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ON CR SUBMANIFOLDS NEARLY AND CLOSELY LORENTZIAN PARA COSYMPLECTIC MANIFOLDS

1. Introduction

Section 2 contains a review of basic concepts related to a Lorentzian para contact manifold. The notion of Lorentzian para contact manifold is an extension of the notion of almost contact metric manifold, studied in [2]. In section 3, some basic results related to Lorentzian para cosymplectic, nearly Lorentzian para cosymplectic have been obtained. In section 4, integrability conditions autoparallelness and nearly autoparallelness of the distribution $D^1 \oplus \{U\}$ on submanifold have also been obtained. An integrability of distributions $D^0, D^0 \oplus \{U\}$ and $D^1 \oplus D^0$ is also studied. In section 5, totally umbilical and totally geodesic submanifolds are discussed. Section 6 is devoted to totally para contact umbilical and totally paraa contact geodesic related to Lorentzian para contact manifold.

2. Preliminaries

An n dimensional differentiable manifold V_n on which there are defined a tensor field F of type (1,1) a vector field U, a 1-form u and a Lorentzian metric g satisfying for arbitrary vector fields X, Y, Z

(1) $\quad \overline{\overline{X}} = X + u(x)U, \quad u(U) = -1, \quad u(U) = 0, \quad u \circ F = 0,$

(2) $\quad g(\overline{X}, \overline{Y}) = g(X,Y) + u(X)u(Y),$

where $\overline{X} = FX$,

(3) $\quad g(X, FY) = g(FX, Y), \quad g(X,U) = u(X) \quad \forall X, Y \in V_n$

is called a Lorentzian para contact (L.P. contact) manifold and the structure (F, U, u, g) is an L.P.contact structure [6].

An almost para contact metric manifold is called a L.P. cosymplectic manifold [2] if

$$(\overline{\nabla}_X F)Y = 0.$$

An almost para contact metric manifold is called a nearly L.P. cosymplectic if F is a Killing, that is,

$$(\overline{\nabla}_X F)Y + (\overline{\nabla}_Y F)X = 0, \tag{4}$$

where $\overline{\nabla}$ is the operator of covariant differentiation with respect to g. On nearly L.P. cosymplectic manifold, U is a Killing vector field. That is,

$$g(\overline{\nabla}_X U, Y) + g(\overline{\nabla}_Y U, X) = 0, \quad \forall X, Y \in TV_n.$$

An almost para contact metric manifold is called a closely L.P. cosymplectic if F is a Killing and u is a closed. On a closely L.P. cosymplectic manifold we have

$$\overline{\nabla}_U F = 0, \overline{\nabla} U = 0, \overline{\nabla} u = 0.$$

Let V_m be a submanifold of a Riemannian manifold V_n with a Riemannian metric g. Then Guass and Wiengarten formulae are given respectively by,

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in TV_m, \tag{5}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \quad N \in T^{\perp} V_m, \tag{6}$$

where $\overline{\nabla}, \nabla, \nabla^{\perp}_X$ are Riemannian, induced Riemannian and induced normal connection in V_n, V_m and the normal bundle $T^{\perp}V_m$ of V_m respectively and h is the second fundamental form related to A by

$$g(h(X, Y)N) = g(A_N X, Y).$$

F is a (1,1) tensor field of on V_m, for $X \in TV_m$ and $N \in T^{\perp}V_m$ we have [4]

$$\begin{aligned} (\overline{\nabla}_X F)Y &= ((\nabla_X P)Y - A_{QY} X - th(X, Y)) \\ &\quad + ((\nabla_X Q)Y + h(X, PY) - fh(X, Y)), \\ (\overline{\nabla}_X F)N &= ((\nabla_X t)N - A_{fN} Y - PA_N X) \\ &\quad + ((\nabla_X f)N + h(X, tN) - QA_N X), \end{aligned} \tag{7}$$

