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Ivan Chajda, Helmut Länger

SAI-LATTICES AND RINGOIDS

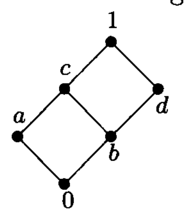
Abstract. The natural bijective correspondence between Boolean algebras and Boolean rings is generalized from Boolean algebras to lattices with 0 every principal ideal of which has an antitone involution. The corresponding ring-like structures are called ringoids. Among them orthorings are characterized by a simple axiom. It is shown that congruences on ringoids are determined by their kernels and that ringoids are permutable at 0.

There is a long series of papers generalizing the natural bijective correspondence between Boolean algebras and Boolean rings (see [1]) to more general structures (cf. [2], [3], [5]–[11] and [13]). The aim of this paper is to generalize this correspondence from Boolean algebras to lattices with 0 every principal ideal of which has an antitone involution. First we define this class of lattice-like structures.

An *antitone involution* on an interval $[0, a]$ of a poset with 0 is a mapping $x \mapsto x^a$ from $[0, a]$ to itself such that $(x^a)^a = x$ as well as $x \leq y$ implies $y^a \leq x^a$ for all $x, y \in [0, a]$.

DEFINITION 1. A *lattice with sectionally antitone involutions* (SAI-*lattice*, for short) is an algebra $(L, \vee, \wedge, (^a; a \in L), 0)$ where $(L, \vee, \wedge, 0)$ is a lattice with 0 and for each $a \in L$, ^{a} is an antitone involution on $([0, a], \leq)$.

EXAMPLE 2. The lattice with the Hasse diagram



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where x^y is given by the following table

$x\backslash y$	0	a	b	c	d	1
0	0	a	b	c	d	1
a		0		b		d
b			0	a	b	c
c				0		b
d					0	a
1						0

is a distributive SAI-lattice which is not complemented.

REMARK 3. If $(L, \vee, \wedge, ({}^a; a \in L), 0)$ is an SAI-lattice and $a \in L$ then the de Morgan laws hold in $([0,a], \vee, \wedge, {}^a, 0, a)$, i. e. for all $b, c \in [0,a]$

$$(b \vee c)^a = b^a \wedge c^a \text{ and}$$
$$(b \wedge c)^a = b^a \vee c^a.$$

Now we define the ring-like structures corresponding to SAI-lattices.

DEFINITION 4. A *ringoid* is an algebra $(R, +, \cdot, 0)$ of type $(2,2,0)$ having the property that for all $x, y \in R$ there exists a $z \in R$ with $xz = x$ and $yz = y$ and satisfying the following axioms:

$$(xy)z = x(yz),$$
$$xy = yx,$$
$$xx = x,$$
$$x0 = 0,$$
$$(xy + y)y = xy + y,$$
$$(xyz + z)(yz + z) + z = yz \text{ and}$$
$$xyz + ((xz + z)(yz + z) + z) = xz + yz.$$

REMARK 5. Since for every ringoid $\mathcal{R} = (R, +, \cdot, 0)$, $(R, \cdot, 0)$ is a semilattice with 0, $\mathcal{R}$ may be considered as a partially ordered set $(R, \leq, 0)$ with smallest element 0 where for every $x, y \in R$, $x \leq y$ is defined by $xy = x$.

LEMMA 6. *Let $(R, +, \cdot, 0)$ be a ringoid, $a, b, c \in R$ and $a, b \in [0, c]$. Then $x \mapsto x + c$ is an antitone involution on $([0,c], \leq)$ and $(a+c)(b+c)+c$ is the supremum $a \vee b$ of a and b in $(R, \leq)$.*

Proof. If $d, e \in [0, c]$ and $d \leq e$ then

$$(d + c)c = (dc + c)c = dc + c = d + c,$$
$$(d + c) + c = (d + c)(d + c) + c = (ddc + c)(dc + c) + c = dc = d \text{ and}$$
$$(e + c)(d + c) = ((dec + c)(ec + c) + c) + c = ec + c = e + c.$$

This shows that $x \mapsto x + c$ is an antitone involution on $([0, c], \leq)$. Hence $f := (a + c)(b + c) + c$ is the supremum of a and b in $([0, c], \leq)$. Let g be an upper bound of a and b in $(R, \leq)$. Then $a \leq f$ and $a \leq g$ and hence $a \leq f \wedge g$. Analogously, $b \leq f \wedge g$. Since $f \wedge g \leq f \leq c$, $f \wedge g$ is an upper bound of a and b in $([0, c], \leq)$. Since f is the supremum of a and b in $([0, c], \leq)$, we obtain $f \leq f \wedge g \leq g$. This shows that f is the supremum of a and b in $(R, \leq)$. ∎

Now we are ready to formulate and prove the fact that there exists a natural bijective correspondence between SAI-lattices and ringoids.

