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SLIGHTLY β-CONTINUOUS MULTIFUNCTIONS

Abstract. In this paper, we introduce and study upper and lower slightly β-continuous multifunctions as a generalization of upper (lower) semicontinuous, upper (lower) α-continuous, upper (lower) precontinuous, upper (lower) quasi-continuous, upper (lower) γ-continuous, upper (lower) β-continuous multifunctions and slightly β-continuous functions. Some characterizations and several properties concerning upper (lower) slightly β-continuous multifunctions are obtained. Furthermore, the relationships between upper (lower) slightly β-continuous multifunctions and other related multifunctions are also discussed.

1. Introduction

Continuity of functions and various types of stronger and weaker forms of continuity of functions are the basic topics in the general topology. In the several branches of mathematics, many authors have researched various stronger and weaker forms of the continuity of functions. A great number of papers dealing with such functions have appeared, and many of them have been extended to the setting of multifunctions. Most of them in ordinary topology have been studied in the setting of multifunctions such as α-continuity [13], precontinuity [24], quasi-continuity [23], β-continuity [26, 27] and γ-continuity [3].

The aim of the present paper is to define upper (lower) slightly β-continuous multifunctions which are generalize upper (lower) semicontinuous, upper (lower) α-continuous, upper (lower) precontinuous, upper (lower) quasi-continuous, upper (lower) γ-continuous, upper (lower) β-continuous multifunctions and slightly β-continuous functions. Also we obtain several characterizations of upper (lower) slightly β-continuous multifunctions and basic properties of such multifunctions. Moreover, the relationships between

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upper (lower) slightly β-continuous multifunctions and other related multifunctions are investigated.

2. Preliminaries

In this paper, spaces (X, τ) and (Y, υ) (or simply X and Y) always mean topological spaces on which no seperation axioms are assumed unless explicitly stated. For a subset A of (X, τ), $cl(A)$ and $int(A)$ represent the closure of A with respect to τ and the interior of A with respect to τ, respectively.

A subset A is said to be α-open [14] (resp. semi-open [10], preopen [12], β-open [1] or semi-preopen [4], b-open [5] or γ-open [9] or sp-open [8]) if $A \subset int(cl(int(A)))$ (resp. $A \subset cl(int(A))$, $A \subset int(cl(A))$, $A \subset cl(int(cl(A)))$, $A \subset cl(int(A)) \cup int(cl(A))$). The family of all α-open (resp. semi-open, preopen, β-open, γ-open, clopen) sets of X containing a point $x \in X$ is denoted by $\alpha O(X, x)$ (resp. $SO(X, x)$, $PO(X, x)$, $\beta O(X, x)$, $\gamma O(X, x)$, $CO(X, x)$). The family of all α-open (resp. β-open, clopen) sets of X is denoted by $\alpha O(X)$ (resp. $\beta O(X)$, $CO(X)$).

The complement of a β-open set is said to be β-closed [12].

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [6, 7] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X : F(x) \subset B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$. For each $A \subset X$, $F(A) = \bigcup_{x \in A} F(x)$. Then F is said to be a surjection if $F(X) = Y$, or equivalently if for each $y \in Y$ there exists an $x \in X$ such that $y \in F(x)$.

Moreover, $F : X \to Y$ is called upper semi continuous (resp. lower semi continuous) if $F^+(V)$ (resp. $F^-(V)$) is open in X for every open set V of Y [21].

For a multifunction $F : X \to Y$, the graph multifunction $G_F : X \to X \times Y$ is defined as follows $G_F(x) = \{x\} \times F(x)$ for every $x \in X$ and the subset $\{\{x\} \times F(x) : x \in X\} \subset X \times Y$ is called the multigraph of F and is denoted by $G(F)$.

DEFINITION 1. A multifunction $F : X \to Y$ is said to be:

1. Upper almost continuous [22, 24, 29] or upper precontinuous [24] (resp. upper quasi-continuous [24], upper α-continuous [13], upper β-continuous [26, 27], upper γ-continuous [3]) at $x \in X$ if for each open set V of Y containing $F(x)$, there exists $U \in PO(X, x)$ (resp. $U \in SO(X, x)$, $U \in \alpha O(X, x)$, $U \in \beta O(X, x)$, $U \in \gamma O(X, x)$) such that $F(U) \subset V$.

2. Lower almost continuous [22, 24, 29] or lower precontinuous [24] (resp. lower quasi-continuous [24], lower α-continuous [13], lower β-continuous [26, 27], lower γ-continuous [3]) at $x \in X$ if for each open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists $U \in PO(X,x)$ (resp. $U \in SO(X,x)$, $U \in \alpha O(X,x)$, $U \in \beta O(X,x)$, $U \in \gamma O(X,x)$) such that $F(u) \cap V \neq \emptyset$ for every $u \in U$.

3. Upper (lower) almost continuous or upper (lower) precontinuous (resp. upper (lower) quasi-continuous, upper (lower) α-continuous, upper (lower) β-continuous, upper (lower) γ-continuous) if it has this property at each point of X.

DEFINITION 2. A function $f : (X,\tau) \to (Y,\sigma)$ is said to be slightly β-continuous [18] if for each point $x \in X$ and each clopen set V containing $f(x)$ there exists a β-open set U of X containing x such that $f(U) \subset V$.

