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FIXED POINTS OF ASYMPTOTICALLY REGULAR NONCOMPATIBLE MAPS

Abstract. The concept of R-weakly commutativity of type A for single-valued mapping is extended to multivalued mappings. The structure of common fixed points and coincidence points of a pair of R-weakly commuting multivalued mappings of type A is also discussed.

1. Introduction and preliminaries

The study of fixed points of multivalued mappings satisfying some contractive type conditions has been a very active topic in the last three decades. The interest on this subject was enhanced after the publication of a paper by Nadler [11]. Since then there has been a lot of activity in this area and a number of generalizations of Nadler's results have appeared. Most of the fixed point theorems existing in the mathematical literature deal with compatible mappings. So, it would be a natural question: what about the mappings which are not compatible. In this paper, we shall investigate such mappings. The compatible single valued mappings were introduced by Jungck [5, 6] as a generalization of commuting mappings. Rashwan [16], Beg and Azam [2] and Kaneko and Sessa [8] extended independently the concept of compatibility for single valued mappings to the setting of single valued and multivalued mappings. Recently Pathak, Cho and Khang [15] introduced the concept of R-weakly commuting mappings of type A and showed that they are not compatible. The notion of R-weak commutativity was originally defined by Pant [12] and then in [13, 14], he proved some fixed point theorems for noncompatible mappings. The aim of this paper is to obtain some common fixed point and coincidence point theorems for a pair of R-weakly commuting multivalued mappings of type A. We may mention that using the idea

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of Shahzad [19, 20], it is possible to obtain applications of our results to the best approximation theory.

Let X be a metric space with a metric d. Then, for $x \in X$ and $A \subseteq X$, $d(x, A) = \inf\{d(x, y) : y \in A\}$. We denote by $CB(X)$ the class of all nonempty bounded closed subsets of X and by $K(X)$ the class of all nonempty compact subsets of X. Let H be the Hausdorff metric with respect to d, that is,

$$H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}$$

for every $A, B \in CB(X)$. It is well known that if X is a complete metric space then so is the metric space $(CB(X), H)$. Let $T : X \to CB(X)$ be a mapping. A point $p \in X$ is said to be a fixed point of $T : X \to CB(X)$ if $p \in Tp$. Let $f : X \to X$ be a mapping. The point p is called a coincidence point of f and T if $fp \in Tp$. A mapping $\phi : (0, \infty) \to [0, 1)$ is said to have the property (P) if, for each t in the domain of ϕ, there exist $\delta(t) > 0$ and $s(t) < 1$ such that $0 \leq r - t < \delta(t)$ implies $\phi(r) \leq s(t) < 1$ (cf., [3], [17]). It is readily seen that the property (P) is equivalent to saying that $\lim_{r \to t^+} \sup \phi(r) < 1$ for all $t > 0$. The mappings $f : X \to X$ and $T : X \to CB(X)$ are called compatible [5] if $fTx \in CB(X)$ for all $x \in X$ and $H(fTx_n, Tfx_n) \to 0$ whenever $\{x_n\}$ is a sequence in X such that $Tx_n \to A \in CB(X)$ and $fx_n \to t \in A$. The mappings $f, g : X \to X$ are called R-weakly commuting of type A_g if, for all $x \in X$, there exists some positive real number R such that $d(ffx, gfx) \leq Rd(fx, gx)$.

EXAMPLE 1.1 ([14]). Let $X = [2, 20]$ and d the usual metric on X. Define $f, g : X \to X$ by

$$fx = \begin{cases} 2 \text{ if } & x = 2, \\ 6 \text{ if } & 2 < x \leq 5, \\ 2 \text{ if } & x > 5. \end{cases}$$

$$gx = \begin{cases} 2 & \text{if } x = 2, \\ 12 & \text{if } 2 < x \leq 5, \\ \frac{x+1}{3} & \text{if } x > 5. \end{cases}$$

Then f and g are R-weakly commuting of type A_g but they are not compatible.