where

$$\begin{aligned} FX = PX + QX, &\quad PX \in TV_m, \quad QX \in T^{\perp}V_m, \\ FN = tN + fN, &\quad tN \in TV_m, \quad fN \in T^{\perp}V_m, \end{aligned}$$

where PX, and tN are tangential parts, while QX and fN are normal parts of FX and FN respectively,

$$\begin{aligned} (\nabla_X P)Y &= \nabla_X PY - P\nabla_X Y, \\ (\nabla_X Q)Y &= \nabla^{\perp}_X QY - Q\nabla_X Y, \\ (\nabla_X t)N &= \nabla_X tN - tQ\nabla^{\perp}_X N, \\ (\nabla_X f)N &= \nabla^{\perp}_X fN - fQ\nabla^{\perp}_X N. \end{aligned}$$

The submanifold V_m is said to be totally geodesic in V_n if $h = 0$ and totally umbilical in V_n if

$$h(X, Y) = g(X, Y)K.$$

For a distribution D on V_m is said to be D-totally geodesic if

$$h(X, Y) = 0 \quad \forall X, Y \in D.$$

For a distribution D on V_m is said to be D-totally umbilical if we have

$$h(X, Y) = g(X, Y)k,$$

where K is a normal vector field $\forall X, Y \in D$. V_m is said to be (D, E)-mixed totally geodesic if $h(X, Y) = 0 \forall X \in D$ and Y in E.

Let D and E be two distributions defined on a manifold $V_m D$ is said to be a E-parallel if we have $\nabla_X y \in D \forall x \in E$ and $Y \in D$. If D is said D-parallel then it is called autoparallel. D is said to be X-parallel if we have $\nabla_X Y \in D \forall X \in TV_m$ and $Y \in D$. D is siad to be parallel if $\forall X \in TV_m$ and $Y \in D$, $\nabla_X Y \in D$.

If a distribution D on V_m is autoparallel then it is a integrable, and by Guass formula D is totally geodesic ni V_m. If D is parallel then orthogonal complementary distribution $D^\perp$ is also parallel.

A submanifold V_m of an almost L.P. contact matric manifold V_n with $U \in TV_m$ is called a CR-submanifold of V_n if for each $x \in V_m$, $T_X V_m = D_x^1 \oplus D_x^0 \oplus \{U\}_x$, where,

$$\begin{aligned} D_x^1 &= \operatorname{Ker}(Q|_{\{U\}}^{\perp})x \\ &= \{X_X \in \{U\}_X^{\perp} ||X_X|| = ||PX_X||\} = T_X V_m \cap F(T_X V_m), \\ D_x^0 &= \operatorname{Ker}(P|_{\{U\}}^{\perp})x \\ &= \{X_X \in \{U\}_X^{\perp} ||X_X|| = ||QX_X||\} = T_X V_m \cap F(T_X^{\perp} V_m). \end{aligned}$$

The condition $T_X V_m = D_x^1 \oplus D_x^0 \oplus \{U\}_X$ implies that $P^3 - P = 0$[7] on V_m and hence Dim $(D_x^1) = \operatorname{Rank}(P_X)$ is independent $\forall x \in V$ and so is D_x^0.

Now we have $TV_m = D_x^1 \oplus D_x^0 \oplus \{U\}_X$, these distribution are also differentiable we have

$$T^\perp V_m = \overline{D}^1 \oplus \overline{D}^0,$$

where

$$\overline{D}^1 = \ker(t) = T^\perp V_m \cap F(T \perp V_m), \quad \overline{D}^0 = \ker(f) = T^\perp V_m \cap F(TV_m),$$
$$QD^0 = \overline{D}^0, \quad t\overline{D}^0 = D^0.$$

3. Some results

Let V_m be a submanifold of a nearly L.P. cosymplectic manifold, tangent to U. By virtue of the eqation (4) and the equation (7) we have,

$$(8)\quad ((\nabla_X P)Y + (\nabla_Y P)X - A_{QY}X - A_{QX}Y - 2th(X,Y) + (\nabla_X Q)Y + (\nabla_Y Q)X + h(X,PY) + h(PX,Y) - 2fh(X,Y)) = 0.$$

