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# NEW INEQUALITIES OF ČEBYŠEV TYPE FOR DOUBLE INTEGRALS

**Abstract.** In this paper, we establish new inequalities of Čebyšev type involving functions of two independent variables by using certain integral identities.

## 1. Introduction

In 1882, P. L. Čebyšev [2] proved the following classical inequality

$$(1.1)\quad \left|\frac{1}{b-a}\int_a^b f(x)g(x)dx-\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right)\right| \leq \frac{1}{12}(b-a)^2\|f'\|_\infty\|g'\|_\infty,$$

provided $f, g$ are absolutely continuous functions defined on $[a, b]$ and $f', g' \in L_\infty[a, b]$.

Since the publication of [2], a number of researchers have given various generalizations, extensions and variants of the above inequality, see [4] and also some of the recent papers appeared in RGMIA Research Report Collection. The main purpose of this paper is to establish new inequalities similar to the inequality (1.1), involving functions of two independent variables and their partial derivatives and double integrals. The analysis used in the proofs is based on the integral identities proved in [1] and [3].

## 2. Statement of results

Let $R$ denotes the set of real numbers and $\Delta = [a, b] \times [c, d]$, $a, b, c, d \in R$. The partial derivatives of a function $z(x, y)$ defined on $\Delta$ are denoted by $D_1 z(x,y) = \frac{\partial}{\partial x} z(x,y)$, $D_2 z(x,y) = \frac{\partial}{\partial y} z(x,y)$, $D_2 D_1 z(x,y) = \frac{\partial^2}{\partial y \partial x} z(x,y)$. We denote by $C(\Delta)$ the class of continuous functions $z : \Delta \to R$

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for which $D_1 z(x,y), D_2 z(x,y), D_2 D_1 z(x,y)$ exist and are continuous on $\Delta$ and belong to $L_\infty(\Delta)$. For any function $z(x,y) \in L_\infty(\Delta)$ we define $\|z\|_\infty = \sup_{(x,y)\in\Delta} |z(x,y)|$. For convenience, we introduce the following notations to simplify the details of presentation :

$$k = (b-a)(d-c),$$

$$H_1(x) = \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right],$$

$$H_2(y) = \left[\frac{1}{4}(d-c)^2 + \left(y - \frac{c+d}{2}\right)^2\right],$$

$$F(x,y) = (d-c)\int_a^b f(t,y)\,dt + (b-a)\int_c^d f(x,s)\,ds - \int_a^b\int_c^d f(t,s)\,dsdt,$$

$$G(x,y) = (d-c)\int_a^b g(t,y)\,dt + (b-a)\int_c^d g(x,s)\,ds - \int_a^b\int_c^d g(t,s)\,dsdt,$$

$$\begin{aligned} A(x,y) =\ & \|D_1 f\|_\infty (d-c) H_1(x) \\ & + \|D_2 f\|_\infty (b-a) H_2(y) + \|D_2 D_1 f\|_\infty H_1(x) H_2(y), \\ B(x,y) =\ & \|D_1 g\|_\infty (d-c) H_1(x) + \|D_2 g\|_\infty (b-a) H_2(y) \\ & + \|D_2 D_1 g\|_\infty H_1(x) H_2(y), \end{aligned}$$

for $f, g \in C_1(\Delta)$ and $p : [a,b]^2 \to R$, $q : [c,d]^2 \to R$ are Peano kernels given by

$$p(x,t) = \begin{cases} t-a & if\ \ t \in [a,x] \\ t-b & if\ \ t \in (x,b], \end{cases}$$

$$q(y,s) = \begin{cases} s-c & if\ \ s \in [c,y] \\ s-d & if\ \ s \in (y,d], \end{cases}$$

and set

$$\begin{aligned} L[h(x,y)] = & \int_a^b\int_c^d p(x,t) D_1 h(t,s)\,dsdt + \int_a^b\int_c^d q(y,s) D_2 h(t,s)\,dsdt \\ & + \int_a^b\int_c^d p(x,t) q(y,s) D_2 D_1 h(t,s)\,dsdt, \end{aligned}$$

$$M[h(x,y)] = \int_a^b\int_c^d p(x,t) q(y,s) D_2 D_1 h(t,s)\,dsdt,$$

for some suitable function $h$ defined on $\Delta$.

The proofs of our results are based on the following integral identities proved in [3] and [1].

