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# IDENTITIES WITH TWO AUTOMORPHISMS ON SEMIPRIME RINGS

**Abstract.** In this paper we investigate identities with two automorphisms on semiprime rings. We prove the following result: Let $T, S : R \to R$ be automorphisms where $R$ is a 2-torsion free semiprime ring satisfying the relation $T(x)x = xS(x)$ for all $x \in R$. In this case the mapping $x \mapsto T(x) - x$ maps $R$ into its center and $T = S$.

## 1. Preliminaries

Throughout, $R$ will represent an associative ring with center $Z(R)$. A ring $R$ is $n$-torsion free, where $n > 1$ is an integer, in case $nx = 0$, $x \in R$ implies $x = 0$. As usual the commutator $xy - yx$ will be denoted by $[x, y]$. We shall frequently use the commutator identities $[xy, z] = [x, z]\,y + x\,[y, z]$ and $[x, yz] = [x, y]\,z + y\,[x, z]$. We denote by $I$ the identity mapping on a ring $R$. Recall that $R$ is prime if $aRb = (0)$ implies $a = 0$ or $b = 0$, and is semiprime if $aRa = (0)$ implies $a = 0$. An additive mapping $D : R \to R$, where $R$ is an arbitrary ring, is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$. We denote by $C$ the extended centroid of a semiprime ring $R$ and by $Q$ Martindale ring of quotients. For the explanation of the extended centroid of a semiprime ring $R$ and the Martindale ring of quotients we refer the reader to [1]. A mapping $f : R \to R$ is called centralizing on $R$ if $[f(x), x] \in Z(R)$ holds for all $x \in R$; in the special case when $[f(x), x] = 0$ holds for all $x \in R$, the mapping $f$ is said to be commuting on $R$. The history of commuting and centralizing mappings goes back to 1955 when Divinsky [6] proved that a simple Artinian ring is commutative if it has a commuting nontrivial automorphism. Two years later Posner [9] has proved that the existence of a nonzero centralizing derivation on a prime ring forces the ring

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to be commutative. Luh [7] generalized the Divinsky result, we have just mentioned above, to arbitrary prime rings. Mayne [8] has proved that in case there exists a nontrivial centralizing automorphism on a prime ring, then the ring is commutative. A result of Brešar [2], which states that every additive commuting mapping of prime ring $R$ is of the form $x \mapsto \lambda x + \zeta(x)$ where $\lambda$ is an element of $C$ and $\zeta : R \to C$ is an additive mapping, should be mentioned. A mapping $f : R \to R$ is called skew-centralizing if $f(x)x + xf(x) \in Z(R)$ holds for all $x \in R$; in particular, if $f(x)x + xf(x) = 0$ holds for all $x \in R$, then it is called skew-commuting on $R$. Brešar [3] has proved that if $R$ is a 2-torsion free semiprime ring and $f : R \to R$ is an additive skew-commuting mapping on $R$, then $f = 0$.

## 2. Results

Let us start with the following result proved by Brešar [4].

THEOREM A ([4], Corollary 4.9). *Let $R$ be a prime ring and let $f, g : R \to R$ be additive mappings satisfying the relation*

$$f(x)x + xg(x) = 0 \tag{1}$$

*for all $x \in R$. In this case there exist $a \in Q$ and an additive mapping $\varsigma : R \to C$ such that $f(x) = xa + \varsigma(x)$, $g(x) = -ax - \varsigma(x)$ holds for all $x \in R$.*

Let us point out that the identity (1) generalizes both concepts, the concept of commuting and the concept of skew-commuting mappings.

Theorem A was the inspiration for Theorem 1 below.

THEOREM 1. *Let $R$ be a 2-torsion free semiprime ring. Suppose there exist automorphisms $T, S : R \to R$ such that $T(x)x = xS(x)$ holds for all $x \in R$. In this case $T - I$ maps $R$ into $Z(R)$ and $T = S$.*

For the proof of Theorem 1 we need the result below.

