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SOME RELATIONS INCLUDING VARIOUS LINEAR OPERATORS

Abstract. Making use of the Carlson-Schaffer linear operator, some subclasses of analytic functions are studied. Some relations including various linear operators are given.

1. Introduction

We denote by $\mathcal{A}_p$ the class of functions f of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad (p \in \mathcal{N} = \{1, 2, 3, \dots\}), \tag{1}$$

which are *analytic* in $\mathcal{U} = \mathcal{U}(1)$, where $\mathcal{U}(r) = \{z : |z| < r\}$.

A function f belonging to the class $\mathcal{A}_p$ is said to be *p-valently starlike of order* α *in* $\mathcal{U}(r)$ if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad (z \in \mathcal{U}(r),\ 0 \le \alpha < 1).$$

We denote by $\mathcal{S}_p^*(\alpha)$ the class of all functions in $\mathcal{A}_p$ which are p-valently starlike of order α in $\mathcal{U}$.

For analytic functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

by $f * g$ we denote the *Hadamard product* or *convolution* of f and g, defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

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For complex parameters a, b, c $(c \neq 0, -1, -2, \dots)$ we define the *hypergeometric function* ${}_2F_1(a, b; c; z)$ by

$$(2) \qquad {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{z^n}{n!} \quad (z \in \mathcal{U}),$$

where (λ, n) is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(\lambda, n) = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0) \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (n \in \mathcal{N}). \end{cases}$$

The power series (2) converges in the unit disk $\mathcal{U}$. For $\Re\mathrm{e}\, c > \Re\mathrm{e}\, b > 0$ the hypergeometric function has the following integral representation

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 u^{b-1}(1 - u)^{c-b-1}(1 - uz)^{-a} du.$$

Using the incomplete Beta function $\phi_p(a, c; z)$ defined by

$$\phi_p(a, c; z) = z^p \, {}_2F_1(a, c; z),$$

Carlson and Shaffer [2] consider a linear operator

$$\mathcal{L}(a, c) : \mathcal{A}_p \to \mathcal{A}_p,$$

defined by the convolution:

$$\mathcal{L}(a, c) f(z) = \phi_p(a, c; z) * f(z), \quad f \in \mathcal{A}_p.$$

The Carlson-Shaffer operator maps $\mathcal{A}_p$ into itself. If $a \neq 0, -1, -2, \dots$, then $\mathcal{L}(a, c)$ has a continuous inverse $\mathcal{L}(c, a)$ and $\mathcal{L}(c, a)$ maps $\mathcal{A}_p$ into $\mathcal{A}_p$ injectively. Also, if $c > a > 0$, then

$$\mathcal{L}(a, c) f(z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)} \int_0^1 u^{c-2}(1 - u)^{c-a-1} f(uz) du.$$

We observe that, for a function f of the form (1), we have

$$\mathcal{L}(a, c) f(z) = \sum_{n=p}^{\infty} \frac{(a, n - p)}{(c, n - p)} a_n z^n.$$

Thus, after some calculations, we obtain

$$(3) \qquad a\mathcal{L}(a + 1, c) f(z) = z \left[\mathcal{L}(a, c) f(z)\right]' - (p - a)\, \mathcal{L}(a, c) f(z).$$

In particular, we denote

$$\mathcal{D}^{\lambda} f(z) = \mathcal{L}(\lambda + p, p) \quad (\lambda > -p),$$

which implies that

$$(4) \qquad \mathcal{D}^n f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{(p,n)} \quad (n \in \mathcal{N}).$$

The linear operator $\mathcal{D}^\lambda f(z)$ $(p = 1)$ was introduced by Ruscheweyh [16]. Moreover, the Carlson-Shaffer operator includes other linear operators, which were considered in earlier works, as (for example) the linear operators introduced by Bernardi-Libera-Livingston ([1], [12], and [13]), Owa [15] (see also [19]), and Srivastava and Owa [18].

Let $\mathcal{V}_p(a, c; \alpha)$ denote the class of functions $f \in \mathcal{A}_p$ such that

$$(5) \qquad \mathrm{Re}\left\{ a\frac{\mathcal{L}(a+1,c)f(z)}{\mathcal{L}(a,c)f(z)} + p - a \right\} > \alpha \quad (z \in \mathcal{U}).$$

Moreover, by $\mathcal{W}_p(a, c; \alpha)$ we denote the class of functions $f \in \mathcal{A}_p$ such that $zf'(z) \in \mathcal{V}_p(a, c; \alpha)$.

