DEMONSTRATIO MATHEMATICA
Vol. XXXIX No 2 2006

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AUTONOMOUS DIFFERENTIAL INCLUSIONS SHARING THE FAMILIES OF TRAJECTORIES

Abstract. We give a sufficient condition for equality of sets of trajectories of two differential inclusions with right-hand sides Borel measurable with respect to the state variable, not necessarily bounded and possibly containing the origin.

1. Introduction and a lemma on eliminating the set where the derivative of an absolutely continuous function is zero

The change of variables in ordinary differential equations is usually one of the first topics treated in standard textbooks on the subject. It has also been done for differential inclusions (see [3], Chapter 2, par. 9).

By a solution of a differential inclusion

$$\dot{x} \in F(x) \tag{1}$$

where $x \in \mathbb{R}^d$, $F(x) \subset \mathbb{R}^d$, we mean an absolutely continuous map $x : [a,b] \to \mathbb{R}^d$ for which $\dot{x}(t) \in F(x(t))$ a.e. in $[a,b]$. Trajectory corresponding to such solution is the set $\{x(t) : t \in [a,b]\} \subset \mathbb{R}^d$. We shall speak of it also as of a trajectory of differential inclusion. In fact the definition of trajectory is valid for any map $x : [a,b] \to \mathbb{R}^d$, not only for solutions. While speaking of maps we shall usually write $x(\cdot)$ reserving the symbol $x(t)$ for the value of that map at t.

We are interested in the following problem – given two differential inclusions (1) and $\dot{y} \in G(y)$ under what possibly weak assumptions they have the same families of trajectories. Our attempts are focused on the following objectives. First, the regularity of F and G as weak as possible – we treat the case of Borel measurability. Next, the sets $F(x)$, $G(x)$ can be unbounded.

Key words and phrases: differential inclusions, trajectories, Borel measurability.
2000 *Mathematics Subject Classification*: Primary 34A60, Secondary 49J53, 54C60.
This research was supported by a grant from the Faculty of Mathematics & Information Science of Warsaw University of Technology.

And last, the origin does not have to be separated from those sets, it may belong to any of them (or none).

If two absolutely continuous and injective maps $x(\cdot)$ and $y(\cdot)$ have the same trajectories, $x(t) = y(s)$ for a pair t, s and $\dot{x}(t)$, $\dot{y}(s)$ exist none of them equal 0 then $\dot{y}(s) = \lambda\dot{x}(t)$ for some $\lambda \neq 0$. When $x(\cdot)$ and $y(\cdot)$ follow the trajectory in the same direction then $\lambda > 0$. This suggests that the right-hand sides of differential inclusions should generate the same cones (or opposite) if they are to generate the same trajectories. However, even for differential equations $\dot{x} = f(x)$, $\dot{y} = g(y)$ with continuous right-hand sides which generate the same cones (half-lines in that case) the sets of trajectories may be different as shows the following simple example.

EXAMPLE 1.1. *Consider differential equations in* $\mathbb{R}^1$

$$\dot{x} = |x|, \qquad \dot{y} = \sqrt{|y|}.$$

The interval $[-1, 1]$ *is a trajectory of the second equation but not of the first one.*

The example given above suggests that the behavior of $F(\cdot)$ is important near to arguments x for which $0 \in F(x)$ or 0 is very close to $F(x)$. This may be omitted by assuming, as in [3], that 0 is separated from $F(x)$. We want, on the contrary, to allow 0 not to be separated.

If 0 is in $F(x)$ we will have solutions $x(\cdot)$ with $\dot{x}(t) = 0$ on sets of positive measure. We give now a lemma which shows that it is possible then to eliminate this set without disturbing the trajectory. This property is essentially known however the proof requires some subtle tools from the theory of real functions and we sketch it for completness. The operation consists in "squeezing" in some sense the interval on which this absolutely continuous function is defined so that the points where its derivative is 0 are eliminated. The method is similar to those used in [4] and references given therein but applied in a different manner.

LEMMA 1.1. *Let* $x : [a, b] \to \mathbb{R}^d$ *be a nonconstant, absolutely continuous map and* $\gamma \subset \mathbb{R}^d$ *its trajectory. There is* $\alpha > 0$ *and a strictly increasing function* $\tau : [0, \alpha] \to [a, b]$ *such that the map* $y(s) = x(\tau(s))$ *is absolutely continuous, its trajectory coincides with* γ *and*

$$\dot{y}(s) = \dot{x}(\tau(s)) \neq 0 \quad \text{a.e. in} \quad [0, \alpha].$$

Before sketching the proof we remark that as a rule the function $\tau(\cdot)$ is not absolutely continuous and may be discontinuous. Nevertheless the composition $y(\tau(\cdot))$ will be absolutely continuous.

