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ON WEAKLY PRIMAL IDEALS (I)

Abstract. Let R be a commutative ring with non-zero identity. We say that an element $a \in R$ is weakly prime to an ideal I of R if $0 \neq ra \in I$ $(r \in R)$ implies that $r \in I$. If I is a proper ideal of R and $w(I)$ is the set of elements of R that are not weakly prime to I, then we define I to be weakly primal if the set $P = w(I) \cup \{0\}$ form an ideal. In this case we also say that I is a P-weakly primal ideal. This paper is devoted to study the weakly primal ideals of a commutative ring. The relationship among the families of weakly prime ideals, primal ideals, and weakly primal ideals of a ring R is considered.

1. Introduction

In this paper all rings are commutative rings with non-zero identity. Primal ideals in a commutative ring with non-zero identity have been introduced and studied by Ladislas Fuchs in [3] (also see [4]). Weakly prime ideals in a commutative ring have been introduced and studied by D. D. Anderson and E. Smith in [1]. Here we study the weakly primal ideals of a commutative ring. The weakly primal, weakly prime and primal ideals are different concepts. In this paper we consider the relationship among the families of weakly prime ideals, primal ideals and weakly primal ideals of a commutative ring R. A number of results concerning weakly primal ideals and examples of weakly primal ideals are given. We shortly summarize the content of the paper. In Theorem 1, we give two other characterizations of weakly primal ideals. We observe in Theorem 3 that every weakly prime ideal is weakly primal, but a weakly primal ideal need not be weakly prime (see Example 1). In Proposition 4, we prove that if I is a P-weakly primal ideal of R, then P is weakly prime. Using these, we observe in Theorem 5 that if I is a P-weakly primal ideal of R that P is not prime, then $I^2 = 0$, $I\sqrt{0} = 0$ and $\sqrt{0} = \sqrt{I}$. We also prove, in section 2, see Theorem 12, that there exists a one-to-one correspondence between the P-weakly primal ideals of R and $S^{-1}P$-weakly primal ideals of $S^{-1}R$.

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A weakly primal ideal need not be primal (see sec. 2), but we prove in Theorem 14, every non-zero weakly primal ideal of a decomposable commutative ring is primal. We also prove, in section 3, Theorem 16, that if I is a weakly primal ideal of a commutative ring R that is not primal, then $I^2 = 0$. A primal ideal need not be weakly primal (see Example 2), but we prove in Proposition 18, an ideal over an integral domain is primal if and only if it is weakly primal. Using this, we observe in Theorem 20 that in a prufer domain of finite character every non-zero ideal is the intersection of a finite number of weakly primal ideals.

Now we define the concepts that we will need. An ideal I of a ring R is called primal if the elements of R that are not prime to I form an ideal: this ideal is always a prime ideal, called the adjoint ideal P of I (see [3]). In this case we also say that I is a P-primal ideal. Here an element $r \in R$ is called prime to I if $rs \in I$ ($s \in R$) implies $s \in I$. We define a proper ideal P of R to be weakly prime if $0 \neq ab \in P$ implies $a \in P$ or $b \in P$ (see [1]). An ideal I of R is said to be irreducible if I is not the intersection of two ideals of R that properly contain it. An integral domain R is said to be finite character if every non-zero element is contained but in finite number of maximal ideals. If I and J are ideals of R, the ideal $\{r \in R : rI \subseteq J\}$ will be denoted by $(J :_R I)$. Then $(0 :_R I)$ is the annihilator of I. A regular element in a ring R is any non-zero-divisor, i.e., any element $a \in R$ such that $(0 :_R a) = 0$. Let N be an R-submodule of M. Then N is pure in M if any finite system of equations over N which is solvable in M also solvable in N. So if N is pure in M, then $IN = N \cap IM$ for each ideal I of R. An R-module is absolutely pure if it is pure in every module that contains it as a submodule. An element $a \in R$ is said to be regular if there exists $b \in R$ such that $a = a^2b$, and R is said to be regular if each of its elements is regular. An important property of regular rings is that every module is absolutely pure (see [5]).

2. Weakly prime ideals

We first recall the definition of weakly primal ideals of arbitrary commutative rings R with non-zero identity as introduced in Abstract.