PROPOSITION 3.1. *Let V_m be a submanifold of a nearly L.P. cosymplectic manifold. If $U \in TV_m$ then $\forall X,Y \in TV_m$ we have,*

$$(9)\quad ((\nabla_X P)Y + (\nabla_Y P)X - A_{QY}X - A_{QX}Y - 2th(X,Y)) = 0,$$

$$(10)\quad ((\nabla_X Q)Y + (\nabla_Y Q)X + h(X,PY) + h(PX,Y) - 2fh(X,Y)) = 0.$$

Proof. Equating tangential and normal parts of the equation (8), we have the results.

PROPOSITION 3.2. *Let V_m be a submanifold of a nearly L.P. cosymplectic manifold. If $U \in TV_m$ then $\forall X,Y \in TV_m$ we have,*

$$(11)\quad \overline{\nabla}_X FY - \overline{\nabla}_Y FX - F[X,Y] = 2((\nabla_X P)Y - A_{QY}X - th(X,Y))Y + 2((\nabla_X Q) + h(X,PY) - fh(X,Y)) = 0.$$

The result is obvious and hence omitted.

THEOREM 3.3. *Let V_m be a submanifold of a nearly L.P. cosymplectic manifold. If $U \in TV_m$ then $\forall X,Y \in TV_m$ we get,*

$$(12)\quad P[X,Y] = -\nabla_X PY - \nabla_Y PX + A_{QY}X + A_{QX}Y + 2P\nabla_X Y + 2th(X,Y),$$

$$(13)\quad Q[X,Y] = -\nabla^{\perp}_X QY - \nabla^{\perp}_Y QX - h(X,PY) - h(PX,Y) + 2Q\nabla_X Y + 2fh(X,Y) = 0.$$

Proof. By virtue of the equation (7) and (11) we get,

$$(\nabla_X PY - P\nabla_X Y - \nabla_Y PX + P\nabla_Y X - A_{QY}X + A_{QX}Y - 2\nabla_X PY + 2P\nabla_X Y + 2A_{QY}X + 2th(X,Y)) + (\nabla^{\perp}_X QY - \nabla^{\perp}_Y QX - Q\nabla_X Y - Q\nabla_Y X + h(X,PY) - h(PX,Y) - 2\nabla^{\perp}_X QY + 2Q\nabla_X Y - 2h(X,PY) + 2fh(X,Y)) = 0.$$

Now equating tangential parts and normal parts we have desired results.

PROPOSITION 3.4. *Let V_m be a submanifold of a nearly L.P. cosymplectic manifold. Then (P,U,u,g) is a nearly L.P. cosymplectic structure on the distribution $D^1 \oplus \{U\}$, if $th(X,Y) = 0, \forall X,Y \in D^1 \oplus \{U\}$.*

Proof. Using $D^1 \oplus \{U\} = \mathrm{Ker}(Q)$ and $P^2 + tQ = I + u \otimes U$ we obtain $P^2 = I + u \otimes U$ on $D^1 \oplus \{U\}$. We also get $PU = 0$, $u(U) = -1, u \cdot P = 0$. Using $D^1 \oplus \{U\} = \mathrm{Ker}(Q)$ and $th(X,Y) = 0$ in the equation (9) we have,

$$(\nabla_X P)Y + (\nabla_Y P)X = 0 \forall X, Y \in D^1 \oplus \{U\},$$

which proves our assertion.

THEOREM 3.5. *Let V_m be a CR-submanifold of a nearly L.P. cosymplectic manifold, we have,*

(a) *if $D^0 \oplus \{U\}$ is auto parallel then*

$$A_{QY}X + A_{QX}Y + 2th(X, Y) = 0 \forall X, Y \in D^0 \oplus \{U\},$$

(b) *if $D^1 \oplus \{U\}$ is auto parallel then*

$$h(X, PY) + h(PPX, Y) = 2fh(X, Y) \forall X, Y \in D^1 \oplus \{U\}.$$

Proof. Using the equation (9) and autoparallelness of $D^0 \oplus \{U\}$, we get (a) and using the equation (10) and autoparallelness of $D^1 \oplus \{U\}$ we get (b).