The proof would be obvious if the set

$$\Lambda = \{t \in [a, b] \,:\, \dot{x}(t) = 0\}$$

was composed of a finite family of intervals (up to a set of measure zero, of course). It may happen, however, that Λ contains an infinite family of disjoint intervals or even a set of positive measure such that for every interval $[\bar{a},\bar{b}] \subset [a,b]$ the measure of $[\bar{a},\bar{b}] \setminus \Lambda$ is strictly positive - like the Cantor set with positive measure.

Proof of Lemma 1.1. Put $\sigma(t) = m([a,t] \setminus \Lambda)$ for $t \in [a,b]$ and let $\alpha = \sigma(b)$. We define a new function $\tau : [0,\alpha] \to \mathbb{R}$.

$$\tau(s) = \begin{cases} \sup\{t \in [a,b] : \sigma(t) < s\} & \text{for } 0 < s \le \alpha \\ 0 & \text{for } s = 0 \end{cases}$$

$\tau(\cdot)$ is strictly increasing, left continuous on $(0,\alpha]$ and right continuous at 0. Note also that $\sigma(\tau(s)) = s$ for all $s \in [0,\alpha]$.

We prove first that the trajectory of function $y(\cdot)$ defined by $y(s) = x(\tau(s))$ is γ. Take some $t \in [0,a]$. Let $u = x(t) \in \gamma$, $s = \sigma(t)$ and

$$t_- = \inf\{t' \le t : m([t',t] \setminus \Lambda) = 0\}.$$

As $\sigma(t) = \sigma(t_-) = s$, $\dot{x} = 0$ on $[t_-,t]$ and $\tau(s) = t_-$ so we get

$$y(s) = x(\tau(s)) = x(t_-) = x(t) = u.$$

We prove now the absolute continuity of $y(\cdot)$. Remark first that the function $\|\dot{x}(\cdot)\|$ is the density of an absolutely continuous measure on $[0,a]$.

Fix $\varepsilon > 0$ and let $\delta > 0$ be such that for any $T \subset [0,a]$ with $m(T) \le \delta$ we have $\int_T \|\dot{x}(t)\|\, dm(t) \le \varepsilon$. Let now $[\alpha_i,\beta_i] \subset [0,\alpha]$, for $i = 1,\dots,k$, be pairwise disjoint with $\sum(\beta_i - \alpha_i) \le \delta$. For every interval $[\alpha_i,\beta_i]$ we have $\beta_i - \alpha_i = \sigma(\tau(\beta_i)) - \sigma(\tau(\alpha_i)) = m([\tau(\alpha_i),\tau(\beta_i)] \setminus \Lambda)$ and

$$\begin{aligned} m\Big(\bigcup_{i=1}^{k}([\tau(\alpha_i),\tau(\beta_i)]) \setminus \Lambda\Big) &= \sum_{i=1}^{k} m([\tau(\alpha_i),\tau(\beta_i)] \setminus \Lambda) \\ &= \sum_{i=1}^{k}(\beta_i - \alpha_i) \le \delta \end{aligned}$$

which in turn permits to write

$$\sum_{i=1}^{k} \|y(\beta_i) - y(\alpha_i)\| = \sum_{i=1}^{k} \|x(\tau(\beta_i) - x(\tau(\alpha_i)\| \le \int\limits_{(\bigcup[\tau(\alpha_i),\tau(\beta_i)])\setminus\Lambda} \|\dot{x}(t)\|\, dm(t) \le \varepsilon.$$

To end we prove that $\dot{y}(s) = \dot{x}(\tau(s)) \ne 0$ a.e. in $[0,\alpha]$. Let $\widetilde{T}$ be the set of density points of $\{t \in [0,a] : \dot{x}(t) \text{ exists}\} \setminus \Lambda$. The function $\sigma(\cdot)$ is injective on $\widetilde{T}$ and $m(\sigma(\widetilde{T})) = \alpha$.

