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FIXED POINT THEOREM FOR SEQUENCES OF MAPS

Abstract. In this paper, we prove a common fixed point theorem for sequences of maps under the condition of compatible mappings on complete metric space. We extend and generalize several fixed point theorems on complete metric space.

1. Introduction

In this paper (X,d) denotes a complete metric space and $B(X)$ stands for the set of all bounded subsets of X. The function δ of $B(X)\times B(X)$ into $[0,\infty)$ is defined as

$$\delta(A,B)=\sup\{d(a,b):a\in A,b\in B\}$$

for all A,B in $B(X)$. If $A=\{a\}$ is singleton, we write $\delta(A,B)=\delta(a,B)$ and if $B=\{b\}$, then we put $\delta(A,B)=\delta(a,b)$. It is easily seen that

$$\begin{aligned}
&\delta(A,B)=\delta(B,A)\geq 0,\\
&\delta(A,B)\leq\delta(A,C)+\delta(C,B),\\
&\delta(A,A)=\operatorname{diam}A,\\
&\delta(A,B)=0 \text{ implies } A=B=\{a\}
\end{aligned}$$

for all A,B,C in $B(X)$. We recall some definitions and a basic lemmas of Fisher [2] and Imdad et al [3]. Let $\{A_n : n=1,2,\ldots\}$ be a sequence of subsets of X. We say that the sequence $\{A_n\}$ converges to a subset A of X if each point a in A is the limit of a convergent sequence $\{a_n\}$ with a_n in A_n for $n=1,2,\ldots$ and if for any $\varepsilon>0$, there exists an integer N such that $A_n\subseteq A_\varepsilon$ for $n>N$, A_ε being the union of all open spheres with centers in A and radius ε. The following lemmas hold.

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LEMMA 1. [2] *If* $\{A_n\}$ *and* $\{B_n\}$ *are sequences of bounded subsets of* (X,d) *which converge to the bounded subsets* A *and* B *respectively then the sequence* $\{\delta(A_n,B_n)\}$ *converges to* $\delta(A,B)$.

LEMMA 2. [3] *If* $\{A_n\}$ *is a sequence of bounded sets in the complete metric space* (X,d) *and if* $\lim_{n\to\infty}\delta(A_n,\{y\})=0$ *for some* $y\in X$ *then* $\{A_n\}\to\{y\}$.

A set-valued mapping of F of X into $B(X)$ is continuous at the point x in X if whenever $\{x_n\}$ is a sequence of points of X converging to x, the sequence $\{Fx_n\}$ in $B(X)$ converges to Fx. F is said to be continuous in X if it is continuous at each point x in X. We say that z is a fixed point of F if z is in Fz.

DEFINITION 1. [4] Let (X,d) be a metric space. Let $S:X\to X$ and $F:X\to B(X)$. F and S are compatible if and only if $SFx\in B(X)$ for $x\in X$ and $\delta(SFx_n,FSx_n)\to 0$ whenever $\{x_n\}$ is a sequence in X such that $Sx_n\to t$ and $Fx_n\to\{t\}$ for some $t\in X$.

PROPOSITION 3. [4] *Let* (X,d) *be a complete metric space. Suppose* $S:X\to X$, $F:X\to B(X)$ *and* S *and* F *are compatible. If* $\{Su\}=Fu$ *for some* $u\in X$, *then* $FSu=SFu$.

2. Fixed point theorem

Let (X,d) be a metric space. Let $\{S_r\}$ be a sequence of self maps on X and $\{F_r\}$ be a sequence of maps on X into $B(X)$ satisfying

$$F_r(X)\subseteq S_r(X). \tag{2.1}$$

We define a sequence of points $\{x_n\}$ as follows. For $x_0\in X$ arbitrary, let $x_1\in X$, guaranteed by (2.1), be such that $S_rx_1\in F_rx_0$. Having defined $x_n\in X$, let $x_{n+1}\in X$ be such that $S_rx_{n+1}\in F_rx_n$.

