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Tadeusz Jagodziński, Wiesław Sasin

WEIL HOMOMORPHISM IN NON-COMMUTATIVE DIFFERENTIAL SPACES

In this paper we construct Weil homomorphism in locally free module over a non-commutative differential space, which is a generalization of Sikorski differential space [6]. We consider real case, but the complex case can be done analogously.

1. Preliminaries

Let (M, C) be a differential space [6], $\mathfrak{A}$ a noncommutative unital algebra such that the center $Z(\mathfrak{A})$ is isomorphic with $\mathbf{R}$. We assume that algebra $\mathfrak{A}$ is finite dimensional with a basis $e_1, \ldots, e_m$. Typically $\mathfrak{A}$ could be a matrix algebra. A function $a : M \to \mathfrak{A}$ is said to be smooth if $a = \sum_{i=1}^{m} f^i e_i$ with $f^i \in C$ for $i = 1, \ldots, m$. Let A be the algebra of all smooth functions defined on M with values in $\mathfrak{A}$. The center $Z(A)$ of the algebra A is of the form $Z(A) = C \cdot \mathbf{1}$, where $\mathbf{1}$ is the unit of $\mathfrak{A}$.

The pair (M, A) is called a non-commutative differential space. Now we present some geometrical notions in such spaces.

A linear mapping $v : A \to \mathfrak{A}$ satisfying the Leibniz rule

$$v(a \cdot b) = v(a) \cdot b(p) + a(p) \cdot v(b)$$

for every $a, b \in A$, is said to be a tangent vector to (M, A) at the point $p \in M$.

The linear space of all tangent vectors to (M, A) at $p \in M$ will be denoted by $T_p(M, A)$. It is easy to observe that for every $v \in T_p(M, A)$, $v(\mathbf{1}) = 0$ and consequently

$$v(k) = v(k \cdot \mathbf{1}) = 0.$$

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We can consider $k \in \mathbf{R}$ as an element of A by the embedding $\mathbf{R} \subset A$, $k \to k \cdot \mathbf{1}$.

A mapping $F : (M, A) \to (N, B)$ between two differential non-commutative differential spaces is said to be smooth if $F^*B \subset A$ (or equivalently $b \circ F \in A$ for every $b \in B$).

For any $u \in T_p(M, C)$ we define $\overline{u} \in T_p(M, A)$ by $\overline{u}(a) = \sum_{i=1}^{m} u(f^i) e_i$, for $a \in A$, $a = \sum_{i=1}^{m} f^i e_i$.

Analogously, for any derivation $w \in Der(\mathfrak{A})$ we define $\overline{w} \in T_p(M, A)$ by $\overline{w}(a) = \sum_{i=1}^{m} f^i(p) w(e_i)$.

Let us denote by $T(M, A) = \cup_{p \in M} T_p(M, A)$, the disjoint sum of tangent spaces. We obviously have the projection $\pi : T(M, A) \to M$ given by $v \mapsto p$, where $v \in T_p(M, A)$.

A mapping $X : M \to T(M, A)$ such that $\pi \circ X = id_M$ is said to be a tangent vector field to the noncommutative differential space (M, A).

For any $a \in A$ we define the action of X on A as a $\mathfrak{A}$ -valued function on M, $Xa : M \to \mathfrak{A}$ given by $(Xa)(p) = X(p)(a)$for $p \in M$.

The set of all smooth tangent vector fields to (M, A) will be denote by $\mathbf{V}(A)$. In this set we naturally introduce the $Z(A)$ - module structure. We define addition and multiplication in the following way

$$(X + Y)(p) = X(p) + Y(p),$$

$$(\alpha \cdot X)(p) = \alpha(p) X(p)$$

for $p \in M$, $X, Y \in \mathbf{V}(A)$, $\alpha \in Z(A)$.

A non-commutative differential space (M, A) is said to be of a constant differential dimension n if and only if $Z(A)$ - module $\mathbf{V}(A)$ is locally free of rank n.