Let I be an ideal of R. An element $a \in R$ is called weakly prime to I if $0 \neq ra \in I$ ($r \in R$) implies that $r \in I$. 0 is always weakly prime to I. Also, every element prime to I is weakly prime to I, but the converse is not true. For example, let $R = Z/24Z$ and consider the ideal $I = 8Z/24Z$. Clearly, $\bar{6}$ is weakly prime to I, but it is not prime to I (since $\bar{12}.\bar{6} = 0 \in I$ with $\bar{12} \notin I$). A proper ideal I of R is called weakly primal if the set $P = w(I) \cup \{0\}$ form an ideal: this ideal is called the weakly adjoint ideal P of I. Let R be a commutative ring which is not an integral domain. Then 0 is a 0-weakly primal ideal of R (by definition), so a weakly primal ideal need not be primal.

Let R be a commutative ring, I an ideal of R and A a subset of R. We say that A satisfies $(*)$ if A is exactly the set of elements of R that are not weakly prime to I. Our starting point is to give two other characterizations of weakly primal ideals:

THEOREM 1. *Let I and P be proper ideals of a commutative ring R. Then the following statements are equivalent.*

(i) I is a P-weakly primal ideal of R.

(ii) For $x \notin P - \{0\}$, $(I :_R x) = I \cup (0 :_R x)$, and for $0 \neq x \in P$, $I \cup (0 :_R x) \subsetneqq (I :_R x)$.

(iii) For $x \notin P - \{0\}$, $(I :_R x) = I$ or $(I :_R x) = (0 :_R x)$, and for $0 \neq x \in P$, $I \subsetneqq (I :_R x)$ and $(0 :_R x) \subsetneqq (I :_R x)$.

Proof. $(i) \Rightarrow (ii)$ Let I be a P-weakly primal ideal of R. Then $P - \{0\}$ satisfies $(*)$. First suppose that $x \notin P - \{0\}$, so x is weakly prime to I. Let $r \in (I :_R x)$. If $rx \neq 0$, then x weakly prime to I gives $r \in I$. If $rx = 0$, then $r \in (0 :_R x)$. So $(I :_R x) \subseteq I \cup (0 :_R x)$. As the reverse containment holds for any ideal I, we have equality.

Next, assume that $0 \neq x \in P$, so x is not weakly prime to I; hence there exists $r \in R - I$ such that $0 \neq rx \in I$. thus $r \in (I :_R x) - (I \cup (0 :_R x))$, as required.

$(ii) \Rightarrow (iii)$ Let $x \notin P - \{0\}$. It is well known that if an ideal is the union of two ideals, then it is equal to one of them. Moreover, if $0 \neq x \in P$, then by (ii) we have $I \subsetneqq (I :_R x)$ and $(0 :_R x) \subsetneqq (I :_R x)$.

$(iii) \Rightarrow (i)$ By (iii), $P - \{0\}$ satisfies $(*)$. Thus I is P-weakly primal. ∎

LEMMA 2. *Let I be a proper ideal of a commutative ring R. Then the following hold:*

(i) If I is a P-weakly primal ideal of R, then $I \subseteq P$.

(ii) If I is a 0-weakly primal ideal of R, then $I = 0$.

Proof. (i) Let $0 \neq a \in I$. As $0 \neq 1_R a \in I$ with $1_R \notin I$, we get a is not weakly prime to I; hence $I \subseteq P$.

(ii) This follows from (i). ∎

EXAMPLE 1. Let $R = Z/8Z$ and consider the ideals $I = 4Z/8Z$ and $P = 2Z/8Z$ of R. Then I is not weakly prime ideal of R since $0 \neq \bar{2}.\bar{2} \in I$, but $\bar{2} \notin I$ (see [1]). Now we show that I is a P-weakly primal ideal of R. It is enough to show that $P - \{0\}$ satisfies $(*)$. Let $\bar{0} \neq \bar{a} = 2k + 8Z \in P$. If k is an odd number, then $\bar{0} \neq \bar{2}.\bar{a} \in I$, but $\bar{2} \notin I$, and if k is an even number, then $\bar{0} \neq \bar{a}.\bar{1} \in I$, but $\bar{1} \notin I$; hence $\bar{a}$ is not weakly prime to I. On the other hand, if $\bar{b} = c + 8Z \notin P$, then c is an odd number. If $\bar{0} \neq \bar{b}.\bar{m} \in I$ for some $\bar{m} = s + 8Z \in R$, then $4 \mid cs$, so $4 \mid s$ since $(4, c) = 1$; hence $\bar{m} \in I$. Thus I is a P-weakly primal ideal of R. Note that this example provides an instance of an ideal which is weakly primal but not weakly prime.