THEOREM 7. *On every set A the formulas*

$$x + y := (x \wedge y)^{x \vee y}$$
$$xy := x \wedge y$$

and

$$x \vee y := (x + z)(y + z) + z$$
$$x \wedge y := xy$$
$$x^y := x + y$$

(*where z is an arbitrary element of A satisfying $xz = x$ and $yz = y$*) *induce mutually inverse bijections between the set of all* SAI-*lattices on A and the set of all ringoids on A.*

Proof. Let $b, c, d \in A$. First assume $(A, \vee, \wedge, ({}^a; a \in A), 0)$ to be an SAI-lattice and define

$$x + y := (x \wedge y)^{x \vee y} \text{ and}$$
$$xy := x \wedge y$$

for all $x, y \in A$. Then

$b(b \vee c) = b \wedge (b \vee c) = b,$

$c(b \vee c) = c \wedge (b \vee c) = c,$

$(bc)d = (b \wedge c) \wedge d = b \wedge (c \wedge d) = b(cd),$

$bc = b \wedge c = c \wedge b = cb,$

$bb = b \wedge b = b,$

$b0 = b \wedge 0 = 0,$

$(bc + c)c = (b \wedge c)^c = bc + c,$

$(bcd + d)(cd + d) + d = ((b \wedge c \wedge d)^d \wedge (c \wedge d)^d)^d = ((c \wedge d)^d)^d = c \wedge d = cd,$

$bcd + ((bd + d)(cd + d) + d)$

$= (b \wedge c \wedge d \wedge ((b \wedge d)^d \wedge (c \wedge d)^d)^d)^{(b \wedge c \wedge d) \vee ((b \wedge d)^d \wedge (c \wedge d)^d)^d}$

$$= (b \wedge c \wedge d \wedge ((b \wedge d) \vee (c \wedge d)))^{(b \wedge c \wedge d) \vee (b \wedge d) \vee (c \wedge d)}$$
$$= (b \wedge c \wedge d)^{(b \wedge d) \vee (c \wedge d)} = bd + cd.$$

Moreover, if $bd = b$ and $cd = c$ then $b, c \leq d$ and hence

$$(b + d)(c + d) + d = (b^d \wedge c^d)^d = b \vee c \text{ and}$$
$$b + d = b^d.$$

Conversely, assume $(A, +, \cdot, 0)$ to be a ringoid and define

$$x \vee y := (x + z)(y + z) + z,$$
$$x \wedge y := xy \text{ and}$$
$$x^y := x + y$$

for all $x, y \in A$ where z is an arbitrary element of A satisfying $xz = x$ and $yz = y$. Then $(A, \cdot, 0) = (A, \wedge, 0)$ is a meet-semilattice with 0 the corresponding partial order relation $\leq$ of which is defined by $x \leq y$ if $xy = x$ $(x, y \in A)$. According to Lemma 6, ^{a} is an antitone involution on $([0, a], \leq)$ for every $a \in A$ and $b \vee c$ is the supremum of b and c in $(A, \leq)$. Finally, if $bd = b$ and $cd = c$ then

$$(b \wedge c)^{b \vee c} = bc + ((b + d)(c + d) + d)$$
$$= bcd + ((bd + d)(cd + d) + d) = bd + cd = b + c. \blacksquare$$

EXAMPLE 8. The operation tables of the ringoid corresponding to the SAI-lattice defined in Example 2 look as follows:

+	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	0	c	b	1	d
b	b	c	0	a	b	c
c	c	b	a	0	c	b
d	d	1	b	c	0	a
1	1	d	c	b	a	0

and

·	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	b	b	b	b
c	0	a	b	c	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

LEMMA 9. *Every ringoid* $\mathcal{R} = (R, +, \cdot, 0)$ *satisfies the identities*

$$x + y = y + x,$$
$$x + x = 0,$$
$$x + 0 = x \text{ and}$$
$$(xy + x) + x = xy.$$

Moreover, $x = y$ *is equivalent to* $x + y = 0$.

