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TANGENCY AND ORTHOGONALITY IN METRIC SPACES

Abstract. We consider an abstract definition of tangency in metric spaces and study some of its properties. We introduce also a particular structure on metric spaces and define, with respect this structure, the notion of tangency and orthogonality. Some properties of continuous curves in such spaces are investigated.

1. Introduction

The "metric geometry" is studied since the beginning of the twentieth century, and many concepts coming from the theory of linear spaces have been extended to metric spaces (see e.g. [1], [5], [11]). Moreover, H. Buseman, in [4], has pointed out the importance of this geometry for the Finsler spaces.

On the other hand, the extension of notions typical of the linear case can be difficult even in simple situations. For example, the definition of orthogonality and the definition of tangent line to continuous curves in a normed space gives interesting and nontrivial problems.

In the second section we consider particular operations defined for subsets of a metric space and related in a natural way of the notion of tangency between sets that extends the elementary notion of tangency at a point of the graph of a real differentiable function.

In the third section we introduce an abstract structure on a metric space which allows to define a notion of tangent and normal line at a point of a continuous curves in the metric space. Moreover we can give a definition of generalized curvature at a point of a continuous curve. In the last section we add some further considerations on the curvature.

Roughly speaking, the abstract structure defined in the second section over the metric space corresponds, in some particular situations, to consideration of each point of the metric space as the "center" of a system of paths.

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Many problems can be subsequentely considered, for example: the comparison with analogous definitions on linear metric spaces, the application to continuous plane curve and general abstract definitions of "differentiable functions between metric spaces".

The original definition of tangency is investigated from different authors (see [7] [15], [16], [8],[9],[10], [13], [14]), but the aim is very different.

2. Abstract operations

Let (X, d) be a metric space and A, B be nonempty, compact (or locally compact) subsets of X. Assume that $x_0 \in A \cap B$ is an accumulation point of A. We define the following functions:

$$\underline{D}_{x_0}(A, B) = \liminf_{A\setminus\{x_0\}\ni x\to x_0} \frac{d(x, B)}{d(x, x_0)}; \tag{2.1}$$

$$\overline{D}_{x_0}(A, B) = \limsup_{A\setminus\{x_0\}\ni x\to x_0} \frac{d(x, B)}{d(x, x_0)}; \tag{2.2}$$

where $d(x, B) = \inf\{d(x, y) | \; y \in B\}$.

When $\underline{D}_{x_0}(A, B) = \overline{D}_{x_0}(A, B)$, we write $D_{x_0}(A, B)$; in general, one has

$$0 \leq \underline{D}_{x_0}(A, B) \leq \overline{D}_{x_0}(A, B) \leq 1. \tag{2.3}$$

If the point x_0 is an accumulation point for $A \bigcap B$, we have in general

$$\underline{D}_{x_0}(A, B) \neq \underline{D}_{x_0}(B, A), \qquad \overline{D}_{x_0}(A, B) \neq \overline{D}_{x_0}(B, A).$$

Some estimates between $\underline{D}_{x_0}(A, B)$, $\underline{D}_{x_0}(B, A)$, $\overline{D}_{x_0}(A, B)$, and $\overline{D}_{x_0}(B, A)$ are given in the next proposition.

PROPOSITION 2.1. *Let A, B be nonempty, compact (or locally compact) sets of the metric space X; let x_0 be an accumulation point for A and B. The following inequalities hold:*

$$|\underline{D}_{x_0}(A, B) - \underline{D}_{x_0}(B, A)| \leq \underline{D}_{x_0}(A, B) \cdot \underline{D}_{x_0}(B, A), \tag{2.4}$$

$$\underline{D}_{x_0}(A, B) - \overline{D}_{x_0}(B, A) \leq \underline{D}_{x_0}(A, B) \cdot \overline{D}_{x_0}(B, A). \tag{2.5}$$

Moreover, we have also:

$$\overline{D}_{x_0}(B, A) = 0 \implies \underline{D}_{x_0}(A, B) = 0,$$
$$\underline{D}_{x_0}(A, B) = 0 \iff \underline{D}_{x_0}(B, A) = 0.$$

Proof. Let (b_n) be a sequence in B such that $b_n \neq x_0$ for all $n \in \mathbb{N}$ and $\lim_n d(b_n, x_0) = 0$, $\underline{D}_{x_0}(B, A) = \lim_n \frac{d(b_n,A)}{d(b_n,x_0)}$.