We now introduce the following definition.

DEFINITION 1.2. The mappings $f : X \to X$ and $T : X \to CB(X)$ are said to be R-weakly commuting of type A_T at $z \in X$ if, there exists some positive real number R such that

$$d(ffz, Tfz) \leq Rd(fz, Tz).$$

Here f and T are R-weakly commuting of type A_T on X if the above inequality holds for all $z \in X$.

If T is a single valued self mapping of X this definition of R-weak commutativity reduces to that of Pathak, Cho and Kang [15].

EXAMPLE 1.3. Let $X = [1, \infty)$ with the usual metric. Define $f : X \to X, T : X \to CB(X)$ by $fx = 2x$ and $Tx = [1, 2x + 1]$ for all $x \in X$. Let $\{x_n\}$ is a sequence in X, such that $x_n \to 1$. Then

$$\lim_{n\to\infty} fx_n = 2 \in [1,3] = \lim_{n\to\infty} Tx_n,$$
$$\lim_{n\to\infty} H(fTx_n, Tfx_n) \neq 0 \text{ and } d(ffx, Tfx) = 0.$$

Therefore the mappings f and T are R-weakly commuting of type A_T but they are not compatible.

EXAMPLE 1.4. Let $X = [0, \infty)$ be endowed with usual metric d. Let for all $x \in X, Tx = [1, 2]$ and

$$fx = \begin{cases} 1 + \frac{1}{2}x \text{ if } & x \in [0, 1], \\ 1 & \text{if } x \in (1, \infty). \end{cases}$$

Then $Tfx = [1, 2]$ and

$$ffx = \begin{cases} \frac{3}{2} \text{ if } & x = 0, \\ 1 \text{ if } & 0 < x \leq 1, \\ \frac{3}{2} \text{ if } & 1 < x < \infty. \end{cases}$$

Therefore f and T are R-weakly commuting of type A_T. Now suppose that $\{x_n\}$ is a sequence in X such that $x_n \to 0$. Then $\lim_{n\to\infty} fx_n = 1 \in \lim_{n\to\infty} Tx_n$. On the other hand $\lim_{n\to\infty} H(fTx_n, Tfx_n) \neq 0$ and thus f and T are not compatible.

We shall require the following well-known facts (cf., [1], [11]).

LEMMA 1.5. *If $A, B \in CB(X)$ with $H(A, B) < \epsilon$, then, for each $a \in A$, there exists an element $b \in B$ such that $d(a, b) < \epsilon$.*

LEMMA 1.6. *If $\{A_n\}$ is a sequence in $CB(X)$ and $\lim_{n\to\infty} H(A_n, A) = 0$ for $A \in CB(X)$. If $x_n \in A_n$ and $\lim_{n\to\infty} d(x_n, x) = 0$ then $x \in A$.*

If, for $x_0 \in X$, there exists a sequence $\{x_n\}$ in X such that $fx_n \in Tx_{n-1}$ then $O_f(x_0) = \{fx_n : n = 1, 2, \ldots\}$ is said to be orbit for $(T; f)$ at x_0. If, for $x_0 \in X$, there exists a sequence $\{x_n\}$ in X such that every Cauchy sequence of the form $O_f(x_0)$ converges in X, then X is called $(T; f)$-orbitally complete. The mapping T is called asymptotically regular at $x_0 \in X$ if for any sequence $\{x_n\}$ in X and each sequence $\{y_n\}$ in X such that $y_n \in Tx_{n-1}$, $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$. For details, we refer to [21].

2. Main results

We are now in a position to state and prove our first result.