3. Slightly β-continuous multifunctions

DEFINITION 3. A multifunction $F : X \to Y$ is said to be:

1. Upper slightly β-continuous at $x \in X$ if for each clopen set V of Y containing $F(x)$, there exists $U \in \beta O(X)$ containing x such that $F(U) \subset V$.

2. Lower slightly β-continuous at $x \in X$ if for each clopen set V of Y such that $F(x) \cap V \neq \emptyset$, there exists $U \in \beta O(X)$ containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in U$.

3. Upper (lower) slightly β-continuous if it has this property at each point of X.

REMARK 4. For a multifunction $F : X \to Y$ from a topological space (X,τ) to a topological space (Y,υ), the following implications hold:

$$\begin{array}{ccc}
\text{upper/lower semi-continuity} & & \\
\Downarrow & & \\
\text{upper/lower } \alpha\text{-continuity} & \Rightarrow & \text{upper/lower precontinuity} \\
\Downarrow & & \Downarrow \\
\text{upper/lower quasi-continuity} & \Rightarrow & \text{upper/lower } \gamma\text{-continuity} \\
 & & \Downarrow \\
 & & \text{upper/lower } \beta\text{-continuity} \\
 & & \Downarrow \\
 & & \text{upper/lower slightly } \beta\text{-continuity}
\end{array}$$

However the converses are not true in general.

EXAMPLE 5. Let $X = \{a,b,c\}$ and $Y = \{1,2,3,4,5\}$. Let τ and υ be respectively topologies on X and on Y given by $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$

and $\upsilon = \{\emptyset, Y, \{1,2\}, \{3,4\}, \{3,4,5\}, \{1,2,3,4\}\}$. Define the multifunction $F : X \to Y$ by $F(a) = \{1,5\}$, $F(b) = \{1,5\}$ and $F(c) = \{1,3,4\}$. Then F is lower slightly β-continuous multifunction but F is not lower β-continuous.

EXAMPLE 6. Let $X = \{a,b,c\}$ and $Y = \{1,2,3,4,5\}$. Let τ and υ be respectively topologies on X and on Y given by $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$ and $\upsilon = \{\emptyset, Y, \{1,2\}, \{3,4\}, \{3,4,5\}, \{1,2,3,4\}\}$. Define the multifunction $F : X \to Y$ by $F(a) = \{3,4,5\}$, $F(b) = \{1,2\}$ and $F(c) = \{1,2,3\}$. Then F is upper slightly β-continuous multifunction but F is not upper β-continuous.

The other implication are not reversible as shown in [3, 20].

The following theorem state some characterizations of upper slightly β-continuous multifunctions.

THEOREM 7. *Let $F : X \to Y$ be a multifunction from a topological space (X,τ) to a topological space (Y,υ). Then the following statements are equivalent:*

(i) *F is upper slightly β-continuous;*

(ii) *for each $x \in X$ and for each clopen set V such that $x \in F^+(V)$, there exists an β-open set U containing x such that $U \subset F^+(V)$;*

(iii) *for each $x \in X$ and for each clopen set V such that $x \in F^+(Y\backslash V)$, there exists an β-closed set H such that $x \in X\backslash H$ and $F^-(V) \subset H$;*

(iv) *$F^+(V)$ is an β-open set for any clopen set $V \subset Y$;*

(v) *$F^-(V)$ is a β-closed set for any clopen set $V \subset Y$;*

(vi) *$F^-(Y\backslash V)$ is a β-closed set for any clopen set $V \subset Y$;*

(vii) *$F^+(Y\backslash V)$ is a β-open set for any clopen set $V \subset Y$.*

Proof. (i)$\Leftrightarrow$(ii). Clear.

(ii)$\Leftrightarrow$(iii). Let $x \in X$ and let V be a clopen such that $x \in F^+(Y\backslash V)$. By (ii), there exists an β-open set U containing x such that $U \subset F^+(Y\backslash V)$. Then $F^-(V) \subset X\backslash U$. Take $H = X\backslash U$. We have $x \in X\backslash H$ and H is β-closed.

The converse is similar.

(i)$\Leftrightarrow$(iv). Let $x \in F^+(V)$ and let V be a clopen set. From (i), there exists an β-open set U_x containing x such that $U_x \subset F^+(V)$. It follows that $F^+(V) = \bigcup_{x \in F^+(V)} U_x$ and hence $F^+(V)$ is β-open.

The converse can be shown easily.

(iv)$\Rightarrow$(v). Let $V \subset Y$ be a clopen set. We have that $Y\backslash V$ is a clopen set. From (iv), $F^+(Y\backslash V) = X\backslash F^-(V)$ is an β-open set. Then it is obtained that $F^-(V)$ is an β-closed set.

(v)$\Rightarrow$(iv). Similar to the above.

(iv)$\Leftrightarrow$(vi), (v)$\Leftrightarrow$(vii). Since $F^-(Y\backslash V) = X\backslash F^+(V)$ and $F^+(Y\backslash V) = X\backslash F^-(V)$, the proof is clear. ∎

The following theorem state some characterizations of lower slightly β-continuous multifunction.