Let now (a_n) be a sequence in A such that $d(b_n, A) = d(b_n, a_n)$ for all $n \in \mathbb{N}$. Assume first $a_n \neq x_0$ for all $n \in \mathbb{N}$. Then:

$$\begin{aligned}\frac{d(b_n,a_n)}{d(b_n,x_0)} &\geq \frac{d(a_n,B)}{d(b_n,x_0)} = \frac{d(a_n,B)}{d(a_n,x_0)} \cdot \frac{d(a_n,x_0)}{d(b_n,x_0)} \\ &\geq \frac{d(a_n,B)}{d(a_n,x_0)} \cdot \frac{d(x_0,b_n) - d(b_n,a_n)}{d(b_n,x_0)} \\ &= \frac{d(a_n,B)}{d(a_n,x_0)} \cdot \left[1 - \frac{d(b_n,a_n)}{d(b_n,x_0)}\right] = \frac{d(a_n,B)}{d(a_n,x_0)} \cdot \left[1 - \frac{d(b_n,A)}{d(b_n,x_0)}\right].\end{aligned}$$

For $n \to \infty$, the inequality

$$\underline{D}_{x_0}(B,A) \geq \liminf_{A\setminus\{x_0\}\ni a\to x_0} \frac{d(a,B)}{d(a,x_0)} \cdot [1 - \underline{D}_{x_0}(B,A)] \tag{2.6}$$

follows, and then

$$\underline{D}_{x_0}(B,A) \geq \underline{D}_{x_0}(A,B)[1 - \underline{D}_{x_0}(B,A)]. \tag{2.7}$$

If we have $a_n = x_0$ for infinitely many n, then $\underline{D}_{x_0}(B,A) = 1$ and so (2.7) is true.

In a similar way we may prove the following inequality

$$\underline{D}_{x_0}(A,B) \geq \underline{D}_{x_0}(B,A)[1 - \underline{D}_{x_0}(A,B)]. \tag{2.8}$$

Now, (2.4) follows from (2.7) and (2.8), and (2.5) from (2.7) and (2.3). ■

REMARK 2.2. If $D_{x_0}(A,B)$ and $D_{x_0}(B,A)$ exist, then:

$$D_{x_0}(A,B) = 0 \iff D_{x_0}(B,A) = 0 \tag{2.9}$$

and if $D_{x_0}(A,B) \neq 1$ then

$$\frac{D_{x_0}(A,B)}{1 + D_{x_0}(A,B)} \leq D_{x_0}(B,A) \leq \frac{D_{x_0}(A,B)}{1 - D_{x_0}(A,B)}. \tag{2.10}$$

If three sets A, B, C are considered, further estimates hold, as we see in the next proposition.

PROPOSITION 2.3. *Let A, B, C be nonempty, compact (or locally compact) subsets of the metric space X; let x_0 be an accumulation point for A, B and C. We have the following*

$$\underline{D}_{x_0}(A,C) - \underline{D}_{x_0}(A,B) \leq \overline{D}_{x_0}(B,C) \cdot [1 + \underline{D}_{x_0}(A,B)], \tag{2.11}$$

$$\overline{D}_{x_0}(A,C) - \overline{D}_{x_0}(A,B) \leq \overline{D}_{x_0}(B,C) \cdot [1 + \overline{D}_{x_0}(A,B)], \tag{2.12}$$

$$|\underline{D}_{x_0}(A,C) - \underline{D}_{x_0}(A,B)| \leq 2\max\{\overline{D}_{x_0}(B,C), \overline{D}_{x_0}(C,B)\}, \tag{2.13}$$

$$|\overline{D}_{x_0}(A,C) - \overline{D}_{x_0}(A,B)| \leq 2\max\{\overline{D}_{x_0}(B,C), \overline{D}_{x_0}(C,B)\}, \tag{2.14}$$

$$\overline{D}_{x_0}(A,C) + \underline{D}_{x_0}(A,B) \geq \underline{D}_{x_0}(B,C) \cdot [1 - \underline{D}_{x_0}(A,B)], \tag{2.15}$$

$$\underline{D}_{x_0}(A,C) + \overline{D}_{x_0}(A,B) \geq \underline{D}_{x_0}(B,C) \cdot [1 - \overline{D}_{x_0}(A,B)]. \tag{2.16}$$

Proof. Let $(a_n), (b_n)$, and (c_n) be sequences in X such that for every $n \in \mathbb{N}$:

$$a_n \in A \setminus \{x_0\},\ b_n \in B,\ c_n \in C,$$

with

$$d(a_n, B) = d(a_n, b_n),\ d(a_n, C) = d(a_n, c_n)$$

and

$$\lim_n d(a_n, x_0) = 0,\ \lim_n \frac{d(a_n, b_n)}{d(a_n, x_0)} = \underline{D}_{x_0}(A, B),\ \liminf_n \frac{d(a_n, c_n)}{d(a_n, x_0)} \geq \underline{D}_{x_0}(A, C).$$