THEOREM 2.1. *Let X be a metric space. The mappings $f : X \to X, T : X \to CB(X)$ such that $TX \subseteq f(X)$ and*

$$H(Tx, Ty) < \phi(d(fx, fy))d(fx, fy) \tag{1}$$

for every $x, y \in X$ with $x \neq y$, where $\phi : (0, \infty) \to [0, 1)$ is a real function with the property (P). If there exists a point $x_0 \in X$ such that T is asymptotically regular at x_0 and $f(X)$ is $(T; f, x_0)$-orbitally complete then f and T have a coincidence point $z \in X$. Further, if fz is a fixed point of f, then fz is a common fixed point of f and T provided f and T are R-weakly commuting mappings of type A_T at z.

Proof. Let x_0 be a point in X and $y_0 = fx_0$. Since $Tx_0 \subseteq fX$, there exists $x_1 \in X$ such that $y_1 = fx_1 \in Tx_0$. Let $\epsilon = \phi(d(fx_0, fx_1))d(fx_0, fx_1)$. Then by (1) we have $H(Tx_0, Tx_1) < \epsilon$. Now, using Lemma 1.5, we obtain $y_2 \in Tx_1$ such that $d(y_1, y_2) < \epsilon$. It further implies that

$$d(y_1, y_2) < d(fx_0, fx_1).$$

Since $Tx_1 \subseteq fX$, there exists $x_2 \in X$ such that $y_2 = fx_2$. Hence

$$d(fx_1, fx_2) < d(fx_0, fx_1).$$

Continuing in this fashion, we produce a sequence $\{x_n\}$ of points of X such that $fx_n \in Tx_{n-1}$ $(n \geq 1)$ and

$$d(fx_n, fx_{n+1}) < \phi(d(fx_{n-1}, fx_n))d(fx_{n-1}, fx_n) < d(fx_{n-1}, fx_n).$$

Thus $\{d(fx_n, fx_{n+1})\}$ is a decreasing sequence of positive real numbers and, therefore, converges to its greatest lower bound $L \geq 0$. We claim that $L = 0$. Indeed, if $L > 0$, then by the property (P) there exist $\delta(t) > 0$ and $s(t) < 1$ such that $0 \leq r - t < \delta(t)$ implies $\phi(r) \leq s(t)$. Since $d(fx_n, fx_{n+1}) \to L$, for given $\delta(t) > 0$ there exists an integer N such that $0 \leq d(fx_n, fx_{n+1}) - t \leq \delta(t)$ for all $n \geq N$. This yields

$$\phi(d(fx_n, fx_{n+1})) \leq s(t) \qquad \text{for all } n \geq N.$$

Then

$$\begin{aligned} d(fx_n, fx_{n+1}) &< \phi(d(fx_{n-1}, fx_n))d(fx_{n-1}, fx_n) \\ &\leq Md(fx_{n-1}, fx_n) \leq \cdots\cdots\cdots\cdots\cdots \\ &\leq M^n d(fx_0, fx_1) \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

where

$M = \max\{\phi(d(fx_0, fx_1)), \phi(d(fx_1, fx_2)), \ldots, \phi(d(fx_{N-1}, fx_N)), s(t)\} < 1.$

So we have reached a contradiction to the assumption that $L > 0$. Thus

$$\lim_{n\to\infty} d(fx_n, fx_{n+1}) = 0.$$

It further implies that

$$\lim_{n\to\infty} d(fx_n, Tx_n) = 0.$$

We claim that the sequence $\{fx_n\}$ is Cauchy. For, if not, there exist $q > 0$ and subsequences $\{n_i\}$ and $\{m_i\}$ of integers with $n_i < m_i$ such that

$$d(fx_{n_i}, fx_{m_i}) \geq q, \quad d(fx_{n_i}, fx_{n_i-1}) < q \quad \text{for } i = 1, 2, 3, \ldots$$

Now

$$q \leq d(fx_{n_i}, fx_{m_i}) \leq d(fx_{n_i}, fx_{m_i-1}) + d(fx_{m_i-1}, fx_{m_i}).$$

On making $i \to \infty$ we obtain

$$\lim_{i\to\infty} d(fx_{n_i}, fx_{m_i}) = q,$$

since $d(fx_{m_i}, fx_{m_i-1}) < q$. By the property (P), there exist $\delta(q) > 0$ and $s(q) < 1$ such that $0 \leq r - q < \delta(q)$ implies $\phi(r) \leq s(q)$.