THEOREM 8. *Let $F : X \to Y$ be a multifunction from a topological space (X, τ) to a topological space (Y, υ). Then the following statements are equivalent:*

(i) *F is lower slightly β-continuous;*

(ii) *for each $x \in X$ and for each clopen set V such that $x \in F^-(V)$, there exists an β-open set U containing x such that $U \subset F^-(V)$;*

(iii) *for each $x \in X$ and for each clopen set V such that $x \in F^-(Y \backslash V)$, there exists an β-closed set H such that $x \in X \backslash H$ and $F^+(V) \subset H$;*

(iv) *$F^-(V)$ is an β-open set for any clopen set $V \subset Y$;*

(v) *$F^+(V)$ is a β-closed set for any clopen set $V \subset Y$;*

(vi) *$F^+(Y \backslash V)$ is a β-closed set for any clopen set $V \subset Y$;*

(vii) *$F^-(Y \backslash V)$ is a β-open set for any clopen set $V \subset Y$.*

Proof. It can be obtained similarly as Theorem 7. ■

COROLLARY 9 (Noiri [18]). *For a function $f : (X, \tau) \to (Y, \sigma)$, the following are equivalent:*

(1) *f is slightly β-continuous;*

(2) *$f^{-1}(V) \in \beta O(X)$ for each clopen set V in Y;*

(3) *$f^{-1}(V)$ is β-clopen for each clopen set V in Y;*

(4) *for each $x \in X$ and each clopen set V containing $f(x)$, there exists β-clopen set U containing x such that $f(U) \subset V$.*

4. Some properties

We know that a net (x_α) in a topological space (X, τ) is called eventually in the set $U \subset X$ if there exists an index $\alpha_0 \in J$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$.

DEFINITION 10. A sequence (x_n) is said to be β-converge to a point x if for every β-open set V containing x, there exists an index n_0 such that for $n \geq n_0$, $x_n \in V$.

THEOREM 11. *Let $F : X \to Y$ be a multifunction. If F is upper (resp. lower) slightly β-continuous, then for each $x \in X$ and for each net (x_α) which β-converges to x in X and for each clopen set $V \subset Y$ such that $x \in F^+(V)$ (resp. $x \in F^-(V)$), the net (x_α) is eventually in $F^+(V)$ (resp. $F^-(V)$).*

Proof. Let (x_α) be a net which β-converges to x in X and let $V \subset Y$ be any clopen set such that $x \in F^+(V)$. Since F is upper slightly β-continuous multifunction, it follows that there exists an β-open set $U \subset X$ containing x such that $U \subset F^+(V)$. Since (x_α) β-converges to x, it follows that there exists an index $\alpha_0 \in J$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$. From here, we obtain that $x_\alpha \in U \subset F^+(V)$ for all $\alpha \geq \alpha_0$. Thus, the net (x_α) is eventually in $F^+(V)$.

The proof of the lower continuity is similar. ∎

THEOREM 12. *Let $F : X \to Y$ be a multifunction from a topological space (X, τ) to a topological space (Y, υ) and let $F(X)$ be endowed with subspace topology. If F is upper slightly β-continuous multifunction then $F : X \to F(X)$ is upper slightly β-continuous multifunction.*

Proof. Since F is upper slightly β-continuous, $F^+(V \cap F(X)) = F^+(V) \cap F^+(F(X)) = F^+(V)$ is β-open for each clopen subset V of Y. Hence $F : X \to F(X)$ is upper slightly β-continuous multifunction. ∎

Suppose that (X, τ), (Y, υ) and (Z, ω) are topological spaces. It is known that if $F_1 : X \to Y$ and $F_2 : Y \to Z$ are multifunctions, then the multifunction $F_2 \circ F_1 : X \to Z$ is defined by $(F_2 \circ F_1)(x) = F_2(F_1(x))$ for each $x \in X$.

THEOREM 13. *Let (X, τ), (Y, υ), (Z, ω) be topological spaces and let $F : X \to Y$ and $G : Y \to Z$ be multifunctions. If $F : X \to Y$ is upper (lower) β-continuous multifunction and $G : Y \to Z$ is upper (lower) semicontinuous multifunction, then $G \circ F : X \to Z$ is an upper (lower) slightly β-continuous multifunction.*

Proof. Let $V \subset Z$ be any clopen set. From the definition of $G \circ F$, we have $(G \circ F)^+(V) = F^+(G^+(V))$ $((G \circ F)^-(V) = F^-(G^-(V)))$. Since G is upper (lower) semicontinuous multifunction, it follows that $G^+(V)$ $(G^-(V))$ is an open set. Since F is upper (lower) β-continuous multifunction, it follows that $F^+(G^+(V))$ $(F^-(G^-(V)))$ is a β-open set. This shows that $G \circ F$ is a upper (lower) slightly β-continuous multifunction. ∎

REMARK 14. It should be noted that every restriction of a upper (lower) slightly β-continuous multifunction is not necessarily upper (lower) slightly β-continuous.

EXAMPLE 15. Let $X = \{a, b, c, d\}$ and let σ and τ be topologies on X given by $\sigma = \{\emptyset, X, \{a, b\}\}$ and $\tau = \{\emptyset, X, \{a\}, \{b, c, d\}\}$. Define the multifunction $F : (X, \sigma) \to (X, \tau)$ by $F(a) = \{a\}$, $F(b) = \{a, d\}$, $F(c) = \{a, c, d\}$ and $F(d) = \{a\}$. Then F is lower slightly β-continuous multifunction.