In particular, we have

$$\mathcal{W}_p\,(p-1, p; \alpha) = \mathcal{V}_p\,(p, p; \alpha) = \mathcal{S}_p^*(\alpha).$$

Classes $\mathcal{V}_1(a, c; \alpha)$, $\mathcal{W}_1(a, c; \alpha)$ were investigated by Kim and Srivastava [11]. Classes of functions defined by some linear operators were also investigated by (among others) Srivastava *et al.* [6], [7], [8] and [17] (see also [3], [4] and [5]).

In this paper we present inclusions with respect to the parameter a for the classes defined above. Also some relations including the Carlson-Shaffer operator and the Ruscheweyh operator are given.

2. Main results

We shall need the following lemma due to Jack [10].

LEMMA 1. *Let w be a nonconstant function analytic in $\mathcal{U}(r)$ with $w(0) = 0$. If*

$$|w(z_0)| = \max\{|w(z)|\,;\ |z| \le |z_0|\} \quad (z_0 \in \mathcal{U}(r)),$$

then there exists a real number k $(k \ge 1)$, such that

$$z_0 w'(z_0) = k w(z_0).$$

Making use of Jack's Lemma, Eenigenburg, Miller, Mocanu and Reade [9] (see also [14]) proved the following result.

LEMMA 2. *If q is an analytic function in $\mathcal{U}(r)$, $q(0) = p$ and*

$$\mathrm{Re}\left(q(z) + \frac{zq'(z)}{q(z)+\gamma} \right) > \alpha \quad (z \in \mathcal{U}(r),\ 0 \le \alpha < p, \mathrm{Re}\gamma \ge -\alpha),$$

then

$$\mathrm{Re}q(z) > \alpha \quad (z \in \mathcal{U}(r)).$$

Making use of the above lemma, we get the following theorem.

THEOREM 1. *If* $\mathrm{Re}a \geq p - \alpha$, *then*

$$\mathcal{V}_p(a+1, c; \alpha) \subset \mathcal{V}_p(a, c; \alpha).$$

Proof. Let a function f belong to the class $\mathcal{V}_p(a+1, c; \alpha)$. It is sufficient to verify condition (5). If we put

$$R = \sup\{r: \ \mathcal{L}(a,c)f(z) \neq 0, \ z \in \mathcal{U}(r)\},$$

then the function

$$q(z) = a\frac{\mathcal{L}(a+1,c)f(z)}{\mathcal{L}(a,c)f(z)} + p - a \tag{6}$$

is an analytic in $\mathcal{U}(R)$ and $q(0) = p$. Taking the logarithmic derivative of (6) we get

$$\frac{z\left[\mathcal{L}(a+1,c)f(z)\right]'}{\mathcal{L}(a+1,c)f(z)} - \frac{z\left[\mathcal{L}(a,c)f(z)\right]'}{\mathcal{L}(a,c)f(z)} = \frac{zq'(z)}{q(z)+a-p} \quad (z \in \mathcal{U}(R)).$$

Applying (3) and (6) we obtain

$$(a+1)\frac{\mathcal{L}(a+2,c)f(z)}{\mathcal{L}(a+1,c)f(z)} + p - a - 1 = q(z) + \frac{zq'(z)}{q(z)+a-p} \quad (z \in \mathcal{U}(R)). \tag{7}$$

Since $f \in \mathcal{V}_p(a+1, c; \alpha)$, we have

$$\mathrm{Re}\left(q(z) + \frac{zq'(z)}{q(z)+a-p}\right) > \alpha \quad (z \in \mathcal{U}(R)).$$

Lemma 2 now yields

$$\mathrm{Re}q(z) > \alpha \quad (z \in \mathcal{U}(R)). \tag{8}$$

By (6) it suffices to verify that $R = 1$. From (8), (6) and (3) we conclude that $\mathcal{L}(a,c)f(z)$ is $p-$valently starlike in $\mathcal{U}(R)$ and consequently it is $p-$valent in $\mathcal{U}(R)$. Thus we see that $\mathcal{L}(a,c)f(z)$ cannot vanish on $|z| = R$ if $R < 1$. Hence $R = 1$ and this proves the Theorem 1.

THEOREM 2. *If a function* $f \in \mathcal{A}_p$ *satisfies the following inequality:*

$$\left|\frac{\mathcal{L}(a+2,c)f(z)}{\mathcal{L}(a+1,c)f(z)} - 1\right| < \frac{2(p-\alpha)^2 + 3(p-\alpha) - a}{2(a+1)(p-\alpha)} \tag{9}$$

$$(z \in \mathcal{U}, \ 0 \leq \alpha < p, \ p - \alpha \leq a \leq 3(p-\alpha)),$$

then f *belongs to the class* $\mathcal{V}_p(a, c; \alpha)$.