Since $\lim_{i\to\infty} d(fx_{n_i}, fx_{m_i}) = q$, there exists an integer N_0 such that

$$0 \leq d(fx_{n_i}, fx_{m_i}) - q \leq \delta(q) \quad \text{for all } i \geq N_0.$$

So

$$\phi(d(fx_{n_i}, fx_{m_i})) \leq s(q) \quad \text{for all } i \geq N_0.$$

Further,

$$\begin{aligned} d(fx_{n_i}, fx_{m_i}) &\leq d(fx_{n_i}, fx_{n_i+1}) + d(fx_{n_i+1}, fx_{m_i+1}) + d(fx_{m_i+1}, fx_{m_i}) \\ &\leq d(fx_{n_i}, fx_{n_i+1}) + \phi(d(fx_{n_i}, fx_{m_i}))d(fx_{n_i}, fx_{m_i}) + \\ &\qquad\qquad d(fx_{m_i+1}, fx_{m_i}) \\ &\leq d(fx_{n_i}, fx_{n_i+1}) + s(q)d(fx_{n_i}, fx_{m_i}) + d(fx_{m_i+1}, fx_{m_i}). \end{aligned}$$

This inequality on letting $i \to \infty$ implies that $q = s(q)q < q$, a contradiction.

Hence $\{fx_n\}$ is a Cauchy sequence in X. Since $f(X)$ is $(T; f, x_0)$-orbitally complete, $\{fx_n\}$ has a limit say u, in $f(X)$. Therefore, $u = fz$ for some $z \in X$. Now

$$\begin{aligned} d(fz, Tz) &\leq d(fz, fx_n) + d(fx_n, Tz) \\ &\leq d(fz, fx_n) + H(Tx_{n-1}, Tz) \\ &\leq d(fz, fx_n) + \phi(d(fx_{n-1}, fz))d(fx_{n-1}, fz). \end{aligned}$$

Letting $n \to \infty$ the above inequality yields $d(fz, Tz) = 0$. This implies that $fz \in Tz$. Since f and T are R-weakly commuting of type A_T at z, we have

$$d(ffz, Tfz) \leq Rd(fz, Tz).$$

This shows that $ffz \in Tfz$. If $u = fz$ is also a fixed point of f, then $u = fz = fu \in Tu$. Hence $u = fz$ is a common fixed point of f and T.

The following is an example of R-weakly commuting mappings of type A_T satisfying the conditions of Theorem 2.1 and having a common fixed point.

EXAMPLE 2.2. Let $X = [0,1]$ and d the usual metric on X. Define $f : X \to X$ and $T : X \to CB(X)$ by $fx = \frac{1}{2}x^{1/2}$, $Tx = [0, \frac{1}{8}x^{1/2}]$ for all $x \in X$. Then, for any $x \in X$,

$$d(ffx, Tfx) = \frac{3}{8\sqrt{2}}|x^{1/4}|, \quad d(fx, Tx) = \frac{3}{8}|x^{1/2}|$$

that is,

$$d(ffx, Tfx) \leq \frac{1}{\sqrt{2}}d(fx, Tx).$$

Thus the mappings $f : X \to X$ and $T : X \to CB(X)$ are R-weakly commuting of type A_T. Taking the function $\phi(x) = c$, where $1/4 < c < 1$, it is easily seen that f and T satisfy all the conditions of Theorem 2.1 and have a common fixed point $x = 0$. Note that f and T do not satisfy the conditions of theorems in [3], [4], [7], [10] and [11].

THEOREM 2.3. *Let X, f and T satisfy the hypotheses of Theorem 2.1. Suppose $f(X)$ is complete and for each $x, y \in X$,*

$$(2)\quad d(fx, fy) \leq k \max\left\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}\right\},$$

where $0 \leq k < 1$. Then f and T have a common fixed point provided f and T are R-weakly commuting of type A_T on X.