Let $A = \{a, c, d\}$. A is not α-open in (X, σ). If σ_A is the relative topology on A induced by σ, then $F |_A$: $(A, \sigma_A) \to (X, \tau)$ is not lower slightly β-continuous.

EXAMPLE 16. Let $X = \{a, b, c, d\}$ and let σ and τ be topologies on X given by $\sigma = \{\emptyset, X, \{a, b\}\}$ and $\tau = \{\emptyset, X, \{a\}, \{b, c, d\}\}$. Define the multifunction

$F : (X,\sigma) \to (X,\tau)$ by $F(a) = \{a,b\}$, $F(b) = \{d\}$, $F(c) = \{c,d\}$ and $F(d) = \{a,c,d\}$. Then F is upper slightly β-continuous multifunction.

Let $A = \{a,c,d\}$. A is not α-open in (X,σ). If σ_A is the relative topology on A induced by σ, then $F|_A: (A,\sigma_A) \to (X,\tau)$ is not upper slightly β-continuous.

THEOREM 17. *Let $F : X \to Y$ be a multifunction and let $U \in \alpha O(X)$. If F is a lower (upper) slightly β-continuous multifunction, then the restriction multifunction $F|_U: U \to Y$ is a lower (upper) slightly β-continuous multifunction.*

Proof. Suppose that $V \subset Y$ is a clopen set. Let $x \in U$ and let $x \in F^-|_U(V)$. Since F is lower slightly β-continuous multifunction, it follows that there exists $G \in \beta O(X)$ containing x such that $G \subset F^-(V)$. From here we obtain that $x \in G \cap U \in \beta O(U)$ (from Lemma 2.5 in [1]) and $G \cap U \subset F^-|_U(V)$. Thus, we show that the restriction multifunction $F|_U$ is a lower slightly β-continuous.

The proof of the upper slightly β-continuity of $F|_U$ is similar to the above. ■

THEOREM 18. *Let $\{U_\lambda : \lambda \in \Lambda\}$ be a α-open cover of a space X. Then a multifunction $F : X \to Y$ is upper slightly β-continuous (resp. lower slightly β-continuous) if the restriction $F|_{U_\lambda}: U_\lambda \to Y$ is upper slightly β-continuous (resp. lower slightly β-continuous) for each $\lambda \in \Lambda$.*

Proof. We prove only the case for F upper slightly β-continuous.

Let V be any clopen set of Y. Since $F|_{U_\lambda}$ is slightly β-continuous for each $\lambda \in \Lambda$, $(F|_{U_\lambda})^+(V) = F^+(V) \cap U_\lambda$ is β-open in U_λ. By Lemma 2.7 in [1], $(F|_{U_\lambda})^+(V)$ is β-open in X for each $\lambda \in \Lambda$. We obtain that $F^+(V) = \bigcup_{\lambda\in\Lambda}(F|_{U_\lambda})^+(V)$ is β-open in X. Hence F is upper slightly β-continuous. ■

LEMMA 19. *For a multifunction $F : X \to Y$, the following hold:*

(1) $G_F^+(A \times B) = A \cap F^+(B)$,
(2) $G_F^-(A \times B) = A \cap F^-(B)$

for any subsets $A \subset X$ and $B \subset Y$ [16].

THEOREM 20. *Let $F : X \to Y$ be a multifunction from a topological space (X,τ) to a topological space (Y,υ). If the graph multifunction of F is upper slightly β-continuous multifunction then F is upper slightly β-continuous multifunction.*

Proof. Let $x \in X$ and let $V \subset Y$ be a clopen set such that $x \in F^+(V)$. We obtain that $x \in G_F^+(X \times V)$ and that $X \times V$ is a clopen set. Since graph multifunction G_F is upper slightly β-continuous, it follows that there exists an β-open set $U \subset X$ containing x such that $U \subset G_F^+(X \times V)$. Since

$U \subset G_F^+(X \times V) = X \cap F^+(V) = F^+(V)$, we obtain that $U \subset F^+(V)$. Thus, F is upper slightly β-continuous multifunction. ∎

THEOREM 21. *A multifunction $F : X \to Y$ is lower slightly β-continuous if $G_F : X \to X \times Y$ is lower slightly β-continuous.*

Proof. Suppose that G_F is lower slightly β-continuous. Let $x \in X$ and V be any clopen set of Y such that $x \in F^-(V)$. Then $X \times V$ is clopen in $X \times Y$ and $G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \emptyset$. Since G_F is lower slightly β-continuous, there exists a β-open set U containing x such that $U \subset G_F^-(X \times V)$. By the previous lemma, we have $U \subset F^-(V)$. This shows that F is lower slightly β-continuous. ∎

COROLLARY 22 (Noiri [18]). *A function $f : X \to Y$ is slightly β-continuous if the graph function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$, is slightly β-continuous.*

THEOREM 23. *Suppose that (X, τ) and (X_α, τ_α) are topological spaces where $\alpha \in J$. Let $F : X \to \prod_{\alpha \in J} X_\alpha$ be a multifunction from X to the product space $\prod_{\alpha \in J} X_\alpha$ and let $P_\alpha : \prod_{\alpha \in J} X_\alpha \to X_\alpha$ be the projection multifunction for each $\alpha \in J$ which is defined by $P_\alpha((x_\alpha)) = \{x_\alpha\}$. If F is upper (lower) slightly β-continuous multifunction, then $P_\alpha \circ F$ is upper (lower) slightly β-continuous multifunction for each $\alpha \in J$.*