Letting $F_rx_n=Y_n^r$ we denote by $O(Y_k^r;n)$ the family of sets $\{Y_k^r,Y_{k+1}^r,\dots,Y_{k+n}^r\}$. Let us assume that S_r and F_r satisfy the following conditions for every x,y in X

$$\delta(F_rx,F_ry)\le h\max\{d(S_rx,S_ry),\delta(S_rx,F_rx),\delta(S_ry,F_ry),\delta(S_rx,F_ry),\delta(S_ry,F_rx)\}, \tag{2.2}$$

where $0<h<1$ and for $r\ne p$;

$$\delta(F_rx,F_py)\le\psi(\max\{d(S_rx,S_py),\delta(S_rx,F_py),\delta(S_py,F_rx)) \tag{2.3}$$

where $\psi:[0,\infty)\to[0,\infty)$, is nondecreasing, right continuous and satisfies $\psi(t)<t$ for $t>0$. We begin with some results about $\delta(O(Y_k^r;n))$, the diameter of $O(Y_k^r;n)$ in the form lemmas. We can prove the following lemmas with the proof techniques of Das and Naik [1].

LEMMA 4. *For $k \geq 0$ and $n \in N$ suppose that $\delta(O(Y_k^r;n)) > 0$. Then $\delta(O(Y_k^r;n)) = \delta(Y_k^r, Y_j^r)$, where j is such that $k < j \leq k+n$. Also for $k \geq 1$*

$$\delta(O(Y_k^r;n)) \leq h\delta(O(Y_{k-1}^r;n+1)). \tag{2.4}$$

Proof. For i,j such that $1 \leq i < j$,

$$\begin{aligned}\delta(Y_i^r, Y_j^r) &= \delta(F_r x_i, F_r x_j)\\ &\leq h\max\{d(S_r x_i, S_r x_j), \delta(S_r x_i, F_r x_i), \delta(S_r x_j, F_r x_j),\\ &\qquad \delta(S_r x_i, F_r x_j), \delta(S_r x_j, F_r x_i)\}\\ &\leq h\max\{\delta(F_r x_{i-1}, F_r x_{j-1}), \delta(F_r x_{i-1}, F_r x_i), \delta(F_r x_{j-1}, F_r x_j),\\ &\qquad \delta(F_r x_{i-1}, F_r x_j), \delta(F_r x_{j-1}, F_r x_i)\}\\ &= h\max\{\delta(Y_{i-1}^r, Y_{j-1}^r), \delta(Y_{i-1}^r, Y_i^r), \delta(Y_{j-1}^r, Y_j^r),\\ &\qquad \delta(Y_{i-1}^r, Y_j^r), \delta(Y_{j-1}^r, Y_i^r)\}.\end{aligned}$$

Thus

$$\delta(Y_i^r, Y_j^r) \leq h\delta(O(Y_{i-1}^r; j-i+1)). \tag{2.5}$$

Now $\delta(O(Y_k^r;n) = \delta(Y_i^r, Y_j^r)$, for some i,j satisfying $k \leq i < j \leq k+n$, in view of the fact that the supremum of a finite number of distances is taken.

If $i > k$, then by (2.5)

$$\delta(O(Y_k^r;n)) \leq h\delta(O(Y_{i-1}^r; j-i+1))$$

with $i-1 \geq k$ and $j \leq k+n$, whence

$$\delta(O(Y_k^r;n)) \leq h\delta(O(Y_k^r;n)),$$

a contradiction. This proves the first assertion.

Moreover,

$$\delta(O(Y_k^r;n)) = \delta(Y_k^r, Y_j^r) \leq h\delta(O(Y_{k-1}^r; j-k+1)) \leq h\delta(O(Y_{k-1}^r; n+1)).$$

LEMMA 5. *Under the hypotheses of Lemma 4 we have*

$$\delta(O(Y_k^r;n)) \leq \frac{h^k}{1-h}\delta(Y_0^r, Y_1^r). \tag{2.6}$$

Proof.

$$\begin{aligned}\delta(O(Y_s^r;m)) &= \delta(Y_s^r, Y_j^r)\\ &\leq \delta(Y_s^r, Y_{s+1}^r) + \delta(Y_{s+1}^r, Y_j^r)\\ &\leq \delta(Y_s^r, Y_{s+1}^r) + \delta(O(Y_{s+1}^r; m-1)),\end{aligned}$$

since $j \leq s+m$. Thus

$$\delta(O(Y_s^r;m)) \leq \delta(Y_s^r, Y_{s+1}^r) + h\delta(O(Y_s^r;m)),$$

in view of (2.5). This leads to

$$\delta(O(Y_s^r;m)) \leq \frac{1}{1-h}\delta(Y_s^r, Y_{s+1}^r). \tag{2.7}$$

By repeated application of (2.4) we have

$$\delta(O(Y_k^r;n)) \leq h^k \delta(O(Y_0^r;n+k)),$$

whence (2.6) follows in view of (2.7) with $s=0$, $m=n+k$.

