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SUBMONOIDS OF GENERALIZED HYPERSUBSTITUTIONS

Abstract. In this paper we define the operation $\oplus_G$ on the set of all generalized hypersubstitutions and investigate some algebraic-structural properties of the set of all generalized hypersubstitutions and of some submonoids M of the set of all generalized hypersubstitutions, respectively.

1. Introduction

The concept of a hypersubstitution was introduced by K. Denecke, D. Lau, R. Pöschel and D. Schweigert in [2]. In [7], the author and K. Denecke generalized the concept of a hypersubstitution to a generalized hypersubstitution. This is useful for several applications, such as, to solve the hyperunification problem means to decide whether any two given terms t, t' of the same type are hyperunifiable or not. In a corresponding way we can formulate the generalized hyperunification problems. Our results can be used to solve the hyperunification problem and the generalized hyperunification problem for the type $\tau = (2)$ (see [7]). A generalized hypersubstitution is a mapping from the set of all fundamental operations into the set of all terms of the same language which does not necessarily preserve the arity. Generalized hypersubstitutions can be extended to mappings defined on the set of all terms of the given type. This extension is uniquely determined and allows us to define a multiplication denoted by $\circ_G$, on the set $Hyp_G(\tau)$ of all generalized hypersubstitutions of type τ. The multiplication $\circ_G$ is an example of operation on generalized hypersubstitutions. In [6], the author defined the other binary operation $+_G$ on $Hyp_G(\tau)$ and proved that $(Hyp_G(\tau); +_G, \circ_G)$

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is a left seminear-ring but it is not a right seminear-ring. In this paper we will define another operation on $Hyp_G(\tau)$ and give some algebraic-structural properties of the set of all generalized hypersubstitutions and of some submonoids M of the set of all generalized hypersubstitutions.

2. Generalized hypersubstitutions

In this section, we want to briefly recall some basic concepts of generalized hypersubstitutions that will be referred to the following sections. For more details on generalized hypersubstitutions see [8].

Let $\{f_i \mid i \in I\}$ be an indexed set of operation symbols of type τ where f_i is n_i-ary, $n_i \in \mathbb{N} \setminus \{0\}$, and let $W_\tau(X)$ be the set of all terms built up by elements of the alphabet $X = \{x_1, x_2, \ldots, x_n, \ldots\}$ and operation symbols from $\{f_i \mid i \in I\}$. Generalized hypersubstitutions of type τ are mappings $\sigma : \{f_i \mid i \in I\} \longrightarrow W_\tau(X)$ which do not necessarily preserve the arities. To define the extension $\hat{\sigma}$ of σ to a mapping defined on the set $W_\tau(X)$ of all terms of type τ, we defined inductively the concept of superposition of terms $S^m : W_\tau(X)^{m+1} \longrightarrow W_\tau(X)$ by the following steps:

(i) If $t = x_j, 1 \leq j \leq m$, then
$S^m(x_j, t_1, \ldots, t_m) := t_j$ where $t_1, \ldots, t_m \in W_\tau(X)$.

(ii) If $t = x_j, m < j \in \mathbb{N}$, then
$S^m(x_j, t_1, \ldots, t_m) := x_j$.

(iii) If $t = f_i(s_1, \ldots, s_{n_i})$, then
$S^m(t, t_1, \ldots, t_m) := f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m))$.

Then we have the following proposition.

PROPOSITION 2.1 ([6]). *For arbitrary terms $t, t_1, \ldots, t_m \in W_\tau(X)$,*

$$S^{n_i}(t, S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m)) = S^m(S^{n_i}(t, s_1, \ldots, s_{n_i}), t_1, \ldots, t_m). \quad \blacksquare$$

The generalized hypersubstitution σ can be extended to a mapping $\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X)$ on the set of all terms of type τ by the following steps:

(i) $\hat{\sigma}[x_k] := x_k \in X$,

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$, for an n_i-ary operation symbol f_i where $\hat{\sigma}[t_j]$, $1 \leq j \leq n_i$ are already known.

Let $Hyp_G(\tau)$ be the set of all generalized hypersubstitutions of type τ and let $Hyp(\tau)$ be the set of all usual hypersubstitutions of type τ. We define a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings and $\sigma_1, \sigma_2 \in Hyp_G(\tau)$. Let σ_{id} be the identity hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, \ldots, x_{n_i})$. Then we have the following proposition.