We fix $s_0 \in \sigma(\widetilde{T})$. The inclusion $\tau(s_0) \in \widetilde{T}$ implies that $\tau(\cdot)$ is continuous at s_0. For $s \in [0, \alpha]$ we have $s = m([0, \tau(s)] \setminus \Lambda)$ so taking any $s_1 > s_0$ we we may write

$$s_1 - s_0 = m([\tau(s_0), \tau(s_1)] \setminus \Lambda) = (\tau(s_1) - \tau(s_0))(1 - \varepsilon)$$

where $\varepsilon \to 0$ when $s_1 \to s_0+$ and so

$$\lim_{s_1 \to s_0+} \frac{\tau(s_1) - \tau(s_0)}{s_1 - s_0} = 1.$$

The same reasoning is valid for $s_1 < s_0$ and thus $\dot{\tau}(s) = 1$ for all $s \in \sigma(\widetilde{T})$. The assertion follows from the fact that $x(\cdot)$ has derivative different than 0 at $\tau(s_0)$ and, of course, $\dot{\tau}(s_0) = 1$.

We recall now the definition of measurability of multifunctions in the simple case we need.

DEFINITION 1.1. A multifunction $F : M \to \mathrm{Cl}(\mathbb{R}^d)$ is called measurable with respect to a σ–field $\mathcal{M}$ in M if the set $\{x \in M : F(x) \cap O\}$ belongs to $\mathcal{M}$ for every open $O \subset \mathbb{R}^d$.

We use in the sequel measurability in two cases. First, when $\mathcal{M}$ is the Borel σ–field in $M = \mathbb{R}^d$. Second, when $\mathcal{M}$ is the Lebesgue σ–field in $\mathbb{R}$.

The important feature of a Borel measurable multifunction $F : \mathbb{R}^d \to \mathrm{Cl}(\mathbb{R}^d)$ is that the composition $F(x(t))$, for $x : [a, b] \to \mathbb{R}^d$ continuous, defines also a Borel measurable multifunction (in fact it is enough for $x(\cdot)$ to be Borel measurable). This composition need not be Lebesgue measurable if $F(\cdot)$ is only Lebesgue measurable.

We refer to [2] for detailed discussion of various aspects and consequences of measurability of multifunctions – [1] can also be consulted.

2. Finding a solution with prescribed trajectory

For $K \subset \mathbb{R}^d$ by S_K we denote the cone it generates (with 0 included). The following function defined for $p \neq 0$ will be of use

$$w(K, p) = \sup\left(\left\{\lambda > 0 \; : \; \lambda \cdot \frac{p}{\|p\|} \in K\right\} \cup \{0\}\right).$$

Remark that $w(K, p) = 0$ means that either $K \cap \mathrm{S}_{\{p\}}$ contains only the origin or is empty.

Ω below will be an open subset of $\mathbb{R}^d$.

LEMMA 2.1. *If $x : [0, a] \to \mathbb{R}^d$ is absolutely continuous with values in Ω, $\dot{x}(t) \neq 0$ a.e. and $F : \Omega \to \mathrm{Cl}(\mathbb{R}^d)$ is Borel measurable then the function $\phi(t) = w(F(x(t)), \dot{x}(t))$ is measurable.*

P r o o f. The multifunction $F(x(\cdot))$ is Borel measurable as composition of Borel measurable maps. $\mathrm{S}_{\{\dot{x}(\cdot)\}}$ is measurable (can be easily proved using, for example, Castaing representation theorem, see Theorem III.9 in [2]). The intersection $F(x(\cdot)) \cap \mathrm{S}_{\{\dot{x}(\cdot)\}}$ is thus a measurable multifunction and this implies that for any $\gamma > 0$ the set at the right-hand side of

$$\{t \in [0,a] : w(F(x(t)), \dot{x}(t)) \le \gamma\} = \{t \in [0,a] : F(x(t)) \cap \mathrm{S}_{\{\dot{x}(t)\}} \subset \gamma B_1\}$$

is measurable which proves our assertion.

THEOREM 2.1. *Assume that* $x : [0,a] \to \mathbb{R}^d$ *is absolutely continuous with values in* Ω *and* $F : \Omega \to \mathrm{Cl}(\mathbb{R}^d)$ *is Borel measurable. If*

$$\int_\Gamma \frac{\|\dot{x}(t)\|}{w(F(x(t)), \dot{x}(t))}\, dm(t) < +\infty \tag{2}$$

where $\Gamma = \{t \in [0,a] : \dot{x}(t) \neq 0\}$, *then there is a solution of differential inclusion*

$$\dot{y} \in F(y) \tag{3}$$

whose trajectory is equal to that of $x(\cdot)$.