Proof. Take any $\alpha_0 \in J$. Let V_{α_0} be a clopen set in $(X_{\alpha_0}, \tau_{\alpha_0})$. Then $(P_{\alpha_0} \circ F)^+(V_{\alpha_0}) = F^+(P_{\alpha_0}^+(V_{\alpha_0})) = F^+(V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)$ (respectively, $(P_{\alpha_0} \circ F)^-(V_{\alpha_0}) = F^-(P_{\alpha_0}^-(V_{\alpha_0})) = F^-(V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)$). Since F is upper (lower) slightly β-continuous multifunction and since $V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha$ is a clopen set, it follows that $F^+(V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)$ (respectively, $F^-(V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)$) is β-open in (X, τ). It shows that $P_{\alpha_0} \circ F$ is upper (lower) slightly β-continuous multifunction.

Hence, we obtain that $P_\alpha \circ F$ is upper (lower) slightly β-continuous multifunction for each $\alpha \in J$. ∎

THEOREM 24. *Suppose that for each $\alpha \in J$, (X_α, τ_α), $(Y_\alpha, \upsilon_\alpha)$ are topological spaces. Let $F_\alpha : X_\alpha \to Y_\alpha$ be a multifunction for each $\alpha \in J$ and let $F : \prod_{\alpha \in J} X_\alpha \to \prod_{\alpha \in J} Y_\alpha$ be defined by $F((x_\alpha)) = \prod_{\alpha \in J} F_\alpha(x_\alpha)$ from the product space $\prod_{\alpha \in J} X_\alpha$ to the product space $\prod_{\alpha \in J} Y_\alpha$. If F is upper (lower) slightly β-continuous multifunction, then each F_α is upper (lower) slightly β-continuous multifunction for each $\alpha \in J$.*

Proof. Let $V_\alpha \subset Y_\alpha$ be a clopen set. Then $V_\alpha \times \prod_{\alpha \neq \beta} Y_\beta$ is a clopen set. Since F is upper (lower) slightly β-continuous multifunction, it follows that $F^+(V_\alpha \times \prod_{\alpha \neq \beta} Y_\beta) = F_\alpha^+(V_\alpha) \times \prod_{\alpha \neq \beta} X_\beta$ ($F^-(V_\alpha \times \prod_{\alpha \neq \beta} Y_\beta) = F_\alpha^-(V_\alpha) \times \prod_{\alpha \neq \beta} X_\beta$) is an β-open set. Consequently we obtain that $F_\alpha^+(V_\alpha)$

$(F_\alpha^-(V_\alpha))$ is an β-open set. Thus, we show that F_α is upper (lower) slightly β-continuous multifunction. ∎

For two multifunctions $F_1 : X_1 \to Y_1$ and $F_2 : X_2 \to Y_2$, the product multifunction $F_1 \times F_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is defined as follows: $(F_1 \times F_2)(x_1, x_2) = F_1(x_1) \times F_2(x_2)$ for every $x_1 \in X_1$ and $x_2 \in X_2$.

THEOREM 25. *Suppose that $F_1 : X_1 \to Y_1$, $F_2 : X_2 \to Y_2$ are multifunctions. If $F_1 \times F_2$ is upper (lower) slightly β-continuous multifunction, then F_1 and F_2 are upper (lower) slightly β-continuous multifunctions.*

Proof. Let $K \subset Y_1$, $H \subset Y_2$ be clopen sets. It is known that $K \times H$ is a clopen set and $(F_1 \times F_2)^+(K \times H) = F_1^+(K) \times F_2^+(H)$. Since $F_1 \times F_2$ is upper slightly β-continuous multifunction, it follows that $F_1^+(K) \times F_2^+(H)$ is β-open set. From here, $F_1^+(K)$ and $F_2^+(H)$ are β-open sets. Hence, it is obtained that F_1 and F_2 are upper slightly β-continuous multifunctions.

The proof of the lower slightly β-continuity of F_1 and F_2 is similar to the above. ∎

THEOREM 26. *Suppose that (X,τ), (Y,υ), (Z,ω) are topological spaces and $F_1 : X \to Y$, $F_2 : X \to Z$ are multifunctions. Let $F_1 \times F_2 : X \to Y \times Z$ be a multifunction which is defined by $(F_1 \times F_2)(x) = F_1(x) \times F_2(x)$ for each $x \in X$. If $F_1 \times F_2$ is upper (lower) slightly β-continuous multifunction, then F_1 and F_2 are upper (lower) slightly β-continuous multifunctions.*

Proof. Let $x \in X$ and let $K \subset Y$, $H \subset Z$ be clopen sets such that $x \in F_1^+(K)$ and $x \in F_2^+(H)$. Then we obtain that $F_1(x) \subset K$ and $F_2(x) \subset H$ and from here, $F_1(x) \times F_2(x) = (F_1 \times F_2)(x) \subset K \times H$. We have $x \in (F_1 \times F_2)^+(K \times H)$. Since $F_1 \times F_2$ is upper slightly β-continuous multifunction, it follows that there exists a β-open set U containing x such that $U \subset (F_1 \times F_2)^+(K \times H)$. We obtain that $U \subset F_1^+(K)$ and $U \subset F_2^+(H)$. Thus, we obtain that F_1 and F_2 are upper slightly β-continuous multifunctions.

