is k-fold symmetrical then $f(z,t)$ need not be k-fold symmetrical for each $t \geq 0$. For example, let f be a normalized close-to-starlike mapping with respect to a normalized starlike mapping g. Then

$$f(z,t) = f(z) + (e^t - 1)g(z)$$

is a Loewner chain, and it is clear that if f is close-to-starlike with respect to g, then $f \in S^0(B)$. It is easy to see that if f is k-fold symmetrical but g is not k-fold symmetrical, then $f(z,t)$ is not k-fold symmetrical for $t > 0$.

REMARK 12. Let $f \in H(\mathbb{B}^n)$ be a normalized quasiregular mapping. If there exists a constant $c < 1$ such that

$$(1 - \|z\|^2)\|[Df(z)]^{-1}D^2f(z)(z,\cdot)\| \le c, \; z \in \mathbb{B}^n,$$

then f has a continuous extension to $\overline{\mathbb{B}}^n$, again denoted by f, and

$$F(z) = \begin{cases} f(z) & z \in \overline{\mathbb{B}}^n \\ f\left(\frac{z}{\|z\|}, \log\|z\|\right) & z \notin \overline{\mathbb{B}}^n, \end{cases}$$

where $f(z,t)$ is a Loewner chain defined in (3.2), is a quasiconformal homeomorphism of $\mathbb{C}^n$ onto itself, by [15]. In addition, if f is a k-fold symmetrical mapping, then the extension F is also k-fold symmetrical, i.e. $F(\varepsilon_k z) = \varepsilon_k F(z)$ for $z \in \mathbb{C}^n$.

EXAMPLE 13. Let $\alpha \in [0,1]$ and $\beta \in [0,1/2]$ be such that $\alpha + \beta \le 1$. In the proof of [4, Theorem 2.1], the authors proved that if $f(z_1,t)$ is a Loewner chain on U, then $F_{n,\alpha,\beta}(z,t)$ is a Loewner chain which satisfies the assumptions of Proposition 1 on $\mathbb{B}^n$, where

$$F_{n,\alpha,\beta}(z,t) = \left(f(z_1,t), e^{(1-\alpha-\beta)t}\tilde{z}\left(\frac{f(z_1,t)}{z_1}\right)^{\alpha}(f'(z_1,t))^{\beta}\right) \tag{3.3}$$

for $z = (z_1, \tilde{z}) \in \mathbb{B}^n$. The branches of the power functions are chosen so that

$$\left(\frac{f(z_1,0)}{z_1}\right)^{\alpha}\Bigg|_{z_1=0} = 1 \quad \text{and} \quad (f'(z_1,0))^{\beta}\Big|_{z_1=0} = 1.$$

Moreover, we can easily show that if $f(z_1,t)$ is a k-fold symmetrical Loewner chain, then $F_{n,\alpha,\beta}(z,t)$ is a k-fold symmetrical Loewner chain which satisfies the assumptions of Proposition 1.

EXAMPLE 14. In the following result, we shall denote by $\mathcal{L}S_n$ the set of normalized locally biholomorphic mappings in $\mathbb{B}^n$. Also for $n \ge 1$, let $z' = (z_1, \ldots, z_n)$ so that $z = (z', z_{n+1}) \in \mathbb{C}^{n+1}$. Pfaltzgraff and Suffridge [17] defined the following extension operator $\Phi_n : \mathcal{L}S_n \to \mathcal{L}S_{n+1}$ given by

$$\Phi_n(f)(z) = F(z) = \left(f(z'), z_{n+1}[J_f(z')]^{1/(n+1)}\right), \quad z = (z', z_{n+1}) \in \mathbb{B}^{n+1},$$

where $J_f(z') = \det Df(z')$ for $z' \in \mathbb{B}^n$. On the other hand, in [6] the authors have recently proved that if $f(z',t)$ is a Loewner chain on $\mathbb{B}^n$ such that

$\{e^{-t}f(z',t)\}_{t\geq 0}$ is a normal family on $\mathbb{B}^n$, then $F(z,t)$ is also a Loewner chain such that $\{e^{-t}F(z,t)\}_{t\geq 0}$ is a normal family on $\mathbb{B}^{n+1}$, where

$$F(z,t) = \Big(f(z',t), z_{n+1}e^{t/(n+1)}[J_{f_t}(z')]^{1/(n+1)}\Big), z = (z', z_{n+1}) \in \mathbb{B}^{n+1}, t \geq 0,$$

and $f_t(z') = f(z',t)$. We choose the branch of the power function such that $[J_{f_t}(z')]^{1/(n+1)}\Big|_{z'=0} = e^{nt/(n+1)}$ for $t \geq 0$.

Now, it is easy to see that if $f(z',t)$ is a k-fold symmetrical Loewner chain then $F(z,t)$ is also a k-fold symmetrical Loewner chain.

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