PROPOSITION. *Let $R$ be a 2−torsion free semiprime ring and let $T : R \to R$ be an automorphism. If either $x\,[T(x), x] = 0$ or $[T(x), x]\,x = 0$ holds for all $x \in R$ then $T - I$ maps $R$ into $Z(R)$.*

For the proof of Proposition we shall need two lemmas. Lemma 2 will also be needed in the proof of Theorem 2.

LEMMA 1 ([11], Lemma 1.3). *Let $R$ be a semiprime ring. Suppose there exists $a \in R$ such that $a\,[x, y] = 0$ holds for all pairs $x, y \in R$. In this case $a \in Z(R)$.*

LEMMA 2 ([10], Lemma 3). *Let $R$ be a semiprime ring and let $f : R \to R$ be an additive mapping. If either $f(x)x = 0$ or $xf(x) = 0$ holds for all $x \in R$, then $f = 0$.*

Brešar and Hvala [5] have proved the following result.

THEOREM B. *Let $R$ be a prime ring of characteristic different from two and let $f : R \to R$ be an additive mapping satisfying the relation $f(x)^2 = x^2$ for all $x \in R$, then either $f = I$ or $f = -I$. The result, we have just mentioned, was the inspiration for our second theorem.*

THEOREM 2. *Let $T, S : R \to R$ be automorphisms where $R$ is a 2-torsion free semiprime ring. Suppose that $T(x)S(x) = x^2$ holds for all $x \in R$. In this case $T = S = I$.*

## 3. Proofs

Proof of Proposition. The linearization of the relation below

(2) $$x\,[T(x), x] = 0, \qquad x \in R.$$

gives

(3) $$x\,[T(y), y] + y\,[T(x), x] + x([T(x), y] + [T(y), x]) + y([T(y), x] + [T(x), y]) = 0, \; x, y \in R.$$

Putting in the relation (3) $-x$ for $x$ and comparing the relation so obtained with the relation (3) we obtain

(4) $$x\,[T(x), y] + x\,[T(y), x] + y\,[T(x), x] = 0 \quad x, y \in R.$$

The substitution $xy$ for $y$ in the above relation gives

$$0 = x\,[T(x), xy] + x\,[T(x)T(y), x] + xy\,[T(x), x] = = x^2\,[T(x), y] + xT(x)\,[T(y), x] + xy\,[T(x), x]\,, \qquad x, y \in R.$$

We have therefore

$$x^2\,[T(x), y] + xT(x)\,[T(y), x] + xy\,[T(x), x] = 0, \qquad x, y \in R.$$

Multiplying the relation (4) from the left side by $x$ and subtracting the relation so obtained from the above relation we obtain $xD(x)\,[T(y), x] = 0$, $x, y \in R$, where $D(x)$ denotes $T(x) - x$, which means that we have

(5) $$xD(x)\,[y, x] = 0 \quad x, y \in R.$$

Putting in the above relation $yz$ for $y$, we arrive at

(6) $$xD(x)y\,[z, x] = 0 \quad x, y, z \in R.$$

From the above relation one obtains easily

$$xD(x)y\,[z, w] + xD(w)y\,[z, x] + wD(x)y\,[z, x] = 0 \quad x, y, z, w \in R.$$

Putting in the above relation $[z, w]\,yxD(x)$ for $y$ and applying the relation (5) we obtain $(xD(x)\,[z, w])y(xD(x)\,[z, w]) = 0$, $x, y, z, w \in R$ whence it follows

$$xD(x)\,[z, w] = 0 \quad x, z, w \in R.$$

For fixed $z$ and $w$ we have an additive mapping $x \mapsto xD(x)[z,w]$ on $R$. Therefore, from the above relation it follows according to Lemma 2 that $D(x)[z,w] = 0$, $x, z, w \in R$ which makes it possible to conclude, according to Lemma 1, that $D(x) \in Z(R)$ for any $x \in R$. In other words, $T - I$ maps $R$ into $Z(R)$. Similarly, one obtains that $T - I$ maps $R$ into $Z(R)$ also in case $[T(x), x]\, x = 0$ holds for all $x \in R$. The proof of Proposition is complete.