PROPOSITION 2.2 ([8]). *For arbitrary terms* $t, t_1, \ldots, t_n \in W_\tau(X)$ *and for arbitrary generalized hypersubstitutions* $\sigma, \sigma_1, \sigma_2$ *we have*

(i) $S^n(\hat{\sigma}[t], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) = \hat{\sigma}[S^n(t, t_1, \ldots, t_n)]$,
(ii) $(\hat{\sigma}_1 \circ \sigma_2)\hat{} = \hat{\sigma}_1 \circ \hat{\sigma}_2$. ∎

Now, we can prove the following theorem.

THEOREM 2.3 ([8]). $\underline{Hyp_G(\tau)} = (Hyp_G(\tau); \circ_G, \sigma_{id})$ *is a monoid and the monoid* $\underline{Hyp(\tau)} = (Hyp(\tau); \circ_h, \sigma_{id})$ *of all arity-preserving hypersubstitutions of type* τ *forms a submonoid of* $\underline{Hyp_G(\tau)}$. ∎

3. An algebraic-structural property of generalized hypersubstitutions

In this section, we investigate an algebraic-structural property of the set of all generalized hypersubstitutions. We first recall from [6] the definition of a left (right) seminear-ring.

DEFINITION 3.1. A nonempty set R together with two binary operations, denoted by $+$ and $\cdot$, respectively, is said to be a *left* (*right*) *seminear-ring* if $(R; +)$ and $(R; \cdot)$ are semigroups and satisfies the left (right) distributive law, i.e., for all $a, b, c \in R$, $a \cdot (b + c) = a \cdot b + a \cdot c$ $((a + b) \cdot c = a \cdot c + b \cdot c)$.

In [6], the author defined the binary operation $+_G$ on $Hyp_G(\tau)$ by

$$(\sigma_1 +_G \sigma_2)(f_i) := S^{n_i}(\sigma_2(f_i), \underbrace{\sigma_1(f_i), \ldots, \sigma_1(f_i)}_{n_i \text{ times}}), \forall i \in I.$$

Then we have the following propositions.

PROPOSITION 3.2 ([6]). *For arbitrary generalized hypersubstitutions* σ_1, σ_2 *and* σ_3,

$$((\sigma_1 +_G \sigma_2) +_G \sigma_3)(f_i) = (\sigma_1 +_G (\sigma_2 +_G \sigma_3))(f_i), \forall i \in I. \ ∎$$

PROPOSITION 3.3 ([6]). *For arbitrary generalized hypersubstitutions* σ_1, σ_2 *and* σ_3,

$$(\sigma_1 \circ_G (\sigma_2 +_G \sigma_3))(f_i) = ((\sigma_1 \circ_G \sigma_2) +_G (\sigma_1 \circ_G \sigma_3))(f_i), \forall i \in I. \ ∎$$

So $(Hyp_G(\tau); +_G, \circ_G)$ is a left seminear-ring. But it is not a right seminear-ring because it does not satisfy the right distributive law. As a counterexample, we consider the type $\tau = (2)$, i.e., there is one binary operation symbol f. Let $\sigma_1 : f \mapsto f(x_2, x_3), \sigma_2 : f \mapsto f(x_1, x_2)$ and $\sigma_3 : f \mapsto f(f(x_1, x_1), x_3)$. Then we have $((\sigma_1 +_G \sigma_2) \circ_G \sigma_3)(f) = f(f(x_3, x_3), f(x_3, x_3))$ and $((\sigma_1 \circ_G \sigma_3) +_G (\sigma_2 \circ_G \sigma_3))(f) = f(f(f(x_3, x_3), f(x_3, x_3)), x_3)$. Thus $(\sigma_1 +_G \sigma_2) \circ_G \sigma_3 \neq (\sigma_1 \circ_G \sigma_3) +_G (\sigma_2 \circ_G \sigma_3)$.

Now, we define another binary operation $\oplus_G$ on $Hyp_G(\tau)$ by

$$(\sigma_1 \oplus_G \sigma_2)(f_i) := S^{n_i}(\sigma_1(f_i), \underbrace{\sigma_2(f_i), \ldots, \sigma_2(f_i)}_{n_i \text{ times}}),$$

then we can prove the following propositions.