LEMMA 1 (3, p. 783). *Let $h : \Delta \to R$ be such that the partial derivatives $D_1 h(x,y)$, $D_2 h(x,y)$, $D_2 D_1 h(x,y)$ exist and are continuous on $\Delta$. Then for all $(x,y) \in \Delta$, we have the representation*

$$kh(x,y) - \int_a^b \int_c^d h(t,s)\,ds dt = L[h(x,y)]. \tag{2.1}$$

Proof. We use the following identity, which can be easily proved by integration by parts,

$$g(u) = \frac{1}{\beta - \alpha} \int_\alpha^\beta g(z)\,dz + \frac{1}{\beta - \alpha} \int_\alpha^\beta e(u,z)\, g'(z)\,dz, \tag{2.2}$$

where $e : [\alpha, \beta]^2 \to R$ is given by

$$e(u,z) = \begin{cases} z - \alpha \ \ if \ z \in [\alpha, u] \\ z - \beta \ if \ z \in (u, \beta] \end{cases},$$

and $g$ is absolutely continuous on $[\alpha, \beta]$.

Now, write the identity (2.2) for the partial map $h(.,y)$, $y \in [c,d]$ to obtain

$$h(x,y) = \frac{1}{b-a} \int_a^b h(t,y)\,dt + \frac{1}{b-a} \int_a^b p(x,t)\, D_1 h(t,y)\,dt, \tag{2.3}$$

for all $(x,y) \in \Delta$. Also, if we write (2.2) for any map $h(t,.)$ we get

$$h(t,y) = \frac{1}{d-c} \int_c^d h(t,s)\,ds + \frac{1}{d-c} \int_c^d q(y,s)\, D_2 h(t,s)\,ds, \tag{2.4}$$

for all $(t,y) \in \Delta$. The same formula (2.2) applied for the partial derivative $D_1 h(.,y)$ will produce

$$D_1 h(t,y) = \frac{1}{d-c} \int_c^d D_1 h(t,s)\,ds + \frac{1}{d-c} \int_c^d q(y,s)\, D_2 D_1 h(t,s)\,ds, \tag{2.5}$$

for all $(t,y) \in \Delta$. Substituting (2.4) and (2.5) in (2.3), and using the Fubini's theorem, we get

$$h(x,y) = \frac{1}{b-a} \int_a^b \left[ \frac{1}{d-c} \int_c^d h(t,s)\,ds + \frac{1}{d-c} \int_c^d q(y,s)\, D_2 h(t,s)\,ds \right] dt \tag{2.6}$$

$$+ \frac{1}{b-a} \int_a^b p(x,t) \left[ \frac{1}{d-c} \int_c^d D_1 h(t,s)\,ds + \frac{1}{d-c} \int_c^d q(y,s)\, D_2 D_1 h(t,s)\,ds \right] dt$$

$$= \frac{1}{(b-a)(d-c)} \left[ \int_a^b \int_c^d h(t,s)\,dsdt + \int_a^b \int_c^d q(y,s)\, D_2 h(t,s)\,dsdt \right.$$

$$\left. + \int_a^b \int_c^d p(x,t)\, D_1 h(t,s)\,dsdt + \int_a^b \int_c^d p(x,t)\, q(y,s)\, D_2 D_1 h(t,s)\,dsdt \right].$$

Rewriting (2.6), we get the required identity in (2.1).

LEMMA 2 (1, p. 17). *Let $h : \Delta \to R$ be a continuous mapping on $\Delta$ and $D_2 D_1 h(x,y)$ exists on $(a,b) \times (c,d)$. Then we have the identity*

$$kh(x,y) - H(x,y) = M[h(x,y)], \tag{2.7}$$

*where*

$$H(x,y) = (d-c) \int_a^b h(t,y)\,dt + (b-a) \int_c^d h(x,s)\,ds - \int_a^b \int_c^d h(t,s)\,dsdt.$$