Proof of Theorem 1. We have the relation

$$T(x)x - xS(x) = 0, \quad x \in R. \tag{7}$$

From the relation (7) one obtains $0 = [T(x)x - xS(x), x] = [T(x), x]\, x - x\,[S(x), x]$. We have therefore

$$[T(x), x]\, x - x\,[S(x), x] = 0, \quad x \in R. \tag{8}$$

Linearization of the relation (7) gives

$$T(x)y + T(y)x - xS(y) - yS(x) = 0, \quad x, y \in R. \tag{9}$$

Putting $yx$ for $y$ in the above relation we obtain

$$T(x)yx + T(y)T(x)x - xS(y)S(x) - yxS(x) = 0, \quad x, y \in R. \tag{10}$$

Right multiplication of the relation (9) by $S(x)$ gives

$$T(x)yS(x) + T(y)xS(x) - xS(y)S(x) - yS(x)^2 = 0, \quad x, y \in R. \tag{11}$$

Subtracting (10) from (11) and applying (7) we obtain

$$T(x)y\,(S(x) - x) - y\,(S(x) - x)\,S(x) = 0, \quad x, y \in R. \tag{12}$$

The substitution $xy$ for $y$ in the above relation gives

$$T(x)xy\,(S(x) - x) - xy\,(S(x) - x)\,S(x) = 0, \quad x, y \in R. \tag{13}$$

Left multiplication of the relation (12) by $x$ leads to

$$xT(x)y\,(S(x) - x) - xy\,(S(x) - x)\,S(x) = 0, \quad x, y \in R. \tag{14}$$

Subtracting (14) from (13) we obtain

$$[T(x), x]\, y\,(S(x) - x) = 0, \quad x, y \in R. \tag{15}$$

The substitution $yx$ for $y$ in the above relation gives

$$[T(x), x]\, y \left(xS(x) - x^2\right) = 0, \quad x, y \in R. \tag{16}$$

Right multiplication of the relation (15) by $x$ gives

$$[T(x), x]\, y \left(S(x)x - x^2\right) = 0, \quad x, y \in R. \tag{17}$$

Subtracting (16) from (17) we arrive at

$$[T(x), x]\, y\,[S(x), x] = 0, \quad x, y \in R. \tag{18}$$

Putting in the above relation first $xyx$ for $y$ and using (8) we obtain

$$[T(x), x]\, xy\,[T(x), x]\, x = 0, \quad x, y \in R.$$

Since $R$ is semiprime it follows from the above relation

$$[T(x), x]\, x = 0, \qquad x \in R.$$

Using (8) again we obtain $x\,[S(x), x] = 0$, $x \in R$ as well.

Applying Proposition one can conclude that both mappings $T - I$ and $S - I$ map $R$ into $Z(R)$. In a special case

$$(T(x) - x)\, x = x\,(T(x) - x)\,, \quad x \in R.$$

It follows from the above relation $T(x)x = xT(x) = xS(x)$, $x \in R$. From this relation one obtains

$$x(T(x) - S(x)) = 0, \quad x \in R.$$

Applying Lemma 2 one can conclude that $S = T$, which completes the proof of the theorem.