Theorem 3 and Proposition 4 (see below) are very important facts for us, and will be much reinforced in the remaining section 2 and the section 3.

THEOREM 3. *Let R be a commutative ring. Then every weakly prime ideal of R is weakly primal.*

Proof. let P be a weakly prime ideal of R. We can assume that $P \neq 0$. It suffices to show that $P - \{0\}$ satisfies $(*)$. Let $0 \neq a \in P$. Then as $0 \neq a = 1_R a \in P$ with $1_R \notin P$, we get a is not weakly prime to P. On the other hand, every element $a \notin P - \{0\}$ is weakly prime to P by [1, Theorem 3]. Thus P is weakly primal. ■

PROPOSITION 4. *If I is a P-weakly primal ideal of a commutative ring R, then P is a weakly prime ideal of R.*

Proof. Suppose that $a, b \notin P$; we show that either $ab = 0$ or $ab \notin P$. Assume that $ab \neq 0$ and let $0 \neq rab \in I$ for some $r \in R$. Then by Theorem 1 we have $0 \neq ra \in (I :_R b) = I \cup (0 :_R b)$ where $ra \notin (0 :_R b)$; hence $0 \neq ra \in I$. It then follows from Theorem 1 that $0 \neq r \in (I :_R a) = I \cup (0 :_R a)$, so $r \in I$; hence ab is weakly prime to I, as required. ■

By [1, Theorem 8] and Theorem 3 we have the following corollary:

COROLLARY 1. *(i) Let (R, P) be a quasilocal ring with $P^2 = 0$. Then every proper ideal of R is weakly primal.*

(ii) If $R = F_1 \times F_2$ where F_1 and F_2 are fields, then every proper ideal of R is weakly primal.

(iii) If P and P' are maximal ideals of a commutative ring R with $P \neq P'$, then PP' is weakly primal.

Compare the next result with [1, Theorem 4, Theorem 1 and Corollary 5].

THEOREM 5. *Let R be a commutative ring, I a P-weakly primal ideal of R and J a Q-weakly primal ideal of R. Then the following hold:*

(i) If P is not prime ideal of R, then $I^2 = 0$, $IP = 0$, $I\sqrt{0} = 0$ and $\sqrt{I} = \sqrt{0}$.

(ii) If P and Q are not prime ideals of R, then $IJ = 0$.

Proof. (i) By Lemma 2, Proposition 4 and [1, Theorem 1 and Theorem 3], we get $I^2 \subseteq P^2 = 0$, $IP \subseteq P^2 = 0$ and $I\sqrt{0} \subseteq P\sqrt{0} = 0$. Finally, since $\sqrt{0} \subseteq \sqrt{I}$ is trivial, we will prove the reverse inclusion. As $I^2 = 0$, we get $I \subseteq \sqrt{0}$; hence $\sqrt{I} \subseteq \sqrt{0}$, as needed.

(ii) By Lemma 2, Propositin 4 and [1, Corollary 5], we have $IJ \subseteq PQ = 0$. ■

Now we state and prove a version of Nakayama's lemma.

THEOREM 6. *Let I be a P-weakly primal ideal of a commutative ring R that P is not a prime ideal of R. Then the following hold:*

(i) $I \subseteq J(R)$, where $J(R)$ is the Jacobson radical of R.

(ii) If M is an R-module and $IM = M$, then $M = 0$.

(iii) If M is an R-module and N is a submodule of M such that $IM+N = M$, then $M = N$.

Proof. (i) By Proposition 4, P is a weakly prime ideal of R. Then $I \subseteq J(R)$ by Lemma 2, Theorem 5(i) and [3, Theorem 2.12(i)].

(ii) Since $IM = M$, we have $M = IM = I^2M = 0$ by Theorem 5(i).

(iii) This follows from (ii). ∎

COROLLARY 2. *Let R be a commutative ring. Then the following hold:*

(i) If I is a pure P-weakly primal ideal of R that P is not a prime ideal of R, then $I = 0$.

(ii) If R is regular, then the only weakly primal ideals of R that the weakly adjoint ideals of them are not prime can only be 0.

Proof. (i) Since I is a pure ideal of R, $I^2 = I$, so $I = 0$ by Theorem 5(i).