PROPOSITION 3.4. *For arbitrary generalized hypersubstitutions σ_1, σ_2 and σ_3,*

$$((\sigma_1 \oplus_G \sigma_2) \oplus_G \sigma_3)(f_i) = (\sigma_1 \oplus_G (\sigma_2 \oplus_G \sigma_3))(f_i), \forall i \in I.$$

Proof.

$$\begin{aligned}
&((\sigma_1 \oplus_G \sigma_2) \oplus_G \sigma_3)(f_i) \\
&\quad = S^{n_i}((\sigma_1 \oplus_G \sigma_2)(f_i), \underbrace{\sigma_3(f_i), \ldots, \sigma_3(f_i)}_{n_i \text{ times}}) \\
&\quad = S^{n_i}(S^{n_i}(\sigma_1(f_i), \underbrace{\sigma_2(f_i), \ldots, \sigma_2(f_i)}_{n_i \text{ times}}), \underbrace{\sigma_3(f_i), \ldots, \sigma_3(f_i)}_{n_i \text{ times}}) \\
&\quad = S^{n_i}(\sigma_1(f_i), S^{n_i}(\sigma_2(f_i), \underbrace{\sigma_3(f_i), \ldots, \sigma_3(f_i)}_{n_i \text{ times}}), \ldots, \\
&\qquad S^{n_i}(\sigma_2(f_i), \underbrace{\sigma_3(f_i), \ldots, \sigma_3(f_i)}_{n_i \text{ times}}) \\
&\quad = S^{n_i}(\sigma_1(f_i), \underbrace{(\sigma_2 \oplus_G \sigma_3)(f_i), \ldots, (\sigma_2 \oplus_G \sigma_3)(f_i)}_{n_i \text{ times}}) \\
&\quad = (\sigma_1 \oplus_G (\sigma_2 \oplus_G \sigma_3))(f_i). \blacksquare
\end{aligned}$$

PROPOSITION 3.5. *For arbitrary generalized hypersubstitutions σ_1, σ_2 and σ_3,*

$$(\sigma_1 \circ_G (\sigma_2 \oplus_G \sigma_3))(f_i) = ((\sigma_1 \circ_G \sigma_2) \oplus_G (\sigma_1 \circ_G \sigma_3))(f_i), \ \forall i \in I.$$

Proof.

$$\begin{aligned}
&(\sigma_1 \circ_G (\sigma_2 \oplus_G \sigma_3))(f_i) \\
&\quad = (\hat{\sigma}_1 \circ (\sigma_2 \oplus_G \sigma_3))(f_i) \\
&\quad = \hat{\sigma}_1[S^{n_i}(\sigma_2(f_i), \underbrace{\sigma_3(f_i), \ldots, \sigma_3(f_i)}_{n_i \text{ times}})] \\
&\quad = S^{n_i}(\hat{\sigma}_1[\sigma_2(f_i)], \underbrace{\hat{\sigma}_1[\sigma_3(f_i)], \ldots, \hat{\sigma}_1[\sigma_3(f_i)]}_{n_i \text{ times}}) \\
&\quad = S^{n_i}((\sigma_1 \circ_G \sigma_2)(f_i), \underbrace{(\sigma_1 \circ_G \sigma_3)(f_i), \ldots, (\sigma_1 \circ_G \sigma_3)(f_i)}_{n_i \text{ times}}) \\
&\quad = ((\sigma_1 \circ_G \sigma_2) \oplus_G (\sigma_1 \circ_G \sigma_3))(f_i). \blacksquare
\end{aligned}$$

Thus $(Hyp_G(\tau); \oplus_G, \circ_G)$ is also a left seminear-ring. However, it still does not satisfy the right distributive law. As a counterexample, we consider the type $\tau = (2)$. So there is one binary operation symbol, say f. Let $\sigma_1 : f \mapsto f(x_2, x_3), \sigma_2 : f \mapsto f(x_3, x_2), \sigma_3 : f \mapsto f(f(x_2, x_1), x_3)$. Then we have

