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ON EFFECTIVE NON-AMPLE DIVISORS

Abstract. Let X be a smooth complete algebraic variety. Let $f : X \to Y$ be a morphism from X to another algebraic variety Y, which is neither finite nor constant. Then X admits an effective non-ample divisor. In particular, if X is a smooth complete variety with Picard number one, then every non-constant morphism $f : X \to Y$ is finite and the variety $f(X)$ is projective.

1. Main result

Let X be a smooth complete algebraic variety defined over algebraically closed field k (of arbitrary characteristic). In general we can find algebraic morphisms $f : X \to Y$, which are far from being finite. In particular, even if X is a projective variety, then it is possible that the variety $f(X)$ is not projective. J. Włodarczyk noted (using theory of algebraic groups) that for $X = \mathbb{P}^N$ a morphism $f : X \to Y$ is either finite or constant, in particular (what is interesting here) the variety $f(X)$ is always projective. The aim of this simple note is to generalize this result to the class of all smooth varieties with Picard number one. We start with the following:

THEOREM 1. *Let X be a smooth complete algebraic variety. Let $f : X \to Y$ be a morphism from X to another algebraic variety Y, which is neither finite nor constant. Then X admits an effective, non-ample divisor.*

Proof. We can assume that $f : X \to Y$ is a surjective morphism and Y is not a point. Since the variety X is proper and the morphism f is not finite, there exists a point $y \in Y$ such that the fiber $L = f^{-1}(y)$ has a positive dimension (see [2], Theorem 2.27, p. 142). Let U be an affine open neighborhood of y. Let $R \subset U$ be a non-zero effective divisor in U, which

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is described by a global function $r \in k[U]$ and which does not contain the point y (to construct R it is enough to embed U in some affine space k^N and now to take a sufficiently general hyperplane section). Now let us consider a rational function $f^*(r) \in k(X)$. This function is regular on the set $f^{-1}(U)$ and non-zero on the fiber L. This means that the fiber L is disjoint with the support of a divisor $(f^*(r))$. Note that $H :=$ Supp $(f^*(r))$ is a non-zero effective divisor. The divisor H cannot be ample, because its support is disjoint with a complete subvariety L of positive dimension. ∎

Now we pass to some application of Theorem 1. Let X be an algebraic variety. Let $\equiv$ denote the numerical equivalence (two Cartier divisors D, D' are called numerically equivalent if $D.C = D'.C$ for every curve $C \subset X$). Let $N(X) = Pic(X)/\equiv$. The Theorem of the Base of Néron-Severi asserts that $N(X)$ is of finite rank. By the Picard number of X we mean the rank of $N(X)$. In this note we will deal mainly with varieties with Picard number one. The simplest examples of such varieties are: projective spaces $\mathbb{P}^n$, Grassmannians $G(k,n)$, complete intersections $X \subset \mathbb{P}^n$, where dim $X \geq 3$. First we recall the following result of Kleiman [3] (for the sake of a completeness we give a simple proof):

LEMMA 2. *Let X be a smooth complete algebraic variety with* rank $N(X) = 1$. *Then X is a projective variety.*

Proof. First of all, let us note that every effective divisor has infinite order in $N(X)$. Let us fix an effective divisor $D \in N(X)$. For every effective divisor $Z \in N(X)$, there are numbers $a, b >> 0$, such that $aZ = bD$.

Let $U_1, ..., U_r$ be a finite covering of X by affine open subsets. For every $i = 1, ..., r$, there is a number n_i such that $U_i \subset k^{n_i}$. In particular we have birational mappings $f_i : X - \rightarrow \mathbb{P}^{n_i}$, $i = 1, ..., r$. Let $Z_i = (f_i)^*(O(1))$. Then Z_i is effective and there are positive numbers a_i, b_i, such that $a_i Z_i = b_i D$.

Take $\alpha = \min_{i=1,\dots,r}\{a_i/b_i\}$. For every integral curve $C \subset X$ let $m_P(C)$ be the multiplicity of a point P on C and let $m(C) = \sup_{P \in C}\{m_P(C)\}$. Now we have $(D.C) \geq \alpha m(C)$ (we check this locally in U_i) and by the Seshadri criterion (see [1], Th.7.1, p.37), we obtain that the divisor D is ample. Hence the variety X is projective. ∎

Now we are in a position to prove our main result:

THEOREM 3. *Let X be a smooth complete algebraic variety with* rank $N(X) = 1$. *Let $f : X \rightarrow Y$ be a surjective morphism from X onto another algebraic variety Y. Then either Y is a point or* dim $Y =$ dim X *and f is a finite morphism.*

Proof. By Lemma 2 the variety X is projective. In particular it contains an effective ample divisor D. Since rank $N(X) = 1$ we have that every effective divisor on X is ample (for every effective divisor $Z \in N(X)$, there are numbers $a, b >> 0$, such that $aZ = bD$). Now the result follows directly from Theorem 1. ∎

COROLLARY 4. *Let X be a smooth complete algebraic variety with* rank $N(X) = 1$. *Let $f : X \to Y$ be a surjective morphism from X onto another algebraic variety Y. Then the variety Y is projective.*

Proof. From Lemma 2 we have that X is projective and every effective divisor on X is ample (see the proof of Theorem 3). By Theorem 3 either the morphism f is finite or Y is a point. In the second case there is nothing to prove. In the first case Corollary 4 follows from [1], Proposition 4.4, p.25. ∎

REMARK 5. (a) There is an example of a complete normal variety X with $Pic(X) = \mathbb{Z}$, which is not projective.

(b) There is a normal complete algebraic variety with $Pic(X) = \mathbb{Z}$ and a morphism $f : X \to Y$ from X onto another algebraic variety Y, which is neither finite nor constant.

At the end of these note we give an application of our result. Lazarsfeld (see [4]) proved in 1984 the following theorem:

THEOREM 6. *Let X be a smooth projective variety of dimension n. Assume that there is a surjective and separable morphism $p : \mathbb{P}^n \to X$. Then $X \cong \mathbb{P}^n$.*

From our result it follows that the assumption about projectivity of X can be removed.

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