(ii) This follows from (i) Since every ideal of a regular ring is pure. ∎

THEOREM 7. *let I be a proper ideal of a commutative ring R, and let J be a weakly prime ideal of R with $J \subseteq I$. Then I is a weakly primal ideal of R if and only if I/J is a weakly primal ideal of R/J. In particular, there is a bijective correspondence between the weakly primal ideals of R containing J and the weakly primal ideals of R/J.*

Proof. First suppose that I is a P-weakly primal ideal ideal of R with $J \subseteq I$. Then by Lemma 2 , Proposition 4 and [3, Proposition 2.10(i)], P/J is a weakly prime ideal of R/J. It suffices to show that $P/J - \{0\}$ is exactly the set of elements of R/J that are not weakly prime to I/J.

Let $0 \neq a + J \in P/J$. Then $a \neq 0$ is not weakly prime to I; hence there exists $r \in R - I$ such that $0 \neq ra \in I$. If $0 \neq ra \in J$, then J weakly prime gives $r \in J$ which is a contradiction since $r \notin I$. It follows that $(r + J)(a + J) \neq 0$, so $0 \neq (r + J)(a + J) \in I/J$ with $r + J \notin I/J$ gives $a + J$ is not weakly prime to I/J. Now assume that $b + J$ is not weakly prime to I/J. Then $b + J \neq 0$ and there exists $c + J \in R/J - I/J$ such that $0 \neq cb + J \in I/J$; hence $cb \in I$ with $c \notin I$. So $b \neq 0$ is not weakly prime to I. Therefore, $b + J \in P/J - \{0\}$, and the proof is complete. ∎

Second, suppose that I/J is a P/J-weakly primal ideal of R/J; we show that I is a P-weakly primal ideal of R. By Proposition 4 and [3, Proposition 2.10(ii)], P is a weakly prime ideal of R. It is enough to show that $P - \{0\}$ satisfies $(*)$.

Let $0 \neq a \in P$. By Lemma 2, we can assume that $a \notin J$. As J is a weakly prime ideal and $0 \neq a + J \in P/J$, there exists $r + J \in R/J - I/J$ such that $0 \neq (a + J)(r + J) \in I/J$; hence $0 \neq ra \in I$ with $r \notin I$. Thus a is not weakly prime to I. Now assume that a is not weakly prime to I (so $a \neq 0$); we show that $a \in P$. We can assume that $a \notin I$. Then there is an element $r \in R-I$ such that $0 \neq ra \in I$. Therefore, J weakly prime ideal gives $0 \neq ra+J = (r+J)(a+J) \in I/J$ with $r+J \notin I/J$; hence $a+J \in P/J-\{0\}$ since I/J is P/J-weakly primal. Thus $a \in P$, as required. ∎

For the remainder of this section we continue our program of studying of weakly primal ideal of rings of fractions.

LEMMA 8. *Let S be a multiplicatively closed subset of a commutative ring R. Then the following hold:*

(i) If I is a P-weakly primal ideal of R with $P \cap S = \emptyset$ and $0 \neq a/s \in S^{-1}I$, then $a \in I$.

(ii) If Q is a weakly prime ideal of R with $Q \cap S = \emptyset$ and $0 \neq a/s \in S^{-1}Q$, then $a \in Q$.

Proof. (i) Assume that $0 \neq a/s \in S^{-1}I$ but $a \notin I$. Then $a/s = r/t$ for some $r \in I$ and $t \in S$, so there exists $u \in S$ such that $0 \neq uta = usr \in I$ with $a \notin I$; hence $ut \in S$ is not weakly prime to I which is a contradiction. Thus $a \in I$.

(ii) This follows from (i) and Theorem 3. ∎

PROPOSITION 9. *Let S be a multiplicatively closed subset of a commutative ring R which consists of regular elements and I a P-weakly primal ideal of R such that $P \cap S = \emptyset$. Then the following hold:*

(i) $S^{-1}I$ is a $S^{-1}P$-weakly primal ideal of $S^{-1}R$.

(ii) $I = (S^{-1}I) \cap R$.