Proof. Let $y_n = fy_{n-1} = f^n z, n = 1, 2, \ldots$, where z is a coincidence point of f and T [the existence of z comes from Theorem 2.1]. It follows from (2) that

$$\begin{aligned}
d(y_n, y_{n+1}) &= d(fy_{n-1}, fy_n) \\
&\leq k \max\Big\{d(y_{n-1}, y_n), d(y_{n-1}, fy_{n-1}), d(y_n, fy_n), \\
&\qquad\qquad \frac{d(y_{n-1}, fy_n) + d(y_n, fy_{n-1})}{2}\Big\} \\
&\leq k \max\left\{d(y_{n-1}, y_n), d(y_n, y_{n+1}), \frac{d(y_{n-1}, y_{n+1})}{2}\right\} \\
&\leq kd(y_{n-1}, y_n) \leq \ldots\ldots\ldots\ldots\ldots \leq k^n d(y_1, y_0).
\end{aligned}$$

This shows that $\{y_n\}$ is a Cauchy sequence in X and so $f^n z \to p \in f(X)$. Since f and T are R-weak commuting mappings of type A_T,

$$d(ffz, Tfz) \le Rd(fz, Tz).$$

Since $fz \in Tz$, the above inequality yields

$$f^2 z = ffz \in Tfz.$$

Further, we have

$$d(f^3 z, Tf^2 z) \le Rd(f^2 z, Tfz) = 0.$$

Continuing in this fashion, we get $f^{n+1} z \in Tf^n z$.

Using (2), we have

$$d(fp, f^{n+1}z) \le k \max\Big\{d(p, f^n z), d(p, fp), d(f^n z, f^{n+1} z), \frac{d(p, f^{n+1}z) + d(f^n z, fp)}{2}\Big\}.$$

Taking the limit as $n \to \infty$, we get

$$d(fp, p) \le k \max\Big\{0, d(p, fp), 0, \frac{d(p, fp)}{2}\Big\} = kd(p, fp).$$

Since $0 \le k \le 1$, we have $d(fp, p) = 0$ and so $p = fp$.

Now

$$\begin{aligned} d(p, Tp) &\le d(p, f^{n+1}z) + H(Tf^n z, Tp) \\ &\le d(p, f^{n+1}z) + \phi(d(f^{n+1}z, fp))d(f^{n+1}z, fp) \\ &< d(p, f^{n+1}z) + d(f^{n+1}z, fp). \end{aligned}$$

Letting $n \to \infty$ the above inequality yields $d(p, Tp) = 0$ and so $p \in Tp$.

We now obtain a coincidence point theorem for mulltivalued R-weakly commuting mappings satisfying the Meir-Keeler [9] type contractive condition.

THEOREM 2.4. *Let X be a metric space and take $f : X \to X$ and $T : X \to K(X)$ such that $TX \subseteq fX$ and for given $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that*

(i) *$\epsilon \le d(fx, fy) < \epsilon + \delta$ implies $H(Tx, Ty) < \epsilon$*

and

(ii) *$Tx = Ty$ whenever $fx = fy$.*

If there exists a point $x_0 \in X$ such that T is asymptotically regular at x_0 and $f(X)$ is $(T; f, x_0)$-orbitally complete then f and T have a coincidence point $z \in X$. Further, if fz is a fixed point of f then fz is a common fixed point of f and T, provided f and T are R-weakly commuting mappings of type A_T at z.