The proof of the lower slightly β-continuity of F_1 and F_2 is similar to the above. ∎

DEFINITION 27. Let (X,τ) be a topological space. X is said to be a strongly normal space if for every disjoint closed subsets K and F of X, there exists two clopen sets U and V such that $K \subset U$, $F \subset V$ and $U \cap V = \emptyset$.

EXAMPLE 28. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then (X,τ) is a strongly normal space.

Recall that a multifunction $F : X \to Y$ is said to be punctually closed if, for each $x \in X$, $F(x)$ is closed.

THEOREM 29. *If Y is strongly normal space and $F_i : X_i \to Y$ is upper slightly β-continuous multifunction such that F_i is punctually closed for $i = 1, 2$,* *then a set* $\{(x_1, x_2) \in X_1 \times X_2 : F_1(x_1) \cap F_2(x_2) \neq \emptyset\}$ *is* β*-closed set in* $X_1 \times X_2$.

Proof. Let $A = \{(x_1, x_2) \in X_1 \times X_2 : F_1(x_1) \cap F_2(x_2) \neq \emptyset\}$ and $(x_1, x_2) \in (X_1 \times X_2) \backslash A$. Then $F_1(x_1) \cap F_2(x_2) = \emptyset$. Since Y is strongly normal and F_i is punctually closed for $i = 1, 2$, there exist disjoint clopen sets V_1, V_2 such that $F_i(x_i) \subset V_i$ for $i = 1, 2$. Since F_i is upper slightly β-continuous, $F_i^+(V_i)$ is β-open for $i = 1, 2$. Put $U = F_1^+(V_1) \times F_2^+(V_2)$, then U is β-open and $(x_1, x_2) \in U \subset (X_1 \times X_2) \backslash A$. This shows that $(X_1 \times X_2) \backslash A$ is β-open and hence A is β-closed in $(X_1 \times X_2)$. ■

THEOREM 30. *Let* F *and* G *be upper slightly* β*-continuous punctually closed and upper semi continuous punctually closed multifunctions, respectively, from a topological space* X *to a strongly normal topological space* Y. *Then the set* $K = \{x : F(x) \cap G(x) \neq \emptyset\}$ *is* β*-closed in* X.

Proof. Let $x \in X \backslash K$. Then $F(x) \cap G(x) = \emptyset$. Since F and G are punctually closed multifunctions and Y is a strongly normal space, it follows that there exists disjoint clopen sets U and V containing $F(x)$ and $G(x)$ respectively. Since F and G are upper slightly β-continuous and upper semi continuous, respectively, then the sets $F^+(U)$ and $G^+(V)$ are β-open and open, respectively such that contain x. Let $H = F^+(U) \cap G^+(V)$. Then H is an β-open set containing x and $H \cap K = \emptyset$. Hence, K is β-closed in X. ■

DEFINITION 31. Let $F : X \to Y$ be a multifunction. The multigraph $G(F)$ is said to be β-co-closed if for each $(x, y) \notin G(F)$, there exist β-open set U and clopen set V containing x and y, respectively, such that $(U \times V) \cap G(F) = \emptyset$.

DEFINITION 32. A space X is said to be mildly compact if every clopen cover of X has a finite subcover [30].

DEFINITION 33. A space X is said to be ultra Hausdorff [30] if for each pair of distinct points x and y in X, there exist disjoint clopen sets U and V in X such that $x \in U$ and $y \in V$.

THEOREM 34. *If a multifunction* $F : X \to Y$ *is upper slightly* β*-continuous multifunction such that* $F(x)$ *is mildly compact relative to* Y *for each* $x \in X$ *and* Y *is ultra Hausdorff space, then the multigraph* $G(F)$ *of* F *is* β*-co-closed in* $X \times Y$.

Proof. $(x, y) \notin G(F)$. That is $y \notin F(x)$. Since Y is ultra Hausdorff, for each $z \in F(x)$, there exist disjoint clopen sets $V(z)$ and $U(z)$ of Y such that $z \in U(z)$ and $y \in V(y)$. Then $\{U(z) : z \in F(x)\}$ is clopen cover of $F(x)$ and since $F(x)$ is mildly compact, there exists a finite number of points, say, z_1, z_2, z_3,..., z_n in $F(x)$ such that

$$F(x) \subset \bigcup \{U(z_i) : i = 1, 2, 3, ..., n\}.$$

Put

$$U = \bigcup\{U(z_i) : i = 1, 2, 3, ..., n\} \text{ and } V = \bigcap\{V(y_i) : i = 1, 2, 3, ..., n\}.$$

Then U and V are clopen in Y such that $F(x) \subset U$, $y \in V$ and $U \cap V = \emptyset$. Since F is upper slightly β-continuous multifunction, there exists β-open W containing x such that $F(W) \subset U$. We have $(x, y) \in W \times V \subset (X \times Y)\backslash G(F)$. We obtain that $(W \times V) \cap G(F) = \emptyset$ and hence $G(F)$ is β-co-closed in $X \times Y$. ∎

DEFINITION 35. A space X is said to be β-compact if every β-open cover of X has a finite subcover [2].