One can prove

PROPOSITION 1. *If (M, C) is a differential space of constant differential dimension k, then the non-commutative differential space (M, A) is of the constant differential dimension $k + l$, where l is the dimension of $Der\mathfrak{A}$.*

Proof. If $X_1, \ldots, X_k$ is a local basis of tangent vector fields to (M, C) on an open set U, then we define $\overline{X_i} : U \to \cup_{p \in M} T_p(U, A_U)$, $i = 1, \ldots, n$, by the formula:

$$\overline{X_i}(p)(a) = \sum_{j=1}^{m} X_i(p)\left(f^j\right) e_j,$$

for $p \in U$, where $a = \sum_{i=1}^{n} f^i e_i$, $f^i \in C_U$.

We also prolonge the basis $E_1, \ldots, E_l$ of $Der\mathfrak{A}$ to $\overline{E_1}, \ldots, \overline{E_l} \in \mathbf{V}(A_U)$ by

$$\overline{E_i}(p)(a) = \sum_{i=1}^{m} f^j E_i(e_j), \quad \text{for } p \in U.$$

We obtain a local basis $X_1, \ldots, X_k$, $\overline{E_1}, \ldots, \overline{E_l}$ of $Z(A)$ — module $\mathbf{V}(A)$. Let us denote by $A^k(\mathbf{V}(A), Z(A))$ the $Z(A)$ — module of all skew-symmetric $Z(A) - k-$linear mappings $\omega : \mathbf{V}(A) \times \ldots \times \mathbf{V}(A) \to Z(A)$.

For any $\omega \in A^k(\mathbf{V}(A), Z(A))$ we define its differential $d\omega \in A^{k+1}(\mathbf{V}(A), Z(A))$ by

$$(d\omega)(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k} (-1)^{i+1} X_i(\omega(X_1, \ldots, \widehat{X_i}, \ldots, X_{k+1}))$$
$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{k+1})$$

for $X_1, \ldots, X_{k+1} \in \mathbf{V}(A)$.

Here d is a local $\mathbf{R}$-linear operator and satisfies the standard properties:

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{|\omega_1|} \omega_1 \wedge d\omega_2, \quad d \circ d = 0.$$

Let us put

$$B^k(M) = \{d\theta : \theta \in A^{k-1}(\mathbf{V}(A), Z(A))\}$$

and

$$Z^k(M) = \{\omega \in A^k(\mathbf{V}(A), Z(A)) : d\omega = 0\}.$$

Let $H^k(M) = Z^k(M)/B^k(M)$ be k-th cohomology group of d. For any open set $U \in \tau_C$ let $\widetilde{A}(U)$ be the algebra of smooth $\mathfrak{A}$-valued functions $a : U \to \mathfrak{A}$ defined on U. $\widetilde{A}$ is a sheaf of non-commutative algebras over M. ∎

DEFINITION 2. Let $\mathfrak{y}$ be a sheaf of $Z(A)$- modules over the noncommutative differential space (M, A). A linear connection in $\mathfrak{y}$ is $\mathbf{R}$-linear mapping $D : \mathfrak{y}(M) \to A^1(\mathbf{V}(A), \mathfrak{y}(M))$ satisfying the condition

$$D(\alpha \cdot \eta) = (d\alpha)\eta + \alpha D\eta$$

for any $\eta \in \mathfrak{y}(M)$ and $\alpha \in Z(A)$.

2. Families of connections and curvature matrices

Let D: $\mathfrak{y}(M) \to A^1(\mathbf{V}(A), \mathfrak{y}(M))$ be a linear connection in the sheaf $\mathfrak{y}$. Decomposing 1-forms $D\varepsilon_i \in A^1(\mathbf{V}(\widetilde{A}(U)), \mathfrak{y}(U))$ for $i = 1, 2, \ldots, n$ with respect to a local $Z(U)$-base $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ of the $Z(U)$-module $\mathfrak{y}(U)$,

$U \in \tau_C$ we get

(1) $$D\varepsilon_i = \sum_{j=1}^{n} \theta_i^j(\varepsilon) \cdot \varepsilon_j, \quad i = 1, \ldots, n,$$

where $\theta_i^j(\varepsilon) \in A^1(\widetilde{A}(U)), Z(\widetilde{A}(U))$.