$$\begin{aligned}
&((\sigma_1 \oplus_G \sigma_2) \circ_G \sigma_3)(f)\\
&\quad = (\sigma_1 \oplus_G \sigma_2)\hat{}[\sigma_3(f)]\\
&\quad = (\sigma_1 \oplus_G \sigma_2)\hat{}[f(f(x_2, x_1), x_3)]\\
&\quad = S^2((\sigma_1 \oplus_G \sigma_2)(f), (\sigma_1 \oplus_G \sigma_2)\hat{}[f(x_2, x_1)], x_3)\\
&\quad = S^2(S^2(\sigma_1(f), \sigma_2(f), \sigma_2(f)), S^2((\sigma_1 \oplus_G \sigma_2)(f), x_2, x_1), x_3)\\
&\quad = S^2(S^2(f(x_2, x_3), f(x_3, x_2), f(x_3, x_2)),\\
&\qquad S^2(S^2(f(x_2, x_3), f(x_3, x_2), f(x_3, x_2)), x_2, x_1), x_3)\\
&\quad = S^2(f(f(x_3, x_2), x_3), S^2(f(f(x_3, x_2), x_3), x_2, x_1), x_3)\\
&\quad = S^2(f(f(x_3, x_2), x_3), f(f(x_3, x_1), x_3), x_3)\\
&\quad = f(f(x_3, x_3), x_3),
\end{aligned}$$

and

$$\begin{aligned}
&((\sigma_1 \circ_G \sigma_3) \oplus_G (\sigma_2 \circ_G \sigma_3))(f)\\
&\quad = S^2((\sigma_1 \circ_G \sigma_3)(f), (\sigma_2 \circ_G \sigma_3)(f), (\sigma_2 \circ_G \sigma_3)(f))\\
&\quad = S^2(\hat{\sigma}_1[\sigma_3(f)], \hat{\sigma}_2[\sigma_3(f)], \hat{\sigma}_2[\sigma_3(f)])\\
&\quad = S^2(\hat{\sigma}_1[f(f(x_2, x_1), x_3)], \hat{\sigma}_2[f(f(x_2, x_1), x_3)], \hat{\sigma}_2[f(f(x_2, x_1), x_3)])\\
&\quad = S^2(S^2(\sigma_1(f), \hat{\sigma}_1[f(x_2, x_1)], x_3), S^2(\sigma_2(f), \hat{\sigma}_2[f(x_2, x_1)], x_3),\\
&\qquad S^2(\sigma_2(f), \hat{\sigma}_2[f(x_2, x_1)], x_3))\\
&\quad = S^2(S^2(f(x_2, x_3), S^2(f(x_2, x_3), x_2, x_1), x_3),\\
&\qquad S^2(f(x_3, x_2), S^2(f(x_3, x_2), x_2, x_1), x_3),\\
&\qquad S^2(f(x_3, x_2), S^2(f(x_3, x_2), x_2, x_1), x_3))\\
&\quad = S^2(S^2(f(x_2, x_3), f(x_1, x_3), x_3), S^2(f(x_3, x_2), f(x_3, x_1), x_3),\\
&\qquad S^2(f(x_3, x_2), f(x_3, x_1), x_3))\\
&\quad = S^2(f(x_3, x_3), f(x_3, x_3), f(x_3, x_3))\\
&\quad = f(x_3, x_3).
\end{aligned}$$

Thus, $(\sigma_1 \oplus_G \sigma_2) \circ_G \sigma_3 \neq (\sigma_1 \circ_G \sigma_3) \oplus_G (\sigma_2 \circ_G \sigma_3)$.

4. Algebraic-structural properties of some submonoids

In [3], K. Denecke and Sh. L. Wismath study on M-hyperidentities and M-solid varieties based on submonoids M of the monoid $Hyp(\tau)$. They defined a number of natural such submonoids based on various properties of hypersubstitutions. Now, we will extend these concepts to generalized hypersubstitutions and investigate some algebraic-structural properties of some submonoids M of the set of all generalized hypersubstitutions.

DEFINITION 4.1. Let $\tau = (n_i)_{i\in I}, n_i \in \mathbb{N} \setminus \{0\}$, be a type with an operation symbol f_i, of the arity n_i for each $i \in I$.

A generalized hypersubstitution σ of type τ is called a *projection generalized hypersubstitution* if the term $\sigma(f_i)$ is a variable for each $i \in I$. Let $P_G(\tau)$ be the set of all projection generalized hypersubstitutions of type τ.