THEOREM 3.6. *Let V_m be a submanifold of a nearly L.P. cosymplectic manifold with $U \in TV_m$. If V_m is invariant then V_m is a nearly L.P. cosymplectic manifold. Moreover*

$$h(X, PY) + h(PX, Y) = 2fh(X, Y) \forall X, Y \in TV_m. \tag{14}$$

Proof. From $D^1 \oplus \{U\} = \mathrm{Ker}(Q)$ and the equation (10) we get the equation (14).

4. Integrability conditions

LEMMA 4.1. *Let V_m be a CR-submanifold of a nearly L.P. cosymplectic manifold $\forall X, Y \in D^1 \oplus \{U\}$, we get,*

$$Q[X, Y] = -h(x, PY) - h(PX, Y) + 2Q\nabla_X Y + 2fh(X, Y) \tag{15}$$

or equivalently

$$-h(X, PX) + Q\nabla_X X + fh(X, X) = 0. \tag{16}$$

Proof. Using $D^1 \oplus \{U\} = \mathrm{Ker}(Q)$ and the equation (13) we get the equation (15) and using $X = Y$ in the equation (15) we have the required result.

THEOREM 4.2. *The distribution $D^1 \oplus \{U\}$ on a CR-submanifold of a nearly L.P. cosymplectic manifolds i integrable if and only if*

$$h(X, PY) + h(PX, Y) = 2(Q\nabla_X Y + fh(X, Y)).$$

Proof. From $D^1 \oplus \{U\} = \mathrm{Ker}(Q)$ and using the equation (15) we get the result.

DEFINITION 4.3. Let V_m be a Riemannian manifold with a Riemannian connection ∇. A distribution D on V_m is said to be nerly autoparallel if $\forall X, Y \in D$ we have $(\nabla_X Y + \nabla_Y X) \in D$ or equivalently $\nabla_X X \in D$.

We have

Parallel $\Rightarrow$ Autoparallel $\Rightarrow$ Nearly autoparallel,
Parallel $\Rightarrow$ Integrable,
Autoparallel $\Rightarrow$ Integrable, and
Nearly autoparallel + Integrable $\Rightarrow$ Autoparallel.

THEOREM 4.4. *Let V_m be a CR-submanifold of a nearly L.P. cosymplectic manifold. Then the following relations holds:*

(I) *the distribution $D^1 \oplus \{U\}$ is autoparallel,*
(II) $h(X, PY) + h(PX, Y) = 2fh(X, Y).\forall X, Y \in D^1 \oplus \{U\}$,
(III) $h(X, PX) = fh(X, Y).\forall X \in D^1 \oplus \{U\}$,
(IV) *the distribution $D^1 \oplus \{U\}$ is nearly autoparallel,*

are related by (I)$\Rightarrow$(II)$\Rightarrow$(III)$\Rightarrow$(IV). *In particular if $D^1 \oplus \{U\}$ is integrable, then the above four statements are equivalent.*

Proof. (I)$\Rightarrow$(II) follows from Theorem (3.5)(b). Putting $X = Y$ in (II) we get (II)$\Rightarrow$(III). From (16) we get (III)$\Rightarrow$(IV). This completes the proof of the Theorem.

THEOREM 4.5. *Let V_m be a CR-submanifold of a nearly L.P. cosymplectic manifold, such that V_m is $D^1 \oplus \{U\}$-totally umbilical, then*

(I) *the distribution $D^1 \oplus \{U\}$ is a nearly autoparallel.*

Consequently, the following two statements becomes equivalent:

(II) *the distribution $D^1 \oplus \{U\}$ is a integrable,*
(III) *the distribution $D^1 \oplus \{U\}$ is an autoparallel.*

Proof. If submanifold V_m is $D^1 \oplus \{U\}$-totally geodesic, then $h = 0$. Thus from (16) we get $\nabla_X X = 0$, then the statement (I) holds. Hence from the definition (4.3) we get (II)$\Leftrightarrow$(III).