Proof Theorem 2. We have the relation

(19) $$T(x)S(x) = x^2, \quad x \in R.$$

From the relation above one obtains

(20) $$T(x)S(y) + T(y)S(x) = xy + yx, \quad x, y \in R.$$

Replacing $y$ with $yx$ in the above relation we obtain

(21) $$T(x)S(y)S(x) + T(y)T(x)S(x) = xyx + yx^2, \quad x, y \in R.$$

Using the relations (19) and (20) we obtain from the above relation

$$(xy + yx - T(y)S(x))\, S(x) + T(y)x^2 = xyx + yx^2, \quad x, y \in R.$$

Rearranging the above relation gives

(22) $$xyD(x) + yxD(x) - T(y)\left(S(x)^2 - x^2\right) = 0, \quad x, y \in R,$$

where $D(x)$ stands for $S(x) - x$. In particular for $y = x$ the above relation reduces to

(23) $$2x^2D(x) - T(x)\left(S(x)^2 - x^2\right) = 0, \quad x \in R.$$

Putting $xy$ for $y$ in the relation (22) we obtain

(24) $$x^2yD(x) + xyxD(x) - T(x)T(y)\left(S(x)^2 - x^2\right) = 0, \quad x, y \in R.$$

Multiplying the relation (22) from the left side by $x$ and subtracting the relation so obtained from the above relation we obtain $G(x)T(y)(S(x)^2 - x^2) = 0$, $x, y \in R$, where $G(x)$ denotes $T(x) - x$, which means that we have

$$G(x)y\left(S(x)^2 - x^2\right) = 0, \quad x, y \in R.$$

Putting in the above relation $S(x)yT(x)$ for $y$ we obtain using relations (19) and (23) $xD(x)yx^2D(x) = 0$, $x, y \in R$, and then $x^2D(x)yx^2D(x) = 0$, $x, y \in R$, whence it follows

(25) $$x^2D(x) = 0, \quad x \in R.$$

Because of the relation above the relation (23) reduces to

(26) $$T(x)\left(S(x)^2 - x^2\right) = 0, \quad x \in R.$$

Putting $yx$ for $y$ in the relation (22) we obtain according to (25) and (26)

$$0 = xyxD(x) + yx^2D(x) - T(y)T(x)\left(S(x)^2 - x^2\right) = xyxD(x) \quad x, y \in R.$$

Thus we have $xyxD(x) = 0$, $x, y \in R$, which means that $xD(x)yxD(x) = 0$, $x, y \in R$, whence it follows $xD(x) = 0$, $x \in R$.

From the relation above it follows according to Lemma 2 $D(x) = 0$, $x \in R$. In other words, $S = I$. Now the relation (19) reduces to $T(x)x = x^2$, $x \in R$, which means that $G(x)x = 0$, $x \in R$, whence it follows using Lemma 2 again that $G = 0$. We have therefore $T = I$, which completes the proof of the theorem.

## References

[1] K. I. Beidar, W. S. Martindale III, A. V. Mikhalev, *Rings with Generalized Identities*, Marcel Dekker, Inc. New York 1996.

[2] M. Brešar, *Centralizing mappings and derivations in prime rings*, J. Algebra 156 (1993), 385–394.

[3] M. Brešar, *On skew-commuting mappings of rings*, Bull. Austral. Math. Soc. 47 (1993), 291–296.

[4] M. Brešar, *On generalized biderivations and related maps*, J. Algebra 172 (1995), 764–786.

[5] M. Brešar, B. Hvala, *On additive maps of prime rings*, Bull. Austral. Math. Soc. 51 (1995), 377–381.

[6] I. N. Divinsky, *On commuting automorphisms of rings*, Trans. Roy. Canada Sect. III, 49 (1955), 19–22.

[7] J. Luh, *A note on commuting automorphisms of rings*, Amer. Math. Monthly 77 (1970), 61–62.

[8] J. H. Mayne, *Centralizing automorphisms of prime rings*, Canad. Mat. Bull. 19 (1976), 113–115.

[9] E. C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. 8 (1957), 1093–1100.

[10] J. Vukman, *Identities with derivations and automorphisms on semiprime rings*, Int. J. Math. Math. Sci. 7 (2005), 1031–1038.

[11] B. Zalar, *On centralizers of semiprime rings*, Comment. Math. Univ. Carolinae 32 (1991), 609–614.

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