A generalized hypersubstitution σ of type τ is said to be *leftmost* if for every $i \in I$, the first variable in $\hat{\sigma}[f_i(x_1, \ldots, x_{n_i})]$ is x_1. Let $Left_G(\tau)$ be the set of all leftmost generalized hypersubstitutions of type τ.

A generalized hypersubstitution σ of type τ is said to be *rightmost* if for every $i \in I$, the last variable in $\hat{\sigma}[f_i(x_1, \ldots, x_{n_i})]$ is x_{n_i}. Let $Right_G(\tau)$ be the set of all rightmost generalized hypersubstitutions of type τ.

A generalized hypersubstitution σ of type τ is said to be *outermost* if for every $i \in I$, the first variable in $\hat{\sigma}[f_i(x_1, \ldots, x_{n_i})]$ is x_1 and the last variable is x_{n_i}. Let $Out_G(\tau)$ be the set of all outermost generalized hypersubstitutions of type τ. Note that $Out_G(\tau) = Left_G(\tau) \cap Right_G(\tau)$.

A generalized hypersubstitution σ of type τ is called *regular* if for every $i \in I$, each of the variables $x_1, \ldots, x_{n_i}$ occurs in $\hat{\sigma}[f_i(x_1, \ldots, x_{n_i})]$. Let $Reg_G(\tau)$ be the set of all regular generalized hypersubstitutions of type τ.

A generalized hypersubstitution σ of type τ is called a *pre-generalized hypersubstitution* if for every $i \in I$, the term $\sigma(f_i)$ is not a variable. Let $Pre_G(\tau)$ be the set of all pre-generalized hypersubstitutions of type τ.

PROPOSITION 4.2. *For any type τ, the sets $P_G(\tau) \cup \{\sigma_{id}\}$, $Left_G(\tau)$, $Right_G(\tau)$, $Out_G(\tau), Reg_G(\tau)$, and $Pre_G(\tau)$ are submonoids of $\underline{Hyp_G(\tau)}$.*

Proof. It is clear that the identity hypersubstitution σ_{id} belongs to all of these sets, $P_G(\tau)\cup\{\sigma_{id}\}, Left_G(\tau), Right_G(\tau), Out_G(\tau), Reg_G(\tau)$, and $Pre_G(\tau)$.

Let $\sigma_1, \sigma_2 \in P_G(\tau)\cup\{\sigma_{id}\}$. We have to prove that $\sigma_1 \circ_G \sigma_2 \in P_G(\tau)\cup\{\sigma_{id}\}$. We consider the four cases.

Case 1. If $\sigma_1 \in P_G(\tau)$ and $\sigma_2 = \sigma_{id}$, then

$$\begin{aligned}(\sigma_1 \circ_G \sigma_{id})(f_i) &= \hat{\sigma}_1[\sigma_{id}(f_i)] \\ &= \hat{\sigma}_1[f_i(x_1, \ldots, x_{n_i})] \\ &= S^{n_i}(\sigma_1(f_i), x_1, \ldots, x_{n_i})\end{aligned}$$

$$= \begin{cases} x_j; & \sigma_1(f_i) = x_j, 1 \le j \le n_i, \\ x_k; & \sigma_1(f_i) = x_k, k > n_i. \end{cases}$$

Case 2. If $\sigma_1 = \sigma_{id}$ and $\sigma_2 \in P_G(\tau)$, then

$$(\sigma_{id} \circ_G \sigma_2)(f_i) = \hat{\sigma}_{id}[\sigma_2(f_i)] = \hat{\sigma}_{id}[x_n] = x_n \in X.$$

Case 3. If $\sigma_1 = \sigma_2 = \sigma_{id}$, then

$$\begin{aligned}(\sigma_{id} \circ_G \sigma_{id})(f_i) &= \hat{\sigma}_{id}[\sigma_{id}(f_i)] \\ &= \hat{\sigma}_{id}[f_i(x_1, \ldots, x_{n_i})] \\ &= S^{n_i}(\sigma_{id}(f_i), x_1, \ldots, x_{n_i}) \\ &= f(x_1, \ldots, x_{n_i}) \\ &= \sigma_{id}(f_i).\end{aligned}$$

Case 4. If neither σ_1 nor σ_2 is σ_{id}, then both $\sigma_1(f_i)$ and $\sigma_2(f_i)$ are variables for each $i \in I$. Thus $(\sigma_1 \circ_G \sigma_2)(f_i) = \hat{\sigma}_1[\sigma_2(f_i)] = \hat{\sigma}_1[x_n] = x_n \in X$.