The matrix $\theta(\varepsilon) = (\theta_i^j(\varepsilon))$ of the 1-forms is called the matrix of the connection D with respect to the local $Z(\widetilde{A}(U))$-base ε. In the sequel, for simplicity we will write $Z(U)$ instead of $Z(\widetilde{A}(U))$ and $A^k(U)$ instead $\mathbf{V}(\widetilde{A}(U))$.

There exists an open cover $U = (U_\varepsilon)_{\varepsilon\in\Sigma}$ of M such that every $\varepsilon \in \Sigma$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, is a local $Z(U_\varepsilon)$-basis of $\mathfrak{y}(U_\varepsilon)$.

Definition 3. A family of matrices $\theta = (\theta(\varepsilon))_{\varepsilon\in\Sigma}$, $\theta(\varepsilon) \in M_n(A^1(U_\varepsilon))$ is called a family of connection matrices of a linear Σ connection D with respect to $\sum$.

It satisfies the following transformation law:

(2) $$dg + \theta(\varepsilon)|U_\varepsilon \cap U_{\varepsilon'} \cdot g = g \cdot \theta(\varepsilon)|_{U_\varepsilon \cap U_{\varepsilon'}},$$

for $\varepsilon, \varepsilon' \in \Sigma$ with $U_\varepsilon \cap U_{\varepsilon'} \neq \oslash$, where $g \in GL(n, Z(U_\varepsilon \cap U_{\varepsilon'}))$.

Now let $F^k(\Sigma)$ denote the set of families of matrices $\omega = (\omega(\varepsilon))_{\varepsilon\in\Sigma}$, with $\omega(\varepsilon) \in M_n(A^k(U_\varepsilon))$ satisfying the following transformation law:

$$\omega(\varepsilon')|_{U\cap U'} = \overline{g'}\omega(\varepsilon)|_{U\cap U'}g,$$

whenever $U_e \cap U_{e'} \neq \oslash$. $F^k(\Sigma)$ is $Z(M)$-module for $k \geq 1$.

Definition 4. The family $\Theta \in F^2(\Sigma)$ defined by

(3) $$\Theta(\varepsilon) = d\theta(\varepsilon) + \theta(\varepsilon) \wedge \theta(\varepsilon), \text{ for } \varepsilon \in \Sigma,$$

is called the family of curvative matrices of D with respect to Σ.

For $\omega = (\omega(\varepsilon))_{\varepsilon\in\Sigma} \in F^k(\Sigma)$, we define $D\omega = (D\omega(\varepsilon))_{\varepsilon\in\Sigma}$ to be the family of matrices defined by

(4) $$D\omega(\varepsilon) = d\omega(\varepsilon) + \theta(\varepsilon) \wedge \omega(\varepsilon) - (-1)^k\omega(\varepsilon) \wedge \theta(\varepsilon)$$

for $\varepsilon \in \Sigma$, where $d\omega(\varepsilon) := (d\omega_j^i(\varepsilon))$, $i, j = 1, \ldots, n$, $\varepsilon \in \Sigma$. Of course $D\omega \in F^{k+1}(\Sigma)$.

For any $\chi \in F^k(\Sigma)$ and $\Psi \in F^l(\Sigma)$ and $\varepsilon \in \Sigma$ let us define

$$\chi(\varepsilon) \wedge \psi(\varepsilon) := \Big(\sum_{m=1}^{n} \chi_m^i(\varepsilon) \wedge \psi_j^m(\varepsilon)\Big), \quad i, j = 1, \ldots, n.$$

We define a wedge product $\chi \wedge \psi \in F^{k+l}(\Sigma)$ by

(5) $$(\chi \wedge \psi)(\varepsilon) = \chi(\varepsilon) \wedge \psi(\varepsilon), \text{ for } \varepsilon \in \Sigma.$$

Thus $\chi \wedge \psi$ is the family of $k + l$-forms $\chi \wedge \psi = (\chi(\varepsilon) \wedge \psi(\varepsilon))_{\varepsilon\in\Sigma}$.