Proof. Let a function f belong to the class $\mathcal{A}_p$. Putting

$$q(z) = \frac{p - (2\alpha - p)w(z)}{1 - w(z)} \quad (z \in \mathcal{U}(R)) \tag{10}$$

in (7) we obtain

$$(a+1)\frac{\mathcal{L}(a+2,c)f(z)}{\mathcal{L}(a+1,c)f(z)}+p-a-1=\frac{p-(2\alpha-p)w(z)}{1-w(z)}$$
$$+\frac{(2p-2\alpha-a)\,zw'(z)}{a+(2p-2\alpha-a)w(z)}+\frac{zw'(z)}{1-w(z)}.$$

Consequently, we have

$$\text{(11)}\quad F(z)=w(z)\left\{\frac{zw'(z)}{w(z)}\frac{2p-2\alpha-a}{a+(2p-2\alpha-a)\,w(z)}+\frac{\frac{zw'(z)}{w(z)}+2p-2\alpha}{1-w(z)}\right\},$$

where

$$F(z)=(a+1)\,\frac{\mathcal{L}(a+2,c)f(z)}{\mathcal{L}(a+1,c)f(z)}-a-1.$$

By (3), (6) and (10) it is sufficient to verify that w is analytic in $\mathcal{U}$ and

$$|w(z)|<1\quad(z\in\mathcal{U}).$$

Now, suppose that there exists a point $z_0\in\mathcal{U}(R)$, such that

$$|w(z_0)|=1,\ |w(z)|<1\ \ (|z|<|z_0|).$$

Then, applying Jack's Lemma, we can write

$$z_0w'(z_0)=kw(z_0),\ w(z_0)=e^{i\theta}\quad(k\geq 1).$$

Combining these with (11) , we obtain

$$\begin{aligned}|F(z_0)|&=|w(z_0)|\left|\frac{(2p-2\alpha-a)\,k}{a+(2p-2\alpha-a)\,e^{i\theta}}+\frac{k+2p-2\alpha}{1-e^{i\theta}}\right|\\&\geq \operatorname{Re}\left(\frac{(2p-2\alpha-a)\,k}{a+(2p-2\alpha-a)\,e^{i\theta}}+\frac{k+2p-2\alpha}{1-e^{i\theta}}\right)\\&\geq p-\alpha+k\frac{3\,(p-\alpha)-a}{2\,(p-\alpha)}\geq\frac{2\,(p-\alpha)^2+3\,(p-\alpha)-a}{2\,(p-\alpha)}.\end{aligned}$$

Since this result contradicts (9) we conclude that w is the analytic function in $\mathcal{U}(R)$ and

$$|w(z)|<1\quad(z\in\mathcal{U}\,(R)).$$

Applying the same methods as in the proof of Theorem 1 we obtain $R=1$, which completes the proof of Theorem 2.

THEOREM 3. *If a function $f\in\mathcal{A}_p$ satisfies the following inequality:*

$$\left|\frac{\mathcal{L}(a+2,c)f(z)}{\mathcal{L}(a+1,c)f(z)}-1\right|<\frac{(p-\alpha)^2+(p-\alpha)\,(a+1)}{(a+p-\alpha)\,(a+1)}\quad(z\in\mathcal{U},\ \ a\geq p-\alpha),$$

then

$$\left| a\frac{\mathcal{L}(a+1,c)f(z)}{\mathcal{L}(a,c)f(z)} - a \right| < p-\alpha \quad (z\in\mathcal{U}).$$

Proof. After putting $q(z)=p+(\alpha-p)\,w(z)$ in (10), the proof is analogous to the proof of Theorem 2 and we omits details.

By Theorem 1, 2 and 3 we obtain following three corollaries:

COROLLARY 1. *If* $\operatorname{Re} a \geq p-\alpha$ *and* $b-a$ *is positive integer, then*

$$\mathcal{W}_p(b,c;\alpha)\subset\mathcal{W}_p(a,c;\alpha).$$

COROLLARY 2. *If a function* $f\in\mathcal{A}_p$ *satisfies the following inequality:*

$$\left|\frac{(a+2)\,\mathcal{L}(a+3,c)f(z)+(p-a-2)\,\mathcal{L}(a+2,c)f(z)}{(a+1)\,\mathcal{L}(a+2,c)f(z)+(p-a-1)\,\mathcal{L}(a+1,c)f(z)}-1\right| < \frac{2\,(p-\alpha)^2+3\,(p-\alpha)-a}{2\,(a+1)\,(p-\alpha)}$$
$$(z\in\mathcal{U},\ 0\leq\alpha<p,\ \ p-\alpha\leq a\leq 3\,(p-\alpha)),$$

then f *belongs to the class* $\mathcal{W}_p(a,c;\alpha)$.