Proof. (i) It suffices to show that $S^{-1}P - \{0\}$ satisfies $(*)$. First suppose that $0 \neq a/s \in S^{-1}P$, so $0 \neq a \in P$ by Lemma 8; hence there exists $r \in R-I$ such that $0 \neq ra \in I$. As $(ra)/s \neq 0$ (otherwise there is an element $t \in S$ such that $tra = 0$ which is a contradiction), we get $0 \neq (r/1)(a/s) \in S^{-1}I$ where $r/1 \notin S^{-1}I$ by Lemma 8; hence a/s is not weakly prime to $S^{-1}I$. On the other hand, assume that a/s is not weakly prime to $S^{-1}I$. Then there exists $r/t \in S^{-1}R - S^{-1}I$ such that $0 \neq (a/s)(r/t) \in S^{-1}I$, so $0 \neq ra \in I$ with $r \notin I$ by Lemma 8; hence $0 \neq a \in P$. Thus $a/s \in S^{-1}P - \{0\}$, as needed.

(ii) Since $I \subseteq (S^{-1}I) \cap R$ is clear, we will prove the reverse inclusion. Let $a \in (S^{-1}I) \cap R$. Then $a/1 \in S^{-1}I$, so $a \in I$ by Lemma 8, as needed. ∎

PROPOSITION 10. *Let S be a multiplicatively closed subset of a commutative* *ring R which consists of regular elements and Q a weakly prime ideal of $S^{-1}R$. Then $Q \cap R$ is a weakly prime ideal of R.*

Proof. Suppose that $0 \neq ab \in Q \cap R$, so $(ab)/1 \in Q$. If $(ab)/1 = 0$, then $tab = 0$ for some $t \in S$ which is a contradiction. If $(ab)/1 \neq 0$, then Q weakly prime gives $a/1 \in Q$ or $b/1 \in Q$; hence $a \in Q \cap R$ or $b \in Q \cap R$, as required. ∎

PROPOSITION 11. *Let S be a multiplicatively closed subset of a commutative ring R which consists of regular elements. If I is a Q-weakly primal ideal of $S^{-1}R$, then $I \cap R$ is a $Q \cap R$-weakly primal ideal of R.*

Proof. By Proposition 10, $P = Q \cap R$ is a weakly prime ideal of R. It only remains to show that $P - \{0\}$ is exactly the set of elements non-weakly prime to $I \cap R$. First suppose that $0 \neq a \in P$. Then $0 \neq a/1 \in Q$, so there exists $r/s \in S^{-1}R - I$ such that $0 \neq (r/s)(a/1) \in I$; hence $0 \neq ra \in I \cap R$ with $r \notin I \cap R$. It follows that a is not weakly prime to $I \cap R$. Now assume that b is not weakly prime to $I \cap R$. Then there is an element $s \notin I \cap R$ with $0 \neq sb \in I \cap R$. If $(sb)/1 = 0$, then there exists $t \in S$ such that $tsb = 0$ which is a contradiction. So $0 \neq (s/1)(b/1) \in I$ with $s/1 \notin I$ gives $b/1 \in Q$; hence $b \in P$, and the proof is complete. ∎

THEOREM 12. *Let S be a multiplicatively closed subset of a commutative ring R which consists of regular elements, and let P be a weakly prime ideal of R with $P \cap S = \emptyset$. Then there exists a one-to-one correspondence between the P-weakly primal ideals of R and the $S^{-1}P$-weakly primal ideals of $S^{-1}R$.*

Proof. This follows from Propositions 9, 10, 11 and [6, Lemma 5.24]. ∎

3. Primal ideals

Let R be a commutative ring which is not an integral domain. We recall that 0 is a 0-weakly primal ideal of R, but it is not primal. The following example shows that a primal ideal of R need not be weakly primal.

EXAMPLE 2. Let $R = Z/24Z$ and consider the ideal $I = 8Z/24Z$ of R:

(1) We show that I is not a weakly primal ideal of R. Since $\bar{2}.\bar{4} \in I$ with $\bar{2}, \bar{4} \notin I$, then we get $\bar{2}$ and $\bar{4}$ are not weakly prime to I. As $\bar{2} + \bar{4} = \bar{6}$ is weakly prime to I, we obtain I is not a weakly primal ideal of R.

(2) Set $P = 2Z/24Z$. We show that I is a P-primal ideal of R. It is easy to check that every element of P is not prime to I. Conversely, assume that $\bar{a} \notin P$, so $(a, 8) = 1$. If $\bar{a}.\bar{n} \in I$ for some $\bar{n} \in R$, then $8 \mid n$; hence $\bar{n} \in I$. Therefore, P is exactly the set of elements of R which are not prime to I. Thus I is P-primal.