THEOREM 6. *Let (X,d) be a complete metric space. Let sequences of maps $S_r : X \to X$ and $F_r : X \to B(X)$ satisfy* (2.1), (2.2) *and* (2.3). *Assume that:*

a) *F_r and S_r commute at the coincidence points,*
b) *F_r and S_r are compatible,*
c) *F_r or S_r is continuous.*

Then F_r and S_r have unique common fixed point z in X. Furthermore, $F_r z = \{z\}$.

Proof. If for some n and k, $\delta(O(Y_k^r;n)) = 0$ we have

$$d(S_r x_{k+1}, S_r x_{k+2}) \leq \delta(F_r x_k, F_r x_{k+1}) = \delta(Y_k^r, Y_{k+1}^r) \leq \delta(O(Y_k^r;n)) = 0$$

and so $S_r x_{k+1} = S_r x_{k+2} = F_r x_k = F_r x_{k+1} = \{y_r\}$. Since F_r and S_r commute at the coincidence points and using inequality (2.2) then we have

$$\begin{aligned}
&\delta(F_r y_r, y_r) \leq \delta(F_r S_r x_{k+1}, F_r x_{k+1})\\
&\leq h\max\{d(S_r S_r x_{k+1}, S_r x_{k+1}), \delta(S_r S_r x_{k+1}, F_r S_r x_{k+1}), \delta(S_r x_{k+1}, F_r x_{k+1}),\\
&\qquad \delta(S_r S_r x_{k+1}, F_r x_{k+1}), \delta(S_r x_{k+1}, F_r S_r x_{k+1})\}\\
&\leq h\max\{\delta(S_r F_r x_{k+1}, y_r), \delta(S_r F_r x_{k+1}, F_r S_r x_{k+1}), 0,\\
&\qquad \delta(S_r F_r x_{k+1}, y_r), \delta(y_r, F_r y_r)\}\\
&\leq h\max\{\delta(F_r S_r x_{k+1}, y_r), \delta(S_r F_r x_{k+1}, S_r F_r x_{k+1}), 0,\\
&\qquad \delta(F_r S_r x_{k+1}, y_r), \delta(y_r, F_r y_r)\}\\
&= h\delta(y_r, F_r y_r).
\end{aligned}$$

This is possible only if $\{y_r\} = F_r y_r$, and therefore y_r is a fixed point of F_r. Also

$$d(y_r, S_r y_r) = \delta(y_r, S_r F_r x_{k+1}) = \delta(y_r, F_r S_r x_{k+1}) = \delta(y_r, F_r y_r) = 0.$$

In the present case, and so y_r is a fixed point of S_r, otherwise $\delta(O(Y_k^r;n)) > 0$. Now, passing to the general case, given $\varepsilon > 0$, let $n_0 \in N$ be such that $h^{n_0}\delta(Y_0^r, Y_1^r) < (1-h)\varepsilon$. Thus for $m > n \geq n_0$,

$$d(S_r x_{m+1}, S_r x_{n+1}) \leq \delta(Y_m^r, Y_n^r) \leq \delta(O(Y_{n_0}^r; m-n_0)) < \varepsilon,$$

in view of Lemma 5 and the choice of n_0. Hence $\{S_r x_n\}$ is a Cauchy sequence in a complete metric space and has a limit, say $\{z_r\}$. Using (2.2) we proceed as follows

$$\begin{aligned}\lim_{n\to\infty}\delta(F_r x_n, z_r) &= \lim_{n\to\infty}\delta(F_r x_n, S_r x_{n+1}) \le \lim_{n\to\infty}\delta(F_r x_n, F_r x_n)\\ &\le \lim_{n\to\infty}(h\max\{d(S_r x_n, S_r x_n), \delta(S_r x_n, F_r x_n),\\ &\qquad \delta(S_r x_n, F_r x_n), \delta(S_r x_n, F_r x_n), \delta(S_r x_n, F_r x_n)\})\\ &= h\lim_{n\to\infty}\delta(F_r x_n, z_r)\end{aligned}$$

since $0 < h < 1$, $\lim_{n\to\infty}\delta(F_r x_n, z_r) = 0$. From Lemma 2, the sequence of sets $\{F_r x_n\}$ converges to the set $\{z_r\}$. Consequently, the compatibility implies that $\delta(F_r S_r x_n, S_r F_r x_n) \to 0$ as $n \to \infty$.