THEOREM 36. *Let $F : X \to Y$ be an upper slightly β-continuous surjective multifunction such that $F(x)$ is mildly compact for each $x \in X$. If X is a β-compact space, then Y is mildly compact.*

Proof. Let $\{V_\lambda : \lambda \in \Lambda\}$ be a clopen cover of Y. Since $F(x)$ is mildly compact for each $x \in X$, there exists a finite subset $\Lambda(x)$ of Λ such that $F(x) \subset \bigcup\{V_\lambda : \lambda \in \Lambda(x)\}$. Put

$$V(x) = \bigcup\{V_\lambda : \lambda \in \Lambda(x)\}.$$

Since F is upper slightly β-continuous, there exists a β-open set $U(x)$ of X containing x such that $F(U(x)) \subset V(x)$. Then the family $\{U(x) : x \in X\}$ is a β-open cover of X and since X is β-compact, there exists a finite number of points, say, x_1, x_2, x_3, ...,x_n in X such that $X = \bigcup\{U(x_i) : i = 1, 2, 3, ..., n\}$. Hence we have

$$Y = F(X) = F(\bigcup_{i=1}^{n} U(x_i)) = \bigcup_{i=1}^{n} F(U(x_i)) \subset \bigcup_{i=1}^{n} V(x_i) = \bigcup_{i=1}^{n} \bigcup_{\lambda \in \Lambda(x_i)} V_\lambda.$$

This shows that Y is mildly compact. ∎

Recall that a multifunction $F : X \to Y$ is said to be punctually connected if, for each $x \in X$, $F(x)$ is connected.

DEFINITION 37. A space X is called β-connected provided that X is not the union of two disjoint nonempty β-open sets [15].

THEOREM 38. *Let F be a multifunction from a β-connected topological space X onto a topological space Y such that F is punctually connected. If F is upper slightly β-continuous multifunction, then Y is a connected space.*

Proof. Let $F : X \to Y$ be a upper slightly β-continuous multifunction from a β-connected topological space X onto a topological space Y. Suppose that Y is not connected and let $Y = H \cup K$ be a partition of Y. Then both H and K are open and closed subsets of Y. Since F is upper slightly β-continuous multifunction, $F^+(H)$ and $F^+(K)$ are β-open subsets of X.

In view of the fact that $F^+(H)$, $F^+(K)$ are disjoint and F is punctually connected, $X = F^+(H) \cup F^+(K)$ is a partition of X. This is contrary to the β-connectedness of X. Hence, it is obtained that Y is a connected space. ∎

THEOREM 39. *Let F be a multifunction from a β-connected topological space X onto a topological space Y such that F is punctually connected. If F is lower slightly β-continuous multifunction, then Y is a connected space.*

Proof. Let F be lower slightly β-continuous. Suppose that Y is not connected and let $Y = H \cup K$ be a partition of Y. Then both H and K are open and closed subsets of Y. Then by Theorem 8, $F^+(H)$ is β-closed since H is clopen in Y. Since $X = F^+(H) \cup F^+(K)$ and $F^+(H) \cap F^+(K) = \varnothing$, $F^+(H)$ is β-open. Similarly, $F^+(K)$ is β-open. Consequently, X is not β-connected. This is a contradiction. ∎

COROLLARY 40 (Noiri [18]). *If $f : X \to Y$ is slightly β-continuous surjection and X is β-connected, then Y is connected.*

DEFINITION 41. A space X is said to be β-Hausdorff [11] if for each pair of distinct points x and y in X, there exist disjoint β-open sets U and V in X such that $x \in U$ and $y \in V$.

THEOREM 42. *Let $F : X \to Y$ be a upper slightly β-continuous multifunction and punctually closed from a topological space X to a strongly normal topological space Y and let $F(x) \cap F(y) = \emptyset$ for each distinct pair $x, y \in X$. Then X is a β-Hausdorff space.*

Proof. Let x and y be any two distinct points in X. Then we have $F(x) \cap F(y) = \emptyset$. Since Y is a strongly normal space, it follows that there exists disjoints clopen sets U and V containing $F(x)$ and $F(y)$ respectively. Thus $F^+(U)$ and $F^+(V)$ are disjoint β-open sets containing x and y respectively. Thus, it is obtained that X is β-Hausdorff. ∎

5. Slightly β-continuity and other forms of β-continuity

DEFINITION 43. A multifunction $F : X \to Y$ is said to be:

1. Upper weakly β-continuous [17, 28] (resp. upper almost β-continuous [17, 28]) at $x \in X$ if for each open set V of Y containing $F(x)$, there exists $U \in \beta O(X, x)$ such that $F(U) \subset cl(V)$ (resp. $F(U) \subset int(cl(V))$).

2. Lower weakly β-continuous [17, 28] (resp. lower almost β-continuous [17, 28]) at $x \in X$ if for each open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists $U \in \beta O(X, x)$ such that $F(u) \cap cl(V) \neq \emptyset$ (resp. $F(u) \cap int(cl(V)) \neq \emptyset$) for every $u \in U$.

3. Upper (lower) weakly β-continuous (resp. upper (lower) almost β-continuous) if it has this property at each point of X.