Now, let $[\chi,\psi]=([\chi,\psi](\varepsilon))_{\varepsilon\in\Sigma}$ be the family of matrices defined by

(6) $$[\chi,\psi](\varepsilon)=\chi(\varepsilon)\wedge\psi(\varepsilon)-(-1)^{kl}\psi(\varepsilon)\wedge\chi(\varepsilon),$$

for $\varepsilon\in\Sigma$.

It is easy to see that $[\chi,\psi]=\chi\wedge\psi-(-1)^{kl}\psi\wedge\chi$.

By standard computations [5], [7] one can prove the following identities:

PROPOSITION 5. *Let θ be a family of connection matrices of connection D with respect to Σ and Θ the corresponding family of curvacture matrices. If $\chi\in F^k(\Sigma)$ and $\psi\in F^l(\Sigma)$ then:*

1. $D(\chi\wedge\psi)=D\chi\wedge\psi+(-1)^k\chi\wedge D\psi$,
2. $D\Theta=0$,
3. $D^2\psi=[\Theta,\psi]$.

3. Invariant forms

Let $M_n(\mathbf{R})$ be the set of $n\times n$ matrices with real entries. A k-linear form $P:M_n(\mathbf{R})\times\ldots\times M_n(\mathbf{R})\to\mathbf{R}$ is said to be invariant if

(7) $$P(gA_1g^{-1},\ldots,gA_kg^{-1})=P(A_1,\ldots,A_k),$$

for every $g\in GL(n,\mathbf{R})$ and every $A_1,\ldots,A_k\in M_n(\mathbf{R})$.

We shall denote the $\mathbf{R}$-vector space of all k-linear forms on $M_n(\mathbf{R})$ by $I_k(M_n(\mathbf{R}))$.

Using the usual Einstein summation convention, each matrix $A=(a^i_j)\in M_n(\mathbf{R})$ can be uniquely expressed as a linear combination $A=a^i_jE^i_j$, $i,j=1,\ldots,n$, for the standard basis E^i_j, $i,j=1,\ldots,n$, of $M_n(\mathbf{R})$.

If $A_l=(\overset{l}{a}{}^{i_l}_{j_l})$, $l=1,\ldots,k$ are some matrices of $M_n(\mathbf{R})$ then for any $P\in I_k(M_n(\mathbf{R}))$ we have

(8) $$P(A_1,\ldots,A_k)=P^{i_1\ldots i_k}_{j_1\ldots j_k}\quad \overset{1}{a}{}^{j_1}_{i_1}\ldots\overset{k}{a}{}^{j_k}_{i_k},$$

where

(9) $$P^{i_1\ldots i_k}_{j_1\ldots j_k}=P(E^{i_1}_{j^1},\ldots,E^{i_k}_{j_k}).$$

Now, let $M_n(A^l(U))$ be $Z(U)$-module of all $n\times n$ matrices with entries in $Z(A)$-module l-forms $A^l(U)$, where $U\in\tau_C$ is open subset in M.

We prolonge the action of P to $M_n(A^{l_1}(U))\times\ldots\times M_n(A^{l_k}(U))$ by:

(10) $$P_U(\alpha_1,\ldots,\alpha_k)=P^{j_1\ldots j_k}_{i_1\ldots i_k}\quad \overset{1}{a}{}^{i_1}_{j_1}\wedge\ldots\wedge\ \overset{k}{a}{}^{i_k}_{j_k}\,.$$

It is easy to see that for open $V\subset U$ we have

(11) $$P_U(\alpha_1,\ldots,\alpha_k)\mid_V=P_V(\alpha_1\mid_V,\ldots,\alpha_k\mid_V).$$

$P_V(\alpha_1, \ldots, \alpha_k)$ is $l_1 + \ldots + l_k$ - form satysfying the invariant condition:

(12) $$P_U(g^{-1}\alpha_1 g, \ldots, g^{-1}\alpha_k g) = P_U(\alpha_1, \ldots, \alpha_k)$$

for $g \in GL(n, Z(U))$.