Now we assume that F_r is a continuous in X. Then $\{F_r S_r x_n\}$ converges to the set $\{F_r z_r\}$ and with the use of (2.2), we have

$$\begin{aligned}&\delta(F_r S_r x_{n+1}, F_r x_n)\\ &\le h\max\{d(S_r S_r x_{n+1}, S_r x_n), \delta(S_r S_r x_{n+1}, F_r S_r x_{n+1}), \delta(S_r x_n, F_r x_n),\\ &\qquad \delta(S_r S_r x_{n+1}, F_r x_n), \delta(S_r x_n, F_r S_r x_{n+1})\}\\ &\le h\max\{\delta(S_r F_r x_n, S_r x_n), \delta(S_r F_r x_n, F_r S_r x_{n+1}), \delta(S_r x_n, F_r x_n),\\ &\qquad \delta(S_r F_r x_n, F_r x_n), \delta(S_r x_n, F_r S_r x_{n+1})\}\\ &\le h\max\{\delta(S_r F_r x_n, F_r S_r x_n) + \delta(F_r S_r x_n, F_r x_n) + \delta(F_r x_n, S_r x_n),\\ &\quad \delta(S_r F_r x_n, F_r S_r x_n) + \delta(F_r S_r x_n, F_r S_r x_{n+1}), \delta(S_r x_n, F_r x_{n+1})\},\end{aligned}$$

since $S_r S_r x_{n+1}$ is in $S_r F_r x_n$. Letting $n \to \infty$, using Lemma 1 and the compatibility, we obtain

$$\delta(F_r z_r, z_r) \le h\max\{\delta(F_r z_r, z_r), \delta(F_r z_r, F_r z_r)\}. \tag{2.8}$$

But again using (2.2), we deduce that

$$\begin{aligned}&\delta(F_r S_r x_{n+1}, F_r S_r x_{n+1})\\ &\qquad\le h\max\{d(S_r S_r x_{n+1}, S_r S_r x_{n+1}), \delta(S_r S_r x_{n+1}, F_r S_r x_{n+1})\}\\ &\qquad\le h\max\{\delta(S_r F_r x_n, F_r S_r x_n) + \delta(F_r S_r x_n, F_r S_r x_{n+1})\},\end{aligned}$$

which implies, as $n \to \infty$, by the compatibility that

$$\delta(F_r z_r, F_r z_r) \le h\delta(F_r z_r, F_r z_r).$$

Hence $\delta(F_r z_r, F_r z_r) = 0$ since $h < 1$. From (2.8), it follows that $F_r z_r = \{z_r\}$. Since (2.1) holds, there exists a point w_r in X such that $S_r w_r = z_r$ and using inequality (2.2), we have

$$\begin{aligned}\delta(F_r x_n, F_r w_r) \le h\max\{&d(S_r x_n, z_r), \delta(S_r x_n, F_r x_n),\\ &\delta(z_r, F_r w_r), \delta(S_r x_n, F_r w_r), \delta(z_r, F_r x_n)\},\end{aligned}$$

which implies, as $n \to \infty$, that

$$\delta(z_r, F_r w_r) \le h\delta(z_r, F_r w_r).$$

Thus $\{z_r\} = F_r w_r$ and by Proposition 3 we have

$$d(F_r z_r, S_r z_r) = \delta(F_r S_r w_r, S_r F_r w_r) = 0.$$

It follows that $\{z_r\} = F_r z_r = \{S_r z_r\}$ and thus z_r is also a fixed point of S_r.

Now we assume the continuity of S_r instead of F_r. Then the sequence $\{S_r S_r x_n\}$ converges to the point $S_r z_r$ and the sequence of the sets $\{S_r F_r x_n\}$ converges to the set $\{S_r z_r\}$.

We have

$$\delta(F_r S_r x_n, S_r z_r) \leq \delta(F_r S_r x_n, S_r F_r x_n) + \delta(S_r F_r x_n, S_r z_r)$$

and, as $n \to \infty$, we deduce from the compatibility and Lemma 2 that the sequence of sets $\{F_r S_r x_n\}$ also converges to the set $\{S_r z_r\}$.