COROLLARY 3. *If a function* $f\in\mathcal{A}_p$ *satisfies the following inequality:*

$$\left|\frac{(a+2)\,\mathcal{L}(a+3,c)f(z)+(p-a-2)\,\mathcal{L}(a+2,c)f(z)}{(a+1)\,\mathcal{L}(a+2,c)f(z)+(p-a-1)\,\mathcal{L}(a+1,c)f(z)}-1\right| < \frac{(p-\alpha)^2+(p-\alpha)\,(a+1)}{(a+p-\alpha)\,(a+1)}$$
$$(z\in\mathcal{U},\ \ a\geq p-\alpha),$$

then

$$\left| a\frac{\mathcal{L}'(a+1,c)f(z)}{\mathcal{L}'(a,c)f(z)} - a \right| < p \quad (z\in\mathcal{U}).$$

Putting $c=p,\ a=\lambda+p$ in Theorem 1, 2 and 3 we obtain following three corollaries:

COROLLARY 4. *If a function* f *belongs to the class* $\mathcal{A}_p$ *and*

$$\operatorname{Re}\left\{(\lambda+p+1)\frac{\mathcal{D}^{\lambda+2}f(z)}{\mathcal{D}^{\lambda+1}f(z)}-\lambda-1\right\}>\alpha,\quad (z\in\mathcal{U},\ 0\leq\alpha<p,\ \operatorname{Re}\lambda\geq-\alpha),$$

then

$$\operatorname{Re}\left\{(\lambda+p)\frac{\mathcal{D}^{\lambda+1}f(z)}{\mathcal{D}^{\lambda}f(z)}-\lambda\right\}>\alpha,\qquad (z\in\mathcal{U}).$$

COROLLARY 5. *If a function f belongs to the class $\mathcal{A}_p$ and*

$$\left|\frac{\mathcal{D}^{\lambda+2}f(z)}{\mathcal{D}^{\lambda+1}f(z)}-1\right|<\frac{2\,(p-\alpha)^2+2\,(p-\alpha)-a-\lambda}{2\,(\lambda+p+1)\,(p-\alpha)}$$

$$(z\in\mathcal{U},\ 0\le\alpha<p,\quad -\alpha\le\lambda\le 2p-3\alpha),$$

then

$$\mathrm{Re}\left\{(\lambda+p)\,\frac{\mathcal{D}^{\lambda+1}f(z)}{\mathcal{D}^{\lambda}f(z)}-\lambda\right\}>\alpha\quad(z\in\mathcal{U}).$$

COROLLARY 6. *If a function f belongs to the class $\mathcal{A}_p$ and*

$$\left|\frac{\mathcal{D}^{\lambda+2}f(z)}{\mathcal{D}^{\lambda+1}f(z)}-1\right|<\frac{(p-\alpha)^2+(p-\alpha)\,(\lambda+p+1)}{(2p+\lambda-\alpha)\,(\lambda+p+1)}$$

$$(z\in\mathcal{U},\ 0\le\alpha<p,\ \lambda\ge-\alpha),$$

then

$$(\lambda+p)\left|\frac{\mathcal{D}^{\lambda+1}f(z)}{\mathcal{D}^{\lambda}f(z)}-1\right|<p-\alpha\quad(z\in\mathcal{U}).$$

Putting $\lambda=0$ in Corollary 5 and 6 we obtain the sufficient conditions for starlikeness.

COROLLARY 7. *If a function f belongs to the class $\mathcal{A}_p$ and*

$$\left|\frac{zf''(z)}{f'(z)}-p+1\right|<p-\alpha+1-\frac{p}{2\,(p-\alpha)}\quad(z\in\mathcal{U},\ 0\le\alpha<\frac{2}{3}p),$$

then f belongs to the class $\mathcal{S}^*_p(\alpha)$.

COROLLARY 8. *If a function f belongs to the class $\mathcal{A}_p$ and*

$$\left|\frac{zf''(z)}{f'(z)}-p+1\right|<\frac{(p-\alpha)^2+(p-\alpha)\,(p+1)}{2p-\alpha}\quad(z\in\mathcal{U},\ 0\le\alpha<p),$$

then

$$\left|\frac{zf'(z)}{f(z)}-p\right|<p-\alpha\quad(z\in\mathcal{U}).$$

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