One can prove (see [5]):

PROPOSITION 6. *Let $P \in I_k(M_n(R))$ be invariant k-form. For any $\eta_1 \in F^{l_1}(\Sigma), \ldots, \eta_k \in F^{l_k}(\Sigma)$ there exists exactly one form $P(\eta_1, \ldots, \eta_k)$ of degree $l_1 + \ldots + l_k$ such that*

$$P(\eta_1, \ldots, \eta_k) \,|_U = P_U(\eta_1(U), \ldots, \eta_k(U)).$$

Moreover, the mapping

$$\Phi_P : F^{l_1}(\Sigma) \times \ldots \times F^{l_k}(\Sigma) \to A^{l_1+\ldots+l_k}(M),$$

defined by $\Phi_P(\eta_1, \ldots, \eta_k) := P(\eta_1, \ldots, \eta_k)$ is $Z(A)$-k-linear.

LEMMA 7. *Let $\eta_1 \in F^{l_1}(\Sigma), \ldots, \eta_k \in F^{l_k}(\Sigma)$ and $P \in I_k(M_n(R))$ be arbitrary elements. Then:*

(i) $\sum_{s=1}^{k}(-1)^{l(l_1+\ldots+l_{k-1})}P(\eta_1, \ldots, [\psi, \eta_s], \ldots, \eta_k) = 0$ *for any* $\psi \in F^l(\Sigma)$.

(ii) $dP(\eta_1, \ldots, \eta_k) = \sum_{s=1}^{k}(-1)^{l_1+\ldots+l_{s-1}}P(\eta_1, \ldots, D\eta_s, \ldots, \eta_k)$.

Proof. (i) It is well-known [7] that if $A_1, \ldots, A_k$, $B \in M_n(\mathbf{R})$ and $P \in I_k(M_n(\mathbf{R}))$, then

$$\sum_{s=1}^{k} P(A_1, \ldots, [B, A_s], \ldots, A_k) = 0$$

and consequently

$$\sum_{s=1}^{k} P(A_1, \ldots, B, A_s, \ldots, A_k) - P(A_1, \ldots, A_s B, \ldots, A_k) = 0.$$

Using the above identity one can verify that

$$\sum_{s=1}^{k} P_V(\alpha_1, \ldots, B\alpha_s, \ldots, \alpha_k) - P_V(\alpha_1, \ldots, \alpha_s B, \ldots, \alpha_k) = 0$$

for $V \in \tau_C$, $\alpha_1 \in M_n(A_1(V)), \ldots, \alpha_k \in M_n(A_k(V))$ and $B \in M_n(\mathbf{R})$. Using (11) if we multiply the above equation by an arbitrary $\Phi \in M_n(A_l(V))$, we obtain:

$$\sum_{s=1}^{k}(-1)^{l(l_1+\ldots+l_{s-1})}P_U(\,\alpha_1, \ldots, \Phi B \wedge \alpha_s, \ldots, \alpha_k)$$
$$- \sum_{s=1}^{k}(-1)^{l(l_1+\ldots+l_{s-1})}(-1)^{ll_s}P_U(\alpha_1, \ldots, \alpha_s \wedge \Phi B, \ldots, \alpha_k) = 0.$$

This and the fact the every $\psi \in M_n(A^l(V))$ can be written as $\psi = \psi^i_j E^i_j$ by linearity implies:

$$\sum_{s=1}^{k}(-1)^{l(l_1+\ldots+l_{s-1})}P_U(\alpha_1,\ldots,\psi\wedge\alpha_s,\ldots,\alpha_k)+$$
$$-\sum_{s=1}^{k}(-1)^{l(l_1+\ldots+l_{s-1})}(-1)^{ll_s}P(\alpha_1,\ldots,\alpha_s\wedge\psi,\ldots,\alpha_k)=0,$$

or equivalently

$$\sum_{s=1}^{k}(-1)^{l(l_1+\ldots+l_{s-1})}P_U(\alpha_1,\ldots,[\psi,\alpha_s],\ldots,\alpha_k)=0.$$

Now (i) is evident.