Using the inequality (2.2) and since $S_r S_r x_n$ is in $S_r F_r x_n$, we get

$$\begin{aligned} d(S_r S_r x_{n+1}, S_r x_{n+1}) \\ &\leq \delta(S_r F_r x_n, F_r x_n) \\ &\leq \delta(S_r F_r x_n, F_r S_r x_n) + \delta(F_r S_r x_n, F_r x_n) \\ &\leq \delta(S_r F_r x_n, F_r S_r x_n) + h \max\{d(S_r S_r x_n, S_r x_n), \delta(S_r S_r x_n, F_r S_r x_n), \\ &\quad \delta(S_r x_n, F_r x_n), \delta(S_r S_r x_n, F_r x_n), \delta(S_r x_n, F_r S_r x_n)\}. \end{aligned}$$

As $n \to \infty$, it follows from the compatibility that

$$d(S_r z_r, z_r) \leq h d(S_r z_r, z_r),$$

which implies $S_r z_r = z_r$. Using again the inequality (2. 2), we have

$$\begin{aligned} \delta(F_r z_r, F_r x_n) \leq h \max\{&d(z_r, S_r x_n), \delta(z_r, F_r z_r), \delta(S_r x_n, F_r x_n), \\ &\delta(z_r, F_r x_n), \delta(S_r x_n, F_r z_r)\} \end{aligned}$$

and this implies, as $n \to \infty$, that

$$\delta(z_r, F_r z_r) \leq h\delta(z_r, F_r z_r).$$

Then $\{z_r\} = F_r z_r$ and hence z_r is also fixed point of F_r. In any case, z_r is a common fixed point of F_r and S_r. Suppose that F_r and S_r have another fixed point z_r'. Using the inequality (2.2), we have

$$d(z_r, z_r') = \delta(F_r z_r, F_r z_r') \leq h d(z_r, z_r').$$

This means that $z_r = z_r'$ and therefore z_r is the unique common fixed point of F_r and S_r.

Now for $r \neq p$ we assume that $z_r \neq z_p$. Using the inequality (2.3) we have

$$\begin{aligned} d(z_r, z_p) &= \delta(F_r z_r, F_p z_p) \\ &\leq \psi(\max\{\delta(S_r z_r, S_p z_p), \delta(S_r z_r, F_p z_p), \delta(S_p z_p, F_r z_r)\}) \\ &= \psi(d(z_r, z_p)), \end{aligned}$$

which implies that $d(z_r, z_p) = 0$.This means that for every $r \in N$, $z_r = z$. Thus z is the unique common fixed point of F_r and S_r.

Now, we give an example for Theorem 6.

EXAMPLE 1. Let $X = [0, \infty)$ with the usual metric. Let $S_r : X \to X$ and $F_r : X \to B(X)$ be defined by $S_r x = \frac{rx}{2}$ and $F_r x = [0, \frac{rx}{3}]$ for x in X. S_r and F_r are continuous and $F_r(X) = S_r(X) = X$. If $S_r x_n \to 0$ and $F_r x_n \to 0$, then $\delta(S_r F_r x_n, F_r S_r x_n) \to 0$ and so F_r and S_r are compatible. 0 is the unique coincidence point of F_r and S_r and so F_r, S_r commute at 0. Also for $x \geq y$

$$\delta(F_r x, F_r y) = \frac{rx}{3} = \frac{2}{3}\frac{rx}{2} = \frac{2}{3}\delta(S_r x, F_r x).$$

Thus we have condition (2.2) for $\frac{2}{3} \leq h < 1$ and for $r \neq p$

$$\delta(F_r x, F_p y) = \frac{rx}{3} \text{ or } \frac{py}{3}.$$

If $\frac{rx}{3} \geq \frac{py}{3}$ we have

$$\delta(S_r x, F_p y) = \frac{rx}{2}$$

and if $\frac{rx}{3} < \frac{py}{3}$ we have

$$\delta(S_p y, F_r x) = \frac{py}{2}.$$

Thus for $\psi(t) = \frac{5t}{6}$ we have the condition (2.3). Thus F_r and S_r satisfy the inequality (2.2) and (2.3). Then the sequences of maps F_r and S_r have a unique common fixed point (0) in X.

REMARK 1. By setting $F_r = F$ and $S_r = I$ in Theorem 6, we get Theorem 3.2 of Jungck and Rhoades [4].

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