Assume that $b_n \neq x_0$ for all $n \in \mathbb{N}$. Let (c'_n) be a sequence of elements of C such that $d(b_n, C) = d(b_n, c'_n)$ for all $n \in \mathbb{N}$. Then:

$$\begin{aligned}\frac{d(a_n, c_n)}{d(a_n, x_0)} - \frac{d(a_n, b_n)}{d(a_n, x_0)} &\leq \frac{d(a_n, c'_n)}{d(a_n, x_0)} - \frac{d(a_n, b_n)}{d(a_n, x_0)} \\ &\leq \frac{d(b_n, c'_n)}{d(a_n, x_0)} = \frac{d(b_n, c'_n)}{d(b_n, x_0)} \cdot \frac{d(b_n, x_0)}{d(a_n, x_0)} \\ &\leq \frac{d(b_n, c'_n)}{d(b_n, x_0)} \cdot \left[1 + \frac{d(a_n, b_n)}{d(a_n, x_0)}\right] \\ &= \frac{d(b_n, C)}{d(b_n, x_0)} \cdot \left[1 + \frac{d(a_n, B)}{d(a_n, x_0)}\right].\end{aligned}$$

Then we have

$$\text{(2.17)}\quad \underline{D}_{x_0}(A, C) - \underline{D}_{x_0}(A, B) \leq \limsup_{B\setminus\{x_0\}\ni b \to x_0} \frac{d(b, C)}{d(b, x_0)} [1 + \underline{D}_{x_0}(A, B)].$$

Hence

$$\text{(2.18)}\qquad \underline{D}_{x_0}(A, C) - \underline{D}_{x_0}(A, B) \leq \overline{D}_{x_0}(B, C) \cdot [1 + \underline{D}_{x_0}(A, B)].$$

If $b_n = x_0$ for infinitely many n, since $\underline{D}_{x_0}(A, B) = 1$, the previous inequality is true.

In a similar way, assuming

$$\lim_n \frac{d(a_n, c_n)}{d(a_n, x_0)} = \overline{D}_{x_0}(A, C), \quad \limsup_n \frac{d(a_n, b_n)}{d(a_n, x_0)} \leq \overline{D}_{x_0}(A, B),$$

we prove (2.12).

Since the following condition is also true:

$$\text{(2.19)}\qquad \underline{D}_{x_0}(A, B) - \underline{D}_{x_0}(A, C) \leq \overline{D}_{x_0}(C, B) \cdot [1 + \underline{D}_{x_0}(A, C)]\,,$$

it is easily seen that

$$\text{(2.20)}\quad |\underline{D}_{x_0}(A, B) - \underline{D}_{x_0}(A, C)| \leq 2 \max\{\overline{D}_{x_0}(B, C)\,,\ \overline{D}_{x_0}(C, B)\}$$

and, analogously:

$$\text{(2.21)}\quad |\overline{D}_{x_0}(A, B) - \overline{D}_{x_0}(A, C)| \leq 2 \max\{\overline{D}_{x_0}(B, C)\,,\ \overline{D}_{x_0}(C, B)\}.$$

The last inequalities follow arguing as the previous argument. ∎

REMARK 2.4. From Proposition 2.3 we deduce that the following implications hold:

(i) if $\underline{D}_{x_0}(A,B)=0$ then
$$\underline{D}_{x_0}(A,C)\le \overline{D}_{x_0}(B,C);\quad \underline{D}_{x_0}(B,C)\le \overline{D}_{x_0}(A,C);$$
(ii) if $\overline{D}_{x_0}(A,B)(=\underline{D}_{x_0}(B,A))=0$ then
$$\underline{D}_{x_0}(B,C)\le \underline{D}_{x_0}(A,C);$$
(iii) if $\overline{D}_{x_0}(A,B)=\overline{D}_{x_0}(B,A)=0$ then $\underline{D}_{x_0}(B,C)=\underline{D}_{x_0}(A,C)$; $\underline{D}_{x_0}(C,A)=\underline{D}_{x_0}(C,B)$; $\overline{D}_{x_0}(C,A)=\overline{D}_{x_0}(A,B)$.