(ii) Let $\varepsilon \in \Sigma$. From the very definition $P(\eta_1,\ldots,\eta_k)\mid_{U_\varepsilon}$ we have:

$$dP(\eta_1,\ldots,\eta_k)\mid_{U_\varepsilon}=P^{i_1\ldots i_k}_{j_1\ldots j_k}\sum_{s=1}^{k}(-1)^{l_1+\ldots+l_{s-1}}\overset{1}{\eta}{}^{j_1}_{i_1}(\varepsilon)\wedge\ldots\wedge d\overset{s}{\eta}{}^{j_s}_{i_s}(\varepsilon)\wedge\ldots\wedge\overset{k}{\eta}{}^{j_k}_{i_k}(\varepsilon),$$

that is

$$dP(\eta_1,\ldots,\eta_k)|_{U_\varepsilon}=\sum_{s=1}^{k}(-1)^{l_1+\ldots+l_{s-1}}P_{U\varepsilon}(\eta_1(\varepsilon),\ldots,d\eta_s(\varepsilon),\ldots,\eta_k(\varepsilon)).$$

Then by linearity we have:

$$dP(\eta_1,\ldots,\eta_k)|U_\varepsilon=\sum_{s=1}^{k}(-1)^{l_1+\ldots+l_{s-1}}P_{Ue}(\eta_1(\varepsilon),\ldots,D\eta_s(\varepsilon),\ldots,\eta_k(\varepsilon))+$$
$$-\sum_{s=1}^{k}(-1)^{l_1+\ldots+l_{s-1}}P_{U\varepsilon}(\eta_1(\varepsilon),\ldots,[\Theta,\eta_s](\varepsilon),\ldots,\eta_k(\varepsilon)).$$

From (i) the second term is zero and we obtain:

$$dP(\eta_1,\ldots,\eta_k)\mid_{U_\varepsilon}=\sum_{s=1}^{k}(-1)^{l_1+\ldots+l_s}P_{U\varepsilon}(\eta_1(\varepsilon),\ldots,D\eta_s(\varepsilon),\ldots,\eta_k(\varepsilon))$$

which gives us (ii). ∎

Now let us notice that for every $\omega \in F^k(\Sigma)$ one can define a skew symetric k-linear mapping

$$\omega : \mathbf{V}(A)\times\ldots\times\mathbf{V}(A)\to L(\mathfrak{y}(M),\mathfrak{y}(M)),$$

given by

$$\omega(X_1,\ldots,X_k)(\varepsilon_i)=\omega^j_i(\varepsilon)(X_1,\ldots,X_k)\varepsilon_j,$$

for $i,j=1,\ldots,n$.

One can easily see that ω does not depend on the choice of the family Σ. In the sequell for a connection D in $\mathfrak{y}$ let Θ be the curvature 2-form $\Theta : \mathbf{V}(A) \times \mathbf{V}(A) \to L(\mathfrak{y}(M), \mathfrak{y}(M))$ defined by the family of curvature matrices $(\Theta(\varepsilon))_{\varepsilon \in \Sigma}$. Let $P(\Theta) := P(\Theta, \dots, \Theta)$ be $2k$-form obtaining from the family $(\Theta(\varepsilon))_{\varepsilon \in \Sigma}$ and $P \in I_k(M_n(\mathbf{R}))$.