Proof. Integrating by parts twice we can state :

$$\begin{aligned} (2.8) \quad & \int_a^x \int_c^y (s-a)(t-c)\, D_2 D_1 h(s,t)\,dtds \\ & = (y-c)(x-a)\, h(x,y) - (y-c) \int_a^x h(s,y)\,ds \\ & \quad - (x-a) \int_c^y h(x,t)\,dt + \int_a^x \int_c^y h(s,t)\,dtds, \end{aligned}$$

$$\begin{aligned} (2.9) \quad & \int_a^x \int_y^d (s-a)(t-d)\, D_2 D_1 h(s,t)\,dtds \\ & = (x-a)(d-y)\, h(x,y) - (d-y) \int_a^x h(s,y)\,ds \\ & \quad - (x-a) \int_y^d h(x,t)\,dt + \int_a^x \int_y^d h(s,t)\,dtds, \end{aligned}$$

$$\begin{aligned} (2.10) \quad & \int_x^b \int_y^d (s-b)(t-d)\, D_2 D_1 h(s,t)\,dtds \\ & = (d-y)(b-x)\, h(x,y) - (d-y) \int_x^b h(s,y)\,ds \\ & \quad - (b-x) \int_y^d h(x,t)\,dt + \int_x^b \int_y^d h(s,t)\,dtds, \end{aligned}$$

$$(2.11) \qquad \int_x^b\int_c^y (s-b)(t-c)\,D_2D_1h(s,t)\,dtds$$
$$= (y-c)(b-x)\,h(x,y) - (y-c)\int_x^b h(s,y)\,ds$$
$$-(b-x)\int_c^y h(x,t)\,dt + \int_x^b\int_c^y h(s,t)\,dtds.$$

Adding (2.8)–(2.11) and rewriting we easily deduce (2.7).

Our main results are given in the following theorems.

THEOREM 1. *Let $f, g \in C(\Delta)$. Then*

$$(2.12) \qquad \left|\frac{1}{k}\int_a^b\int_c^d f(x,y)g(x,y)dydx - \left(\frac{1}{k}\int_a^b\int_c^d f(x,y)dydx\right)\left(\frac{1}{k}\int_a^b\int_c^d g(x,y)dydx\right)\right|$$
$$\leq \frac{1}{k^3}\int_a^b\int_c^d A(x,y)\,B(x,y)\,dydx.$$

THEOREM 2. *Let $f, g \in C(\Delta)$. Then*

$$(2.13) \qquad \left|\frac{1}{k}\int_a^b\int_c^d f(x,y)g(x,y)dydx - \frac{1}{k^2}\int_a^b\int_c^d\Big[f(x,y)G(x,y) + g(x,y)F(x,y)\right.$$
$$\left. - \frac{1}{k}F(x,y)G(x,y)\Big]dydx\right|$$
$$\leq \frac{1}{k^3}\left\|D_2D_1f\right\|_\infty \left\|D_2D_1g\right\|_\infty \int_a^b\int_c^d \left[H_1(x)\,H_2(y)\right]^2 dydx.$$

## 3. Proofs of Theorems 1 and 2

From the hypotheses of Theorem 1, we have the following identities (see Lemma 1) :

$$(3.1) \qquad k\,f(x,y) - \int_a^b\int_c^d f(t,s)\,dsdt = L\left[f(x,y)\right],$$

$$(3.2) \qquad k\,g(x,y) - \int_a^b\int_c^d g(t,s)\,dsdt = L\left[g(x,y)\right],$$

for $(x,y) \in \Delta$. Multiplying the left sides and right sides of (3.1) and (3.2) we have

$$(3.3)\quad k^2 f(x,y)\, g(x,y) - k f(x,y) \int_a^b\int_c^d g(t,s)\,dsdt - k\, g(x,y) \int_a^b\int_c^d f(t,s)\,dsdt + \left(\int_a^b\int_c^d f(t,s)\,dsdt\right)\left(\int_a^b\int_c^d g(t,s)\,dsdt\right) = L[f(x,y)]\,L[g(x,y)].$$

Rewriting (3.3) and integrating on $\Delta$ and using the properties of modulus we observe that

$$(3.4)\quad \left|\frac{1}{k}\int_a^b\int_c^d f(x,y)\,g(x,y) - \left(\frac{1}{k}\int_a^b\int_c^d f(x,y)\,dydx\right)\left(\frac{1}{k}\int_a^b\int_c^d g(x,y)\,dydx\right)\right| \le \frac{1}{k^3}\int_a^b\int_c^d |L[f(x,y)]|\,|L[g(x,y)]|\,dydx.$$