Now we investigate when weakly primal ideal of a commutative ring is primal.

LEMMA 13. *Let $R = R_1 \times R_2$ where each R_i is a commutative ring with identity. Then the following hold:*

(i) If I_1 is a primal ideal of R_1, then $I_1 \times R_2$ is a primal ideal of R.

(ii) If I_2 is a primal ideal of R_2, then $R_1 \times I_2$ is a primal ideal of R.

Proof. Suppose that I_1 is a P_1-primal ideal of R_1, so $P_1 \times R_2$ is a prime ideal of R. It suffices to show that $P_1 \times R_2$ is exactly the set of element of R that are not prime to $I_1 \times R_2$. First suppose that $(a,b) \in P_1 \times R_2$. Then a is not prime to I_1, so there exists $r \in R_1 - I_1$ such that $ra \in I_1$. As $(a,b)(r,1) \in I_1 \times R_2$ with $(r,1) \notin I_1 \times R_2$, we get (a,b) is not prime to $I_1 \times R_2$. Now assume that (a,b) is not prime to $I_1 \times R_2$. Then there is an element $(r,s) \in (R - I_1) \times R_2$ such that $(a,b)(r,s) = (ra, bs) \in I_1 \times R_2$, so $ra \in I_1$ with $r \notin I_1$; hence $(a,b) \in P_1 \times R_2$. Thus $I_1 \times R_2$ is a $P_1 \times R_2$-primal ideal of R.

(ii) This proof is similar to that in case (i) and we omit it. ∎

Compare the next result with [1, Theorem 7].

THEOREM 14. *Let $R = R_1 \times R_2$ where each R_i is a commutative ring with identity. If I is a P-weakly primal ideal of R, then either $I = 0$ or I is primal.*

Proof. We may assume that $I = I_1 \times I_2 \neq 0$. Then by Lemma 2 and Proposition 4, $P \neq 0$ and it is weakly prime. It follows from [1, Theorem 7] that either $P = P_1 \times R_2$ or $P = R_1 \times P_2$ and it is a prime ideal of R. First suppose that $P = P_1 \times R_2$. We show that $I_2 = R_2$. Suppose $(0,0) \neq (a,b) \in I$. Then $(0,0) \neq (a,1)(1,b) \in I$ gives $(a,1) \in I$ (since if $(a,1) \notin I$, then $(1,b)$ is not weakly prime to I, so $(1,b) \in P_1 \times R_2$ which is a contradiction); hence $I_2 = R_2$ and $I = I_1 \times R_2$. By Lemma 13, it is enough to show that I_1 is a P_1-primal ideal of R_1.

We show that P_1 is exactly the set of elements of R_1 that are not prime to I_1. Let $a_1 \in P_1$. We can assume that $a_1 \neq 0$. Then $(0,0) \neq (a_1,0) \in P_1 \times R_2$, so there exists $(r_1,r_2) \in R - I$ such that $(0,0) \neq (r_1,r_2)(a_1,0) \in I$. It follows that $r_1a_1 \in I_1$ with $r_1 \notin I_1$; hence a_1 is not prime to I_1. On the other hand, assume that $b_1 \in R_1$ is not prime to I_1; we show that $b_1 \in P_1$. Then there exists $r_1 \in R_1 - I_1$ with $r_1b_1 \in I_1$, so $(0,0) \neq (r_1,1)(b_1,1) \in I$ with $(r_1,1) \notin I$ gives $(b_1,1)$ is not weakly prime to I; hence $(b_1,1) \in P_1 \times R_2$. The case where $P = R_1 \times P_2$ is similar. ∎

PROPOSITION 15. *Let R be a commutative ring, I a P-weakly primal ideal of R and $I^2 \neq 0$. If P is a prime ideal of R, then I is primal.*

Proof. It is enough to show that P is exactly the set of elements of R that is not prime to I. If $a \in P$, then a is not prime to I. Now assume that a is not prime to I; we show that $a \in P$. Then there is an element $r \in R - I$

such that $ra \in I$. If $0 \neq ra \in I$, then a is not weakly prime to I; hence $a \in P$. So assume that $ra = 0$. First suppose that $aI \neq 0$, say $ar_0 \neq 0$ where $r_0 \in I$. Then $0 \neq a(r + r_0) = ar_0 \in I$ with $r + r_0 \notin I$; hence $a \in P$. So we can assume that $aI = 0$. If $rI \neq 0$, then there exists $c \in I$ such that $rc \neq 0$. Then $0 \neq (a+c)r \in I$ with $r \notin I$, so $a \in P$ by Lemma 2. So we can assume that $rI = 0$. Since $I^2 \neq 0$, there exist $a_0, b_0 \in I$ with $a_0 b_0 \neq 0$. Then $0 \neq a_0 b_0 = (a + a_0)(r + b_0) \in I$ with $r + b_0 \notin I$, so $a + a_0 \in P$. Hence $a \in P$, as required. ∎

Compare the next result with [1, Theorem 1].