Hence $\sigma_1 \circ_G \sigma_2 \in P_G(\tau) \cup \{\sigma_{id}\}$.

Let $\sigma \in Out_G(\tau)$ and $t \in W_\tau(X)$. We will prove by induction on the complexity of the term t that the first and the last variable occurring in $\hat{\sigma}[t]$ agree with the first and the last variable, respectively, occurring in t. If $t = x$ is a variable, then $\hat{\sigma}[t] = \hat{\sigma}[x] = x$. If $t = f_i(t_1, \ldots, t_{n_i})$ is a composed term where the first and the last variable occurring in $\hat{\sigma}[t_j]$ agree with the first and the last variable, respectively, occurring in $t_j, 1 \le j \le n_i$. Suppose that the first variable in $\hat{\sigma}[t_1]$ is x_1 and the last variable in $\hat{\sigma}[t_{n_i}]$ is x_{n_i}. Then the first and the last variable in t is x_1 and x_{n_i}, respectively. Since $\sigma \in Out_G(\tau)$, the first and the last variable in $\hat{\sigma}[t] = S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$ is x_1 and x_{n_i}, respectively.

Let $\sigma \in Reg_G(\tau)$ and $t \in W_\tau(X)$. We will prove by induction on the complexity of the term t that the variables occurring in t and $\hat{\sigma}[t]$ are the same. If $t = x$ is a variable, then $\hat{\sigma}[t] = \hat{\sigma}[x] = x$. If $t = f_i(t_1, \ldots, t_{n_i})$ is a composed term where the variables occurring in t_j and $\hat{\sigma}[t_j], 1 \le j \le n_i$ are the same. Since $\hat{\sigma}[t] = S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$ and $\sigma \in Reg_G(\tau)$, the variables occurring in t and $\hat{\sigma}[t]$ are the same.

Now, we can show that the sets $Left_G(\tau), Right_G(\tau), Out_G(\tau)$ and $Reg_G(\tau)$ are closed under the composition operation $\circ_G$. Let σ_1 and σ_2 be two generalized hypersubstitutions, both either leftmost, rightmost, outermost or regular. Then $(\sigma_1 \circ_G \sigma_2)\hat{}[f_i(x_1, \ldots, x_{n_i})] = \hat{\sigma}_1[\hat{\sigma}_2[f_i(x_1, \ldots, x_{n_i})]]$, and it follows from the previous reasons that this product has the corresponding property.

Finally, it is clear that the composition of two pre-generalized hypersubstitutions is again a pre-generalized hypersubstitution.

Hence the sets $P_G(\tau) \cup \{\sigma_{id}\}, Left_G(\tau), Right_G(\tau), Out_G(\tau), Reg_G(\tau)$ and $Pre_G(\tau)$ are submonoids of $Hyp_G(\tau)$. ∎

The next proposition are relationships between submonoids of $Hyp_G(\tau)$.

PROPOSITION 4.3. *Let τ be any type which does not contain a unary operation symbol. The following proper inclusions hold:*

(i) $Reg_G(\tau) \subset Pre_G(\tau)$,
(ii) $Out_G(\tau) \subset Pre_G(\tau)$.

Proof. The proof is straightforward. ∎

THEOREM 4.4. *For any type τ, the sets $P_G(\tau)$, $Left_G(\tau)$, $Right_G(\tau)$, $Out_G(\tau)$, $Reg_G(\tau)$, and $Pre_G(\tau)$ form sub-left seminear-rings of $(Hyp_G(\tau); +_G, \circ_G)$.*

Proof. We will prove that the sets $P_G(\tau)$, $Left_G(\tau)$, $Right_G(\tau)$, $Out_G(\tau)$, $Reg_G(\tau)$, and $Pre_G(\tau)$ are closed under the operation $+_G$.