Now we can prove the folowing generalization of Weil theorem:

THEOREM 8. *Let $\mathfrak{y}$ be locally free sheaf of $Z(A)$ — modules of rank n over non-commutative differential space (M, A) and let θ be a family of connection D in $\mathfrak{y}$. Then for any invariant k-form $P \in I_k(M_n(R))$*

(a) *$dP(\Theta) = 0$, that is $2k$-form $P(\Theta)$ is closed,*
(b) *The cohomology class $[P(\Theta)]$ is independent of the connection D.*

Sketch of proof:

(a) By Lemma 7 we have

$$dP(\Theta) = \sum_{s=1}^{k} P(\Theta, \dots, D\Theta, \dots, \Theta)$$

and $D\Theta = 0$ by Proposition 1. Hence $dP(\Theta) = 0$.

(b) Let θ and θ' be two families of connection matrices of connection D and D' respectively. We may assume that both families are indexed by the same family of local basis Σ. Let $\eta := \theta' - \theta$, so $\eta \in F^1(\Sigma)$. We consider the one-parameter family

$$\theta_t(e) = \theta(e) + t\eta(e) \text{ for } t \in \mathbf{R}.$$

It may be checked that

$$\Theta_t = \Theta + tD\eta + t^2\eta \wedge \eta \text{ for } t \in \mathbf{R} \tag{13}$$

and then

$$D\Theta_t = D\Theta + tD^2\eta + t^2 D(\eta \wedge \eta). \tag{14}$$

Hence using Proposition 1. we have

$$D\Theta_t = t[\Theta_t, \eta] \; for \;\; t \in \mathbf{R}. \tag{15}$$

Using (13) we obtain

$$\frac{d}{dt}\Theta_t = D\eta + 2t\eta \wedge \eta. \tag{16}$$

One can verify that

$$\frac{d}{dt}P(\Theta_t) = kdP\Big(\frac{d}{dt}\Theta_t, \Theta_t, \dots, \Theta_t\Big). \tag{17}$$

From (16)–(17) we obtain

$$\frac{d}{dt}P(\Theta_t) = kdP(D\eta, \Theta_t, \ldots, \Theta_t) + 2ktdP(\eta \wedge \eta, \Theta_t, \ldots, \Theta_t).$$

It follows from Lemma 7 that

$$(18) \quad dP(\eta, \Theta_t, \ldots, \Theta_t) = P(D\eta, \Theta_t, \ldots, \Theta_t) - (k-1)P(\eta, D\Theta_t, \Theta_t, \ldots, \Theta_t).$$

From Lemma 7 we obtain

$$(19) \quad P([\eta, \eta], \Theta_t, \ldots, \Theta_t) - (t-1)P(\eta, [\eta, \Theta_t], \ldots, \Theta_t) = 0.$$

Using (14), (18), (19) we obtain

$$d(kP(\eta, \Theta_t, \ldots, \Theta_t)) = kP(D\eta, \Theta_t, \ldots, \Theta_t) + ktP(\eta \wedge \eta, \Theta_t, \ldots, \Theta_t).$$

Thus

$$(20) \quad \frac{d}{dt}P(\Theta_t) = d(kP(\eta, \Theta_t, \ldots, \Theta_t)).$$

Hence

$$\int_0^1 \frac{d}{dt}P(\Theta_t)dt = \int_0^1 d(kP(\eta, \Theta_t, \ldots, \Theta_t))dt,$$

or equivalently

$$P(\Theta_1) - P(\Theta_0) = d\int_0^1 kP(\eta, \Theta_t, \ldots, \Theta_t)dt.$$

Hence $[P(\Theta')] = [P(\Theta)]$. ∎

COROLLARY 9. *The mapping $w : I^*(M_n(\mathbf{R})) \to H^*(M)$ given by*

$$I^*(M_n(R)) \ni P \longmapsto [P(\Theta)] \in H^*(M),$$

is well defined homomorphism of the graded algebras w is called the Weil homomorphism [4]. *The cohomology class $w(P)$ for $P \in I^*(M_n(\mathbf{R}))$ is called a characteristic class of the sheaf* $\mathfrak{y}$.

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WARSAW UNIVERSITY OF TECHNOLOGY
FACULTY OF MATHEMATICS AND INFORMATION SCIENCE
Pl. Politechniki 1
00-661 WARSAW, POLAND

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