THEOREM 16. *Let I be a weakly primal ideal of a commutative ring R that is not primal. Then $I^2 = 0$. In particular, $\sqrt{I} = \sqrt{0}$*

Proof. Let I be a P-weakly primal ideal of R. If P is not prime then $I^2 = 0$ by Theorem 5. If P is prime, then $I^2 = 0$ by Proposition 15. ∎

THEOREM 17. *Let I be a weakly primal ideal of a commutative ring R that is not primal. Then the following hold:*

(i) $I \subseteq J(R)$, where $J(R)$ is the Jacobson radical of R.

(ii) If M is an R-module and $IM = M$, then $M = 0$.

(iii) If M is an R-module and N is a submodule of M such that $IM+N = M$, then $M = N$.

Proof. The proof is similar to that in the [3, Theorem 2.12] since by Theorem 16, $I^2 = 0$. ∎

COROLLARY 3. *Let R be a commutative ring. Then the following hold:*

(i) If I is a pure weakly primal ideal of R that is not primal, then $I = 0$.

(ii) If R is regular, then the only weakly primal ideals of R that are not primal can only be 0.

PROPOSITION 18. *An ideal I over an integral domain R is primal if and only if it is weakly primal.*

Proof. We can assume that $I \neq 0$. Suppose that I is a P-primal ideal of R; we show that I is weakly primal. It suffices to show that $P - \{0\}$ satisfies $(*)$. First suppose that $a \in P - \{0\}$, so I primal ideal gives $I \subsetneq (I :_R a)$. Assume that a is weakly prime to I and let $b \in (I :_R a)$. We can assume that $b \neq 0$. As $0 \neq ab \in I$, we get $b \in I$; hence $I = (I :_R a)$ which is a contradiction. Thus a is not weakly prime to I. On the other hand, if a is not weakly prime to I, then $a \neq 0$ and a is not prime to I; hence $a \in P - \{0\}$, and the proof is complete.

Conversely, assume that I is a P-weakly primal ideal of R. By Proposition 4, P is weakly prime, so P is a prime ideal of R since R is an integral domain. It is enough to show that P is exactly the set of elements of R that are not prime to I. Clearly, 0 is not prime to I and $0 \in P$. Let $0 \neq a \in P$. Then a is not weakly prime to I; hence it is not prime to I. On the other hand, suppose that a ia not prime to I. We can assume that $a \neq 0$. Then there exists $r \in R - I$ such that $0 \neq ra \in I$, so a is not weakly prime to I; hence $a \in P$, as needed. ■

We believe Lemma 19 is known, but we do not know an appropriate reference, so we include a proof.

LEMMA 19. *Let R be a commutative ring. Then every primary ideal is primal.*

Proof. Let I be a P-primary ideal of R. we show that the set of elements of R that are not prime to I is just P. Suppose that $r \in R$ is not prime to I, so there exists $a \in R - I$ such that $ra \in I$; hence I primary gives $r \in P$. Conversely, assume that $b \notin P$ and let $c \in (I :_R b)$, so $cb \in I$; hence $c \in I$. Thus $I = (I :_R b)$, and the proof is complete. ■

THEOREM 20. *Let R be an integral domain. Then the following hold:*

(i) Every primary ideal (so prime ideal) of R is weakly primal.

(ii) Every irreducible ideal of R is weakly primal.

(iii) If R is a valuation domain, then every proper ideal is weakly primal.

(iv) If R is a Prüfer domain, then an ideal is irreducible if and only if it is weakly primal.

(v) If R is a Prüfer domain of finite character, then a non-zero ideal is the intersection of a finite number of weakly primal ideals.

Proof. This follows from Lemma 19, Proposition 18 and [4, Lemma 2.4, Lemma 2.6 and Theorem 3.2]. ■

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