Let $\sigma_1, \sigma_2 \in P_G(\tau)$. Then both $\sigma_1(f_i)$ and $\sigma_2(f_i)$ are variables for each $i \in I$. Since $(\sigma_1 +_G \sigma_2)(f_i) = S^{n_i}(\sigma_2(f_i), \underbrace{\sigma_1(f_i), \ldots, \sigma_1(f_i)}_{n_i \text{ times}})$ and the terms $\sigma_1(f_i), \sigma_2(f_i)$ are variables, $(\sigma_1 +_G \sigma_2)(f_i)$ is a variable. Thus $\sigma_1 +_G \sigma_2 \in P_G(\tau)$.

Let σ_1 and σ_2 be two generalized hypersubstitutions, both either leftmost, rightmost, outermost or regular. Consider

$$\begin{aligned}
&(\sigma_1 +_G \sigma_2)\hat{}[f_i(x_1, \ldots, x_{n_i})] \\
&= S^{n_i}((\sigma_1 +_G \sigma_2)(f_i), x_1, \ldots, x_{n_i}) \\
&= S^{n_i}(S^{n_i}(\sigma_2(f_i), \underbrace{\sigma_1(f_i), \ldots, \sigma_1(f_i)}_{n_i \text{ times}}), x_1, \ldots, x_{n_i}) \\
&= S^{n_i}(\sigma_2(f_i), \underbrace{S^{n_i}(\sigma_1(f_i), x_1, \ldots, x_{n_i}), \ldots, S^{n_i}(\sigma_1(f_i), x_1, \ldots, x_{n_i})}_{n_i \text{ times}}) \\
&= S^{n_i}(\sigma_2(f_i), \underbrace{\hat{\sigma}_1[f_i(x_1, \ldots, x_{n_i})], \ldots, \hat{\sigma}_1[f_i(x_1, \ldots, x_{n_i})]}_{n_i \text{ times}}).
\end{aligned}$$

Then it follows from Definition 4.1 that $\sigma_1 +_G \sigma_2$ is both either leftmost, rightmost, outermost or regular.

Let $\sigma_1, \sigma_2 \in Pre_G(\tau)$. Then both $\sigma_1(f_i)$ and $\sigma_2(f_i)$ are not variables. Since $(\sigma_1 +_G \sigma_2)(f_i) = S^{n_i}(\sigma_2(f_i), \underbrace{\sigma_1(f_i), \ldots, \sigma_1(f_i)}_{n_i \text{ times}})$ and the terms $\sigma_1(f_i), \sigma_2(f_i)$ are not variables, $(\sigma_1 +_G \sigma_2)(f_i)$ is not a variable. Thus $\sigma_1 +_G \sigma_2 \in Pre_G(\tau)$.

Hence the sets $P_G(\tau), Left_G(\tau), Right_G(\tau), Out_G(\tau), Reg_G(\tau)$, and $Pre_G(\tau)$ form sub-left seminear-rings of $(Hyp_G(\tau); +_G, \circ_G)$. ∎

We can prove in the same manner that these sets form sub-left seminearrings of $(Hyp_G(\tau); \oplus_G, \circ_G)$. Now, we consider another algebraic-structural property of some submonoids M of $(Hyp_G(\tau); +_G, \circ_G)$. We first recall from [5] the definition of a left (right, two sided) ideal.

DEFINITION 4.5. Let $(S; \cdot)$ be a semigroup. A nonempty subset A of S is called a *left ideal* if $SA \subseteq A$, a *right ideal* if $AS \subseteq A$, and a (*two sided*) *ideal* if it is both a left and a right ideal.

PROPOSITION 4.6. *For any type $\tau, P_G(\tau) \setminus Hyp(\tau)$ is a left ideal of $P_G(\tau)$.*

Proof. Let $\sigma_1 \in P_G(\tau) \setminus Hyp(\tau)$ and $\sigma_2 \in P_G(\tau)$. Then $\sigma_1(f_i) = x_j, \exists j > n_i$ and $\sigma_2(f_i) = x_n \in X$. Consider $(\sigma_2 \circ_G \sigma_1)(f_i) = \hat{\sigma}_2[\sigma_1(f_i)] = \hat{\sigma}_2[x_j] = x_j$. Thus $\sigma_2 \circ_G \sigma_1 \in P_G(\tau) \setminus Hyp(\tau)$. So $P_G(\tau) \setminus Hyp(\tau)$ is a left ideal of $P_G(\tau)$. ∎