COROLLARY 4.6. *In totally umbilical CR-submanifold of a nearly L.P. cosymplectic manfold, $D^1 \oplus \{U\}$ is autoparallel.*

Proof. Using Theorem (4.5) we have the result.

LEMMA 4.7. *Let V_m be a CR-submanifold a nearly L.P. cosymplectic manifold, then*

$$3A_{QY}X + A_{QX}Y = P[X, Y], \forall X, Y \in D^0 \oplus \{U\}. \tag{17}$$

Proof. Let $X, Y \in D^0 \oplus \{U\}$, and $Z \in TV_m$ we have from the equation (5) and (6)

$$\begin{aligned} -A_{FX}Z + \nabla^{\perp}_Z FX = \overline{\nabla}_Z FX &= (\overline{\nabla}_Z F)X + F\overline{\nabla}_Z X \\ &= -(\overline{\nabla}_X F)Z + F\nabla_Z X + Fh(Z, X) \end{aligned}$$

so that

$$Fh(Z,X) = -A_{FX}Z + \nabla^{\perp}_{Z}FX + (\overline{\nabla}_X F)Z - F\nabla_Z X,$$

and hence

$$g(Fh(Z,X),Y) = -g(A_{FX}Z,Y) + g((\overline{\nabla}_X F)Z,Y) = -g(A_{FX}Y,Z) + g((\overline{\nabla}_X F)Y,Z).$$

Now we have

$$g(Fh(Z,X),Y) = g(h(Z,X),FY) = g(A_{FY}X,Z).$$

Thus from the above two equations we have

$$g(A_{FY}X,Z) = -g(A_{FY}Y,Z) + g((\overline{\nabla}_X F)Y,Z). \tag{18}$$

Now for $X,Y \in D^0 \oplus \{U\}$, we have

$$\overline{\nabla}_X FY - \overline{\nabla}_Y FX = A_{FX}Y - A_{FY}X + \nabla^{\perp}_X FY - \nabla^{\perp}_Y FX$$

and

$$\overline{\nabla}_X FY - \overline{\nabla}_Y FX = (\overline{\nabla}_X F)Y - (\overline{\nabla}_Y F)X + F[X,Y]$$

from the above two equations we have

$$(\overline{\nabla}_X F)Y - (\overline{\nabla}_Y F)X = A_{FX}Y - A_{FY}X + \nabla^{\perp}_X FY - \nabla^{\perp}_Y FX - F[X,Y].$$

Using the equation (5) and the above equation, we get

$$(\overline{\nabla}_X F)Y = \frac{1}{2}(A_{FX}Y - A_{FY}X + \nabla^{\perp}_X FY - \nabla^{\perp}_Y FX - F[X,Y]).$$

From the above equation and the equation (18) we get the equation (17).

THEOREM 4.8. *Let V_m be CR-submanifold of a nearly L.P. cosymplectic manifold. Then the distribution $D^0 \oplus \{U\}$ is integrable if and only if*

$$3A_{QY}X + A_{QX}Y = 0, \forall X,Y \in D^0 \oplus \{U\}.$$

Proof. From $D^0 \oplus \{U\} = \mathrm{Ker}(P)$ and the equation (17), we have the result and converse is obvious.

THEOREM 4.9. *Let V_m be CR-submanifold of a nearly L.P. cosymplectic manifold. Then the distribution D^0 is an integrable if and only if*

$$3A_{QY}X + A_{QX}Y = 0, \quad \forall X,Y \in D^0.$$

Proof. By definition of D^0 and the equation (17), we get the result.