THEOREM 44. *If a multifunction $F : (X,\tau) \to (Y,\sigma)$ is upper weakly β-continuous, then it is upper slightly β-continuous.*

Proof. Let $x \in X$ and $V \in CO(Y)$ containing $F(x)$. Since F is upper weakly β-continuous, there exists $U \in \beta O(X,x)$ such that $F(U) \subset cl(V) = V$. Therefore, F is upper slightly β-continuous. ∎

THEOREM 45. *If a multifunction $F : (X,\tau) \to (Y,\sigma)$ is lower weakly β-continuous, then it is lower slightly β-continuous.*

Proof. The proof is similar to the proof of the previous theorem. ∎

Recall that a space X is said to be extremally disconnected if the closure of each open set of X is open in X.

THEOREM 46. *If a multifunction $F : (X,\tau) \to (Y,\sigma)$ is upper slightly β-continuous and (Y,σ) is extremally disconnected, then F is upper almost β-continuous.*

Proof. Let $x \in X$ and V be any regular open set of (Y,σ) containing $F(x)$. Then, by Lemma 5.6 of [19] we have $V \in CO(Y)$ since Y is extremally disconnected. Since F is upper slightly β-continuous, there exists $U \in \beta O(X,x)$ such that $F(U) \subset V$. Therefore, F is upper almost β-continuous. ∎

THEOREM 47. *If a multifunction $F : (X,\tau) \to (Y,\sigma)$ is lower slightly β-continuous and (Y,σ) is extremally disconnected, then F is lower almost β-continuous.*

Proof. The proof is similar to the proof of the previous theorem. ∎

REMARK 48. For a multifunction $F : X \to Y$ from a topological space (X,τ) to a topological space (Y,υ), the following diagram hold:

upper/lower slightly β-continuity

$\Uparrow$

upper/lower weakly β-continuity

$\Uparrow$

upper/lower almost β-continuity

$\Uparrow$

upper/lower β-continuity.

Recall that a space is 0-dimensional if its topology has a base consisting of clopen sets.

THEOREM 49. *If a multifunction $F : (X,\tau) \to (Y,\sigma)$ is upper slightly β-continuous, (Y,σ) is 0-dimensional and $F(x)$ is mildly compact relative to Y for each $x \in X$, then F is upper β-continuous.*

Proof. Let $x \in X$ and V be any open set of (Y,σ) containing $F(x)$. Then, by 0-dimensionability of (Y,σ), for each $y \in F(x)$ there exists $W_y \in CO(Y)$ such that $y \in W_y \subset V$. Such $F(x)$ is mildly compact relative to Y, there exists a finite number of points, say, y_1, y_2, ..., $y_n \in F(x)$ such that $W_{y_i} \in CO(Y)$, for each i and $F(x) \subset \bigcup_{i=1}^{n} W_{y_i} \subset V$. Now, put $W = \cup\{W_{y_i} : 1 \leq i \leq n\}$. Then we have $W \in CO(Y)$ and $F(x) \subset W \subset V$. Since F is upper slightly β-continuous, then there exists $U \in \beta O(X,x)$ such that $F(U) \subset W \subset V$. Thus, F is upper β-continuous. ■

LEMMA 50. *Let (Y,σ) be a 0-dimensional topological space. If K is closed in Y and $y \in Y \backslash K$, there exist two disjoint clopen sets containing y and K, respectively.*

Proof. Let $y \in K$ and K be closed in Y. Then $Y\backslash K$ is an open set. Since (Y,σ) is 0-dimensional, there exists a clopen set W such that $y \in W \subset Y\backslash K$. Put $D = Y\backslash W$, then D is clopen, $K \subset D$ and $D \cap W = \varnothing$. ■

LEMMA 51. *Let (Y,σ) be a 0-dimensional topological space and A a subset of Y. Then for every open set D which intersect A, there exists a clopen set D_A such that $A \cap D_A \neq \varnothing$ and $D_A \subset D$.*

Proof. Let $y \in A \cap D$, then $y \notin (Y\backslash D)$. Since $Y\backslash D$ is closed in Y, by the previous lemma there exist two disjoint clopen sets U and V containing y and $Y\backslash D$, respectively. Thus, $y \in U$, $Y\backslash D \subset V$, $U \cap V = \varnothing$ and U, V are clopen sets. Put $D_A = Y\backslash V$, then $y \in D_A$, $A \cap D_A \neq \varnothing$ and $D_A \subset D$. ■

THEOREM 52. *If a multifunction $F : (X,\tau) \to (Y,\sigma)$ is lower slightly β-continuous, (Y,σ) is 0-dimensional, then F is lower β-continuous.*

Proof. Let $x \in X$ and V be any open set of Y such that $F(x) \cap V \neq \varnothing$. By the previous lemma, there exists a clopen set V_x such that $F(x) \cap V_x \neq \varnothing$ and $V_x \subset V$. Since F is lower slightly β-continuous and $F(x) \cap V_x \neq \varnothing$, there exists $U \in \beta O(X,x)$ such that $F(u) \cap V_x \neq \varnothing$ for every $u \in U$. Since $V_x \subset V$, it follows that $F(x) \cap V \neq \varnothing$ for every $u \in U$. Therefore, F is lower β-continuous. ■

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