Proof. In the SAI-lattice corresponding to $\mathcal{R}$ we have

$$
\begin{aligned}
&x + y = (x \wedge y)^{x \vee y} = (y \wedge x)^{y \vee x} = y + x,\\
&x + x = x^x = 0,\\
&x + 0 = 0^x = x \text{ and}\\
&(xy + x) + x = (yx + x) + x = (yx + x)(yx + x) + x\\
&\quad = (yyx + x)(yx + x) + x = yx = xy.
\end{aligned}
$$

Moreover, the following are equivalent:

$$
\begin{aligned}
&x + y = 0,\\
&(x \wedge y)^{x \vee y} = 0,\\
&x \wedge y = x \vee y \text{ and}\\
&x = y. \ \blacksquare
\end{aligned}
$$

In [6] ring-like structures were introduced corresponding in a natural bijective way to lattices with 0 every principal ideal of which is an ortholattice.

DEFINITION 10 (cf.[6]). An *orthoring* is an algebra $(R, +, \cdot, 0)$ of type $(2, 2, 0)$ satisfying the following identities:

$$
\begin{aligned}
&x + y = y + x,\\
&x + 0 = x,\\
&xy = yx,\\
&(xy)z = x(yz),\\
&xx = x,\\
&x0 = 0,\\
&(xy + x) + x = xy,\\
&((x + y) + xy) + xy = x + y,\\
&(xy + x)x = xy + x,\\
&(x + y)xy = 0,\\
&((x + y) + xy)x = x,\\
&((xy + xz) + xyz)x = (xy + xz) + xyz \text{ and}\\
&(xyz + x)(xy + x) = xy + x.
\end{aligned}
$$

According to Theorem 2.1 of [6] every orthoring is a ringoid. Hence the natural question arises when a ringoid becomes an orthoring.

THEOREM 11. *A ringoid* $(R, +, \cdot, 0)$ *is an orthoring if and only if it satisfies the identity* $(x + y)xy = 0$.

Proof. Let $\mathcal{R} = (R, +, \cdot, 0)$ be a ringoid satisfying the identity $(x + y)xy = 0$. Then in the SAI-lattice corresponding to $\mathcal{R}$ the identity

$$(x \wedge y)^{x \vee y} \wedge x \wedge y = 0$$

holds. In case $x \leq y$ this yields $x^y \wedge x = 0$ showing that $([0,a], \vee, \wedge, ^a, 0, a)$ is an ortholattice for every $a \in R$. From Theorem 2.1 of [6] it now follows that $(R, +, \cdot, 0)$ is an orthoring. ∎

In every ringoid we can define another addition which has similar properties as the usual one:

DEFINITION 12. For every ringoid $(R, +, \cdot, 0)$ with corresponding SAI-lattice $(R, \vee, \wedge, (^a; a \in R), 0)$ we define a binary operation $\oplus$ by

$$x \oplus y := (x \wedge y^{x \vee y}) \vee (x^{x \vee y} \wedge y)$$

for all $x, y \in R$.

Some properties of $\oplus$ are summarized in the following lemma:

LEMMA 13. *If $(R, +, \cdot, 0)$ is a ringoid, $(R, \vee, \wedge, (^a; a \in R), 0)$ its corresponding* SAI-*lattice and $b, c \in R$ then* (i)–(vi) *hold:*

(i) $b \oplus c = c \oplus b$,
(ii) $b + c = b \oplus c = b^c$ *if* $b \leq c$,
(iii) $b \oplus c = (b \wedge (c + (b \vee c))) \vee ((b + (b \vee c)) \wedge c)$,
(iv) $b + c = (b \wedge c) \oplus (b \vee c)$,
(v) $(R, \oplus, \cdot, 0)$ *satisfies all axioms of a ringoid except the last one and*
(vi) $(R, \oplus, \cdot, 0)$ *satisfies the last axiom of a ringoid if and only if* $\oplus = +$.

Proof. Let $d \in R$.
(i) $b \oplus c = (b \wedge c^{b \vee c}) \vee (b^{b \vee c} \wedge c) = (c \wedge b^{c \vee b}) \vee (c^{c \vee b} \wedge b) = c \oplus b$.
(ii) If $b \leq c$ then $b + c = b^c$ and $b \oplus c = (b \wedge c^c) \vee (b^c \wedge c) = b^c$.
In the sequel we make frequent use of (ii).
(iii) $b \oplus c = (b \wedge c^{b \vee c}) \vee (b^{b \vee c} \wedge c) = (b \wedge (c + (b \vee c))) \vee ((b + (b \vee c)) \wedge c)$.
(iv) $b + c = (b \wedge c)^{b \vee c} = (b \wedge c) \oplus (b \vee c)$.
(v) $(bc \oplus c)c = (b \wedge c)^c \wedge c = (b \wedge c)^c = bc \oplus c$.
$(bcd \oplus d)(cd \oplus d) \oplus d = ((b \wedge c \wedge d)^d \wedge (c \wedge d)^d)^d = ((c \wedge d)^d)^d = c \wedge d = cd$.
(vi) $bcd \oplus ((bd \oplus d)(cd \oplus d) \oplus d) = bcd \oplus ((bd)^d \wedge (cd)^d)^d = bcd \oplus (bd \vee cd) = (bd \wedge cd)^{bd \vee cd} = bd + cd$. ∎