PROPOSITION 4.7. *For any type $\tau, Out_G(\tau) \setminus Hyp(\tau)$ is a right ideal of $Out_G(\tau)$.*

Proof. Let $\sigma_1 \in Out_G(\tau) \setminus Hyp(\tau)$ and $\sigma_2 \in Out_G(\tau)$. Then the first and the last variable in $\hat{\sigma}_1[f_i(x_1, \ldots, x_{n_i})], \hat{\sigma}_2[f_i(x_1, \ldots, x_{n_i})]$ is x_1 and x_{n_i}, respectively, and there exists at least one variable $x_j, \exists j > n_i$ occurring in $\hat{\sigma}_1[f_i(x_1, \ldots, x_{n_i})]$. Consider

$$(\sigma_1 \circ_G \sigma_2)\hat{}[f_i(x_1, \ldots, x_{n_i})] = \hat{\sigma}_1[\hat{\sigma}_2[f_i(x_1, \ldots, x_{n_i})]].$$

Thus the first and the last variable in $(\sigma_1 \circ_G \sigma_2)\hat{}[f_i(x_1, \ldots, x_{n_i})]$ is x_1 and x_{n_i}, respectively, and x_j also occurs in $(\sigma_1 \circ_G \sigma_2)\hat{}[f_i(x_1, \ldots, x_{n_i})]$. Hence

$$\sigma_1 \circ_G \sigma_2 \in Out_G(\tau) \setminus Hyp(\tau).$$

So $Out_G(\tau) \setminus Hyp(\tau)$ is a right ideal of $Out_G(\tau)$. ∎

PROPOSITION 4.8. *For any type $\tau, Reg_G(\tau) \setminus Hyp(\tau)$ is a right ideal of $Reg_G(\tau)$.*

Proof. Let $\sigma_1 \in Reg_G(\tau) \setminus Hyp(\tau)$ and $\sigma_2 \in Reg_G(\tau)$. Then for every $i \in I$, each of the variables $x_1, \ldots, x_{n_i}$ occurs in $\hat{\sigma}_1[f_i(x_1, \ldots, x_{n_i})], \hat{\sigma}_2[f_i(x_1, \ldots, x_{n_i})]$ and there exists at least one variable $x_j, \exists j > n_i$ occurring in $\hat{\sigma}_1[f_i(x_1, \ldots, x_{n_i})]$. Consider

$$(\sigma_1 \circ_G \sigma_2)\hat{}[f_i(x_1, \ldots, x_{n_i})] = \hat{\sigma}_1[\hat{\sigma}_2[f_i(x_1, \ldots, x_{n_i})]].$$

Thus each of the variables $x_1, \ldots, x_{n_i}$ and x_j occur in $(\sigma_1 \circ_G \sigma_2)\hat{}[f_i(x_1, \ldots, x_{n_i})]$. Hence $\sigma_1 \circ_G \sigma_2 \in Reg_G(\tau) \setminus Hyp(\tau)$. So $Reg_G(\tau) \setminus Hyp(\tau)$ is a right ideal of $Reg_G(\tau)$. ∎

PROPOSITION 4.9. *For any type $\tau, Pre_G(\tau) \setminus Hyp(\tau)$ is a right ideal of $Pre_G(\tau)$.*

Proof. Let $\sigma_1 \in Pre_G(\tau) \setminus Hyp(\tau)$ and $\sigma_2 \in Pre_G(\tau)$. Then the term $\sigma_1(f_i)$ is not a variable and non-arity preserving and $\sigma_2(f_i)$ is not a variable and does not necessarily preserve arity. So there exists at least one variable $x_j, \exists j > n_i$ occurring in the term $\sigma_1(f_i)$. Since $(\sigma_1 \circ_G \sigma_2)(f_i) = \hat{\sigma}_1[\sigma_2(f_i)]$. Then the term $(\sigma_1 \circ_G \sigma_2)(f_i)$ is not a variable and x_j occurs in $(\sigma_1 \circ_G \sigma_2)(f_i)$. Hence $(\sigma_1 \circ_G \sigma_2)(f_i) \in Pre_G(\tau) \setminus Hyp(\tau)$. Thus $Pre_G(\tau) \setminus Hyp(\tau)$ is a right ideal of $Pre_G(\tau)$. ■

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