P r o o f. Thanks to Lemma 1.1 we may assume that the original absolutely continuous map has already been replaced with one having the same trajectory and a.e. nonzero derivative. Together with assumption (2) this means that $0 \neq \dot{x}(t) \in \mathrm{S}_{F(x(t))}$ a.e. in $[0,a]$.

We define in a special way a measurable selection $l(t) \in F(x(t)) \cap \mathrm{S}_{\{\dot{x}(t)\}}$. Split first $[0,a]$ into $T_\infty = \{t \in [0,a] : w(F(x(t)), \dot{x}(t)) = +\infty\}$ and its complement – both are measurable.

1. For $t \in [0,a] \setminus T_\infty$ put
$$l(t) = \frac{w(F(x(t)), \dot{x}(t))}{\|\dot{x}(t)\|} \cdot \dot{x}(t).$$
2. For $t \in T_\infty$ put $l(t) = z$, where $z \in F(x(t)) \cap \mathrm{S}_{\{\dot{x}(t)\}}$, $\|z\| \ge \|\dot{x}(t)\|$ and there is no $z' \in F(x(t)) \cap \mathrm{S}_{\{\dot{x}(t)\}}$ with $\|\dot{x}(t)\| \le \|z'\| < \|z\|$.

In the first case we choose an element in $F(x(t)) \cap \mathrm{S}_{\{\dot{x}(t)\}}$ with the greatest norm. In the second case this is not possible so we choose the point with the smallest norm which is greater than or equal $\|\dot{x}(t)\|$. The measurability of $l(\cdot)$ is proved using standard techniques.

We have the inequalities

$$\int_{[0,a]} \frac{\|\dot{x}(t)\|}{\|l(t)\|}\, dm(t) \le \int_{[0,a]\setminus T_\infty} \frac{\|\dot{x}(t)\|}{w(F(x(t)), \dot{x}(t))}\, dm(t) + m(T_\infty) < +\infty.$$

The function

$$\phi(t) = \int\limits_{[0,t]} \frac{\|\dot{x}(\varrho)\|}{\|l(\varrho)\|}\, dm(\varrho)$$

has thus finite values. It is absolutely continuous and $\dot{\phi}(t) > 0$ a.e. so its inverse $\tau(\cdot)$ is also absolutely continuous (see Lemma 4.1 in the Appendix).

Let $\alpha = \phi(a)$ and put $y(s) = x(\tau(s))$ for $s \in [0, \alpha]$. The composition $x \circ \tau$ is absolutely continuous, the trajectories of $x(\cdot)$ and $y(\cdot)$ are the same and

$$\dot{y}(s) = \dot{x}(\tau(s)) \cdot \dot{\tau}(s) = \dot{x}(\tau(s)) \cdot \frac{\|l(\tau(s))\|}{\|\dot{x}(\tau(s))\|} = l(\tau(s)) \in F(x(\tau(s))) = F(y(s))$$

which ends the proof.

Before describing in the next section some consequences of this theorem let us remark that assumption (2) is automatically satisfied if there is $r > 0$ such that $(rB_1) \cap F(x(t)) = \emptyset$ for all $t \in [0, a]$. This is true, for example, when $F(\cdot)$ is upper semicontinuous and $0 \notin F(x(t))$ for $t \in [0, a]$.

3. Relations between families of trajectories of differential inclusions

Theorem 2.1 permits to formulate a sufficient condition for the set of trajectories of one differential inclusion to be contained in the set of trajectories of another one.

COROLLARY 3.1. *Suppose F, $G : \Omega \to \mathrm{Cl}(\mathbb{R}^d)$ are Borel measurable. If for some positive constant c, all $x \in \Omega$ and $p \neq 0$ the following inequality holds*

$$w(F(x), p) \leq c \cdot w(G(x), p)$$

then every trajectory of differential inclusion $\dot{x} \in F(x)$ is also a trajectory of differential inclusion $\dot{x} \in G(x)$.

Proof. If $\dot{x}(t) \in F(x(t))$ a.e. then for all t for which $\dot{x}(t) \neq 0$

$$\|\dot{x}(t)\| \leq w(F(x(t)), \dot{x}(t)) \leq c \cdot w(G(x(t)), \dot{x}(t)).$$

Condition (2) is thus satisfied for absolutely continuous function $x(\cdot)$ and multifunction $G(\cdot)$ which due to Theorem 2.1 proves our assertion.