Proof. Fix $x_0 \in X$. Since $TX \subseteq fX$ then we can choose $y_1 = fx_1 \in Tx_0$. If $Tx_0 = Tx_1$, choose $y_2 = fx_2 \in Tx_1$ such that $y_1 = y_2$. If $Tx_0 \neq Tx_1$, choose $y_2 = fx_2 \in Tx_1$ such that

$$d(y_1, y_2) \leq H(Tx_0, Tx_1).$$

Such a choice is possible since Tx_1 is compact. In general, choose $y_n = fx_n \in Tx_{n-1}$ such that $y_{n-1} = y_n$ if $Tx_{n-2} = Tx_{n-1}$ and

$$d(y_{n-1}, y_n) \leq H(Tx_{n-2}, Tx_{n-1}) \quad \text{otherwise.}$$

It is clear from (i) that for all $x, y \in X$ with $fx \neq fy$ we have

$$H(Tx, Ty) < d(fx, fy). \tag{3}$$

Then

$$H(Tx_{n-2}, Tx_{n-1}) \leq d(fx_{n-2}, fx_{n-1}) = d(y_{n-2}, y_{n-1}).$$

Since

$$d(y_{n-1}, y_n) \leq H(Tx_{n-2}, Tx_{n-1}),$$

it follows that $\{d(y_n, y_{n+1})\}$ is a decreasing sequence of real numbers and, therefore, converges to its greatest lower bound $r \geq 0$. We claim that $r = 0$. For, if $r > 0$, then given $\delta > 0$ there exists an integer N such that

$$r \leq d(y_n, y_{n+1}) < r + \delta \quad \text{for all } n \geq N.$$

It implies that

$$H(Tx_n, Tx_{n+1}) < r \quad \text{for all } n \geq N.$$

Further,

$$d(y_{n+1}, y_{n+2}) < r \quad \text{for all } n \geq N,$$

a contradiction. Therefore

$$d(fx_n, Tx_n) \leq d(fx_n, Tx_{n+1}) \leq d(y_n, y_{n+1}) \to 0.$$

Using an analogous argument as in the proof of Theorem 1 of Rhoades, Park and Moon [18] it can be seen that $\{y_n\}$ is a Cauchy sequence in X. Therefore there exists $u \in f(X)$ such that $d(fx_n, u) \to 0$. Also $u = fz$ for some $z \in X$. Now, using (3), we have

$$d(fz, Tz) \leq d(fx_n, fz) + d(fx_n, Tz) \leq d(fx_n, fz) + d(fx_{n-1}, fz).$$

This inequality by letting $n \to \infty$ yields $fz \in Tz$.

If $u = fz$ is a fixed point of f then $u = fz = ffz$. Since f and T are R-weakly commuting of type A_T at z we have

$$d(ffz, Tfz) \leq Rd(fz, Tz) = 0.$$

This implies that $u = fu \in Tu$.

THEOREM 2.5. *Let X, f and T satisfy the hypotheses of Theorem 2.4. Suppose $f(X)$ is complete and for each $x, y \in X$,*

$$(4)\quad d(fx, fy) \le k \max\left\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}\right\},$$

where $0 \le k < 1$. Then f and T have a common fixed point provided f and T are R-weakly commuting of type A_T on X.

Proof. The proof is similar to the proof of Theorem 2.3. Instead of using Theorem 2.1, we use Theorem 2.4.

REMARK 2.6. When T is compact-valued, Theorem 2.1 can also be concluded from Theorem 2.4. To see this, let $\epsilon > 0$. Motivated by Xu [22], choose $\delta(\epsilon) > 0$ and $s(\epsilon) < 1$ such that $s(\epsilon)[\epsilon + \delta(\epsilon)] < \epsilon$, $\lim_{r\to\epsilon^+} \sup \phi(r) \le s(\epsilon)$, and $\phi(r) \le s(\epsilon)$ whenever $0 \le r - \epsilon < \delta(\epsilon)$. Then $\epsilon \le d(fx, fy) < \epsilon + \delta(\epsilon)$ implies

$$H(Tx, Ty) < \phi(d(fx, fy))d(fx, fy) < s(\epsilon)[\epsilon + \delta(\epsilon)] < \epsilon.$$

Hence the conclusions of Theorem 2.1 follow from Theorem 2.4.

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