It is easy to observe that

$$\begin{aligned}(3.5)\quad |L[f(x,y)]| &\le \int_a^b\int_c^d |p(x,t)|\,|D_1 f(t,s)|\,dsdt + \int_a^b\int_c^d |q(y,s)|\,|D_2 f(t,s)|\,dsdt \\ &\quad + \int_a^b\int_c^d |p(x,t)|\,|q(y,s)|\,|D_2D_1 f(t,s)|\,dsdt \\ &\le \|D_1 f\|_\infty \int_a^b\int_c^d |p(x,t)|\,dsdt + \|D_2 f\|_\infty \int_a^b\int_c^d |q(y,s)|\,dsdt \\ &\quad + \|D_2D_1 f\|_\infty \int_a^b\int_c^d |p(x,t)|\,|q(y,s)|\,dsdt \\ &= \|D_1 f\|_\infty (d-c)\,H_1(x) + \|D_2 f\|_\infty (b-a)\,H_2(y) \\ &\quad + \|D_2D_1 f\|_\infty H_1(x)\,H_2(y) \\ &= A(x,y).\end{aligned}$$

Similarly, we have

$$(3.6)\quad |L[g(x,y)]| \le B(x,y).$$

Using (3.5) and (3.6) in (3.4) we get the desired inequality in (2.12).

REMARK 1. From (3.3) and using the properties of modulus it is easy to see that the following inequality

$$\left|f(x,y)\,g(x,y) - \frac{1}{k}\left[f(x,y)\int_a^b\int_c^d g(t,s)\,dsdt + g(x,y)\int_a^b\int_c^d f(t,s)\,dsdt\right.\right.$$

$$-\frac{1}{k}\left(\int_a^b\int_c^d f(t,s)\,dsdt\right)\left(\int_a^b\int_c^d g(t,s)\,dsdt\right)\Bigg]\Bigg| \le \frac{1}{k^2}A(x,y)\,B(x,y),$$

holds for $(x,y)\in\Delta$.

From the hypotheses of Theorem 2, we have the following identities ( see Lemma 2) :

$$k\,f(x,y) - F(x,y) = M[f(x,y)], \tag{3.7}$$

$$k\,g(x,y) - G(x,y) = M[g(x,y)], \tag{3.8}$$

for $(x,y)\in\Delta$. Multiplying the left sides and right sides of (3.7) and (3.8) we have

$$\begin{aligned} k^2 f(x,y)\;g(x,y) - k\,f(x,y)\,G(x,y) - k\,g(x,y)\,F(x,y) \\ +F(x,y)\,G(x,y) = M[f(x,y)]\,M[g(x,y)]. \end{aligned} \tag{3.9}$$

Rewriting (3.9) and integrating on $\Delta$ and using the properties of modulus, we have

$$\begin{aligned} \Bigg|\frac{1}{k}\int_a^b\int_c^d f(x,y)g(x,y)dydx - \frac{1}{k^2}\int_a^b\int_c^d \Big[f(x,y)G(x,y) + g(x,y)F(x,y) \\ -\frac{1}{k}F(x,y)\,G(x,y)\Big]dydx\Bigg| \\ \le \frac{1}{k^3}\int_a^b\int_c^d |M[f(x,y)]|\,|M[g(x,y)]|\,dydx. \end{aligned} \tag{3.10}$$

It is easy to observe that

$$\begin{aligned} |M[f(x,y)]| &\le \|D_2D_1f\|_\infty \int_a^b\int_c^d |p(x,t)|\,|q(y,s)|\,dsdt \\ &= \|D_2D_1f\|_\infty H_1(x)\,H_2(y). \end{aligned} \tag{3.11}$$

Similarly, we get

$$|M[g(x,y)]| \le \|D_2D_1g\|_\infty H_1(x)\,H_2(y). \tag{3.12}$$

Using (3.11) and (3.12) in (3.10) we get the required inequality in (2.13).

REMARK 2. From (3.9) one can very easily observe that the following inequality

$$\begin{aligned} \left|f(x,y)g(x,y) - \frac{1}{k}\left[f(x,y)G(x,y) + g(x,y)F(x,y) - \frac{1}{k}F(x,y)G(x,y)\right]\right| \\ \le \frac{1}{k^2}\|D_2D_1f\|_\infty\,\|D_2D_1g\|_\infty\,[H_1(x)\,H_2(y)]^2 \end{aligned}$$

holds for $(x,y)\in\Delta$.

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