DEFINITION 3.1. We say that two differential inclusions are equivalent if their families of trajectories coincide.

Corollary 3.1 permits in an evident way to formulate a sufficient condition for equivalence of two differential inclusions.

THEOREM 3.1. *If F, $G : \Omega \to \mathrm{Cl}(\mathbb{R}^d)$ are Borel measurable and there are positive constants c_1, c_2 such that*

$$(4) \qquad w(F(x),p) \le c_1 \cdot w(G(x),p) \ , \quad w(G(x),p) \le c_2 \cdot w(F(x),p)$$

for all $x \in \Omega$ and $p \neq 0$ then differential inclusions $\dot{x} \in F(x)$ and $\dot{x} \in G(x)$ are equivalent.

Condition given in the above theorem is not necessary for the equality of families of trajectories of two differential inclusions as shows the following example in which it is enough to use ordinary differential equations.

EXAMPLE 3.1. *The families of trajectories of differential equations in $\mathbb{R}$*

$$\dot{x} = 2\sqrt{|x|} \ , \quad \dot{x} = \frac{3}{2}\sqrt[3]{|x|}$$

coincide but

$$\lim_{x\to 0} \frac{\frac{3}{2}\sqrt[3]{|x|}}{2\sqrt{|x|}} = +\infty.$$

Remark that condition (4) implies the equality of cones $\mathrm{S}_{F(x)}$ and $\mathrm{S}_{G(x)}$. Moreover, the values $w(F(x),p)$ and $w(G(x),p)$ for every x and $p \neq 0$ are either both finite or infinite which means that the sets $F(x) \cap \mathrm{S}_{\{p\}}$ and $G(x) \cap \mathrm{S}_{\{p\}}$ are either both bounded or both unbounded. We shall give now a stronger version of sufficient condition for equivalence of differential inclusions where for certain x and $p \neq 0$ one of these sets may be bounded and another not.

THEOREM 3.2. *Suppose $F, G : \Omega \to \mathrm{Cl}(\mathbb{R}^d)$ are Borel measurable and there are positive constants c_1, c_2 such that if $w(F(x),p)$, $w(G(x),p) < +\infty$ then (4) holds. Assume also existence of $\eta > 0$ such that*

$$\text{if } w(F(x),p) = +\infty \text{ then } w(G(x),p) \ge \eta,$$
$$\text{if } w(G(x),p) = +\infty \text{ then } w(F(x),p) \ge \eta.$$

Differential inclusions $\dot{x} \in F(x)$ and $\dot{x} \in G(x)$ are then equivalent.

Proof. Let $x(\cdot)$ be a solution of (1). As before we may assume that $\dot{x}(t) \neq 0$ a.e. We check whether condition (2) is satisfied – the function

$$\psi(t) = \frac{\|\dot{x}(t)\|}{w(G(x(t)),\dot{x}(t))}$$

should be integrable. Put

$$T_1 = \{t : w(G(x(t)),\dot{x}(t)) = +\infty\},$$
$$T_2 = \{t : w(G(x(t)),\dot{x}(t)) < +\infty\,,\ w(F(x(t)),\dot{x}(t) < +\infty\},$$
$$T_3 = \{t : w(G(x(t)),\dot{x}(t)) < +\infty\,,\ w(F(x(t)),\dot{x}(t)) = +\infty\}.$$

These sets are measurable, disjoint and their union is the domain of $x(\cdot)$ (up to a set of measure zero, of course).

1. For $t \in T_1$ is $\psi(t) = 0$.
2. For $t \in T_2$ is $\psi(t) \leq c_1$.
3. For $t \in T_3$ is $\psi(t) \leq \frac{\|\dot{x}(t)\|}{\eta}$.

This means that (2) is satisfied and the trajectory of $x(\cdot)$ is also a trajectory of differential inclusion $\dot{x} \in G(x)$. The opposite inclusion of families of trajectories requires, of course, the same reasoning.

Theorems 3.1 or 3.2 can be used to prove existence of solutions in some special situations when the classical theorems do not match. The way to do this is to find a differential inclusion which satisfies already assumptions of some theorem on existence of solutions and which has the same trajectories as the given one. Let us see one of possible applications of this rule.