THEOREM 4.10. *Let V_m be CR-submanifold of a L.P. cosymplectic manifold. Then the distribution D^0 and $D^0 \oplus \{U\}$ are integrable.*

Proof. The result follows from Theorem (4.8.) and Theorem (4.9).

5. Totally umbilical and totally geodesic submanifolds

LEMMA 5.1. *Let V_m be a submanifold of a closely L.P. cosymplectic manifold, tangent to U. Then the integral curve of U in V_m is geodesic in V_m, and U is an asymptotic direction.*

Proof. Since in a closely L.P. cosymplectic manifold we have $\overline{\nabla}U = 0$. Now in view of the equation (5), we get $h(U,U) = 0$. This completes the proof.

PROPOSITION 5.2. *Let D be a distribution on a submanifold V_m of a closely L.P. cosymplectic manifold such that $U \in TV_m$. If V_m is D-totally umbilical then V_m is D-totally geodesic.*

Proof. For D-totally umbilical we have

$$h(X,Y) = g(X,Y)K, \forall X,Y \in D.$$

A direction U at a point of V_m is an asymptotic direction if normal vector field $K = 0$, which implies that $h(X,Y) = 0$, which shows that V_m is totally geodesic.

PROPOSITION 5.3. *Every totally umbilical submanifold of a closely L.P. cosymplectic manifold, tangent to U, is totally geodesic.*

Proof. The proof follows from Proposition (5.2).

6. Totally Lorentzian para contact umbilical and totally Lorentzian para contact geodesic submanifolds

Let V_m be a submanifold of an almost L.P. contact metric manifolds, tangent to U. In this case $TV_m = \{U\} \oplus \{U\}^{\perp}$, where $\{U\}$ is the distribution spanned by $\{U\}$ and $\{U\}^{\perp}$ is the complementary orthogonal distribution of $\{U\}$ in V_m.

DEFINITION 6.1. A submanifold V_m of an almost L.P. contact metric manifold, tangent to U, is called (1) totally L.P. contact umbilical if it is $\{U\}^{\perp}$ totally umbilical, and (2) totally L.P. contact geodesic if it is $\{U\}^{\perp}$ totally geodesic. The condition of totally L.P. contact umbilical and totally cocntact geodesic is respectively

$$h(F^2X,F^2Y) = g(F^2XF^2Y)K, \quad \forall X,Y \in TV_m, \tag{19}$$

$$h(F^2X,F^2Y) = 0, \quad \forall X,Y \in TV_m, \tag{20}$$

where K is a normal vector field. Using the equation (1) in the equation (19) and (20), we get respectively,

$$h(X,Y) = g(FX,FY)K - u(X)h(Y,U) - u(Y)h(X,U),$$
$$h(X,Y) = -u(X)h(Y,U) - u(Y)h(X,U).$$

THEOREM 6.2. *If V_m is a totally L.P. contact umbilical CR-submanifold of a closely L.P. cosymplectic manifold, then V_m is D^0, D^1-mixed totally geodesic.*

Proof. Now we have $h(X,Y) = g(X,Y)K$, and for $X,Y \in \{U\} \perp h(U,U) = g(U,U)K$. $g(U,U)K = 0$, and using Gauss equation $\Rightarrow K = 0$. Therefore V_m is D^0, D^1 - mixed totally geodesic. This completes our assertions.

THEOREM 6.3. *Let V_m be a totally L.P. contact umbilical CR-submanifold of a closely L.P. cosymplectic manifold, then either $D^0 = \{0\}$ or $\mathrm{Dim}(D^0) = 1$ or the normal vector field k is orthogonal to FD^0.*

Proof. If $\mathrm{Dim}(D^0) > 1$, for each $H \in D^0, \exists X \in D^0$ such that $g(X,H) = 0$ and $||X|| = 0$, then

$$\begin{aligned} g(K,FH) &= g(h(X,X),Fh) = g(A_{FH}X,X) = g(A_{FX}H,X) \\ &= g(h(X,H)FX) = 0. \end{aligned}$$

This gives the desired result.

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