Next we observe that principal filters of ringoids are ringoids, too.

LEMMA 14. *If $\mathcal{R} = (R, +, \cdot, 0)$ is a ringoid and $a \in R$ then $([0,a], +, \cdot, 0)$ is a ringoid, too.*

Proof. If $b, c \in [0, a]$ then in the SAI-lattice corresponding to $\mathcal{R}$ it holds $b + c = (b \wedge c)^{b \vee c} \in [0, b \vee c] \subseteq [0, a]$, $b \vee c \in [0, a]$, $b(b \vee c) = b$ and $c(b \vee c) = c$. The rest of the proof is clear. ∎

Next we show some congruence properties of ringoids. First we prove that in a ringoid $(R, +, \cdot, 0)$ every congruence Θ is uniquely determined by its kernel $[0]\Theta$, i. e. ringoids are *weakly regular* (see [4]).

THEOREM 15. *If* $\mathcal{R} = (R, +, \cdot, 0)$ *is a ringoid,* $a, b \in R$ *and* $\Theta \in \mathrm{Con}\mathcal{R}$ *then* $a \,\Theta\, b$ *if and only if* $ab + a, ab + b \in [0]\Theta$.

Proof. Using Lemma 9 we obtain that if $a \,\Theta\, b$ then

$$ab + a, ab + b \in [aa + a]\Theta = [a + a]\Theta = [0]\Theta$$

and if, conversely, $ab + a, ab + b \in [0]\Theta$ then

$$a = 0 + a \,\Theta\, (ab + a) + a = ab = (ab + b) + b \,\Theta\, 0 + b = b. \ \blacksquare$$

Finally, we want to show that ringoids satisfy a certain congruence condition. For an overview concerning congruence conditions in universal algebra see the monograph [4].

DEFINITION 16 (cf. [4]). An *algebra* $\mathcal{A}$ with element 0 is called *permutable at* 0 if $[0](\Theta \circ \Phi) = [0](\Phi \circ \Theta)$ for all congruences Θ, Φ on $\mathcal{A}$. A *class* $\mathcal{K}$ of algebras of the same type with an equational constant 0 is called *permutable at* 0 if each member of $\mathcal{K}$ has this property.

Though the class of all ringoids does not form a variety this class turns out to be permutable at 0.

THEOREM 17. *The class of all ringoids is permutable at* 0.

Proof. Let $\mathcal{K}$ denote the class of all ringoids and $\mathcal{V}$ the variety of all algebras $(R, +, \cdot, 0)$ of type $(2, 2, 0)$ satisfying the identities $x + x = 0$ and $x + 0 = x$. According to a theorem by H. P. Gumm and A. Ursini ([12]) $\mathcal{V}$ is permutable at 0 if and only if there exists a binary term $t(x, y)$ with $t(x, x) = 0$ and $t(x, 0) = x$ (a so-called *subtractive term*). Obviously, $t(x, y) := x + y$ is a subtractive term of $\mathcal{V}$ and hence $\mathcal{V}$ is permutable at 0. According to Lemma 9, $\mathcal{K}$ is a subclass of $\mathcal{V}$. Hence also $\mathcal{K}$ is permutable at 0. $\blacksquare$

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Ivan Chajda
PALACKÝ UNIVERSITY OLOMOUC
DEPARTMENT OF ALGEBRA AND GEOMETRY
Tomkova 40
77900 OLOMOUC, CZECH REPUBLIC
e-mail: chajda@inf.upol.cz

Helmut Länger
VIENNA UNIVERSITY OF TECHNOLOGY
INSTITUTE OF DISCRETE MATHEMATICS AND GEOMETRY
Wiedner Hauptstrasse 8–10
1040 VIENNA, AUSTRIA
e-mail: h.laenger@tuwien.ac.at

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