THEOREM 3.3. *Suppose $F : \Omega \to \mathrm{Cl}(\mathbb{R}^d)$ is Borel measurable, the cones $\mathrm{S}_{F(x)}$ are closed, convex and contain no straight lines. Moreover the multifunction $\mathrm{S}_{F(\cdot)}$ has closed graph and for some $r > 0$ the intersection $F(x) \cap (rB_1)$ is empty for all $x \in \Omega$. Then for every $x_0 \in \Omega$ the Cauchy problem*

$$\dot{x} \in F(x) \; , \quad x(0) = x_0 \tag{5}$$

has a solution on some interval $[0, a]$.

The proof is done in the following steps. There is a neighborhood $U \subset \Omega$ of x_0 and a hyperplane P not containing 0 such that for $x \in U$ the sets $F(x)$ are contained in the half-space generated by P and not containing 0.

Putting $G(x) = \mathrm{S}_{F(x)} \cap P$ we get a multifunction with nonempty, convex, compact values which is upper semicontinuous. The Cauchy Problem

$$\dot{x} \in G(x) \; , \quad x(0) = x_0$$

has thus a local solution $x(\cdot)$. The inequality

$$\frac{\|\dot{x}(t)\|}{w(F(x(t), \dot{x}(t))} \leq 1$$

is satisfied a.e. in the domain of $x(\cdot)$ so Theorem 2.1 implies existence of a solution of (5).

Let us give at the end some comments on relations to the results contained in Filippov's book [3], Chapter 2, par. 9. Filippov also considers the problem of equality of families of trajectories of two autonomous differential inclusions. He proves in Theorem 3 therein that if $q : \Omega \to \mathbb{R}$ is continuous and strictly positive then differential inclusions $\dot{x} \in F(x)$ and $\dot{x} \in q(x)F(x)$ have the same trajectories. This can be deduced from our Theorem 3.1 in the following way. Suppose $x(\cdot)$ is a solution of $\dot{x} \in F(x)$. Its trajectory is compact and so contained in the interior of some compact set $A \subset \Omega$ on which $q(x) > \gamma$ for some $\gamma > 0$. We may consider restrictions of $F(\cdot)$ and $q(\cdot)$ to A and the condition (4) is then satisfied.

Lemma 3 in the same part of Filippov's book is also near to our Theorem 3.1. It shows how to replace an autonomous differential inclusion with a nonautonomous one but with the reduction of dimension. It could be reformulated to give the equality of families of trajectories of two differential inclusions under the condition that the intersections of cones generated by right-hand sides with some fixed hyperplane are the same. It would require also that all $F(x)$ and $G(x)$ are strongly separated from 0 by some fixed hyperplane. Some regularity of right-hand sides would also be necessary which was not needed in the original formulation. In our case this regularity is Borel measurability (which seems to be a reasonable minimal assumption). Moreover, contrary to Filippov's assumptions we do not need the separation of multifunctions from 0 – it may even belong to $F(x)$ or $G(x)$.

4. Appendix

According to the Referee's suggestion we include here, for convenience of the Reader, the proof of a property used in the proof of Theorem 2.1.

LEMMA 4.1. *Let $\phi : [a,b] \to \mathbb{R}$ be absolutely continuous and $\dot{\phi}(t) > 0$ a.e. in $[a,b]$. Then the inverse function ϕ^{-1} is also absolutely continuous.*

Proof. Suppose ϕ^{-1} is not absolutely continuous and let $\varepsilon > 0$ be such that for any $\delta > 0$ exist pairwise disjoint intervals $[\alpha_1, \beta_1], \ldots, [\alpha_k, \beta_k]$ contained in $[\phi(a), \phi(b)]$ such that $\sum(\beta_i - \alpha_i) < \delta$ and $\sum(\phi^{-1}(\beta_i) - \phi^{-1}(\alpha_i)) \geq \varepsilon$.

We fix now $\alpha > 0$ such that $m(\{t \in [a,b] : \dot{\phi}(t) < \alpha\}) < \varepsilon/2$ and consider $\delta = (\varepsilon \cdot \alpha)/2$. Then

$$\sum(\beta_i - \alpha_i) = \sum(\phi(\phi^{-1}(\beta_i)) - \phi(\phi^{-1}(\alpha_i))$$
$$= \int\limits_{\bigcup[\phi^{-1}(\alpha_i),\phi^{-1}(\beta_i)]} !\dot{\phi}(t)\,dt \geq \alpha \cdot \frac{\varepsilon}{2} > \sum(\beta_i - \alpha_i).$$

This contradiction proves that ϕ^{-1} must be absolutely continuous.

Acknowledgement. The authors are grateful to the anonymous Referee whose remarks helped to improve this text.

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Received May 12, 2005.