We can characterize the condition $D_{x_0}(A,B)=0$ as follows:

PROPOSITION 2.5. *Let A,B nonempty, compact (or locally compact) subsets of a metric space X, and x_0 be an accumulation point for A and B; assume that there exists $D_{x_0}(A,B)$. Then, $D_{x_0}(A,B)=0$ if and only if for every pair of sequences $(a_n)\subset A\setminus\{x_0\}$ and $(b_n)\subset B$ such that $d(a_n,b_n)=d(a_n,B)$ for all $n\in\mathbb{N}$ and $\lim_n d(a_n,x_0)=0$ one has*

$$\frac{d(b_n,x_0)}{d(a_n,x_0)}=1,\qquad \textit{and}\qquad \frac{d(a_n,b_n)}{d(b_n,x_0)}=0. \tag{2.22}$$

Proof. Assume that $D_{x_0}(A,B)=0$ and $(a_n),(b_n)$ are as in the statement. Since
$$\left|\frac{d(b_n,x_0)}{d(a_n,x_0)}-1\right|\le\frac{d(a_n,b_n)}{d(a_n,x_0)}=\frac{d(a_n,B)}{d(a_n,x_0)},$$
the first equality in (2.22) follows. Also, from
$$\frac{d(a_n,b_n)}{d(b_n,x_0)}=\frac{d(a_n,b_n)}{d(a_n,x_0)}\cdot\frac{d(a_n,x_0)}{d(b_n,x_0)}$$
we get the second equality in (2.22).

Conversely, as $\underline{D}_{x_0}(A,B)=\overline{D}_{x_0}(A,B)$, there are sequences $(a_n)\subset A\setminus\{x_0\}$, $(b_n)\subset B$ such that $d(a_n,b_n)=d(a_n,B)$ for all $n\in\mathbb{N}$, $\lim_n d(a_n,x_0)=0$ and
$$\lim_n\frac{d(a_n,b_n)}{d(a_n,x_0)}=D_{x_0}(A,B).$$
By hypothesis, we may assume $b_n\ne x_0$ for any $n\in\mathbb{N}$, hence we have:
$$\frac{d(a_n,b_n)}{d(a_n,x_0)}=\frac{d(a_n,b_n)}{d(b_n,x_0)}\cdot\frac{d(b_n,x_0)}{d(a_n,x_0)}\qquad\forall n\in\mathbb{N}.$$
Then $D_{x_0}(A,B)=\lim_n\frac{d(a_n,b_n)}{d(a_n,x_0)}=0$. ∎

An application of the preceding results which will justify Definition 2.8 below is given in the following example, whose proof is in [12].

EXAMPLE 2.6. As a metric space take $\mathbb{R}^2$, endowed with the usual euclidean metric. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $G_f \subset \mathbb{R}^2$ its graph. Let $p_0 = (x_0, f(x_0)) \in G_f$, and consider the functions $\underline{D}_{p_0}$, $\overline{D}_{p_0}$. For r a straight line through p_0, then, the following conditions are equivalent:

(i) r is the tangent line to G_f at p_0;
(ii) $\overline{D}_{p_0}(r, G_f) = 0$ and $\underline{D}_{p_0}(s, G_f) > 0$ for every line $s \neq r$ through p_0;
(iii) $\overline{D}_{p_0}(G_f, r) = 0$ and $\underline{D}_{p_0}(G_f, s) > 0$ for every line $s \neq r$ through p_0.

It is not clear if the above example holds when the function f is continuous only at x_0. But is very easy to prove the following proposition, still given in [12].

PROPOSITION 2.7. *Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function, let $p_0 = (x_0, f(x_0))$ be in G_f and let r be a straight line through p_0. If $\overline{D}_{p_0}(r, G_f) = 0$ then r is the tangent line to G_f in p_0.*

Let us now show how the functions $\overline{D}$ and $\underline{D}$ can be used to give a definition of tangency in a metric space.

DEFINITION 2.8. Let A, B be nonempty, compact (or locally compact) sets of the metric space X and let x_0 be an accumulation point of A and B. We say that A *is tangent to* B *in* x_0 if and only if $D_{x_0}(A, B) = 0$.

We say that A, B *are tangent in* x_0 if and only if both $D_{x_0}(A, B)$ and $D_{x_0}(B, A)$ exist and $D_{x_0}(A, B) = D_{x_0}(B, A) = 0$.

3. Line-structured metric spaces

In this section we introduce an abstract structure on a metric space X by assigning to each point $x \in X$ a family $\mathcal{R}(x)$ of subsets of X whose properties are similar to those of the straight lines in the euclidean space. This allows us to give an abstract definition of tangent and normal lines to a given curve in X.

DEFINITION 3.1. Let (X, d) be an abstract metric space. We say $(X, d, \mathcal{R})$ is a *line-structured metric space* (L-SMS for short) or a metric space with line elements if for every $x \in X$, $\mathcal{R}(x)$ is a family of locally compact subsets of X such that:

(3.1) $$x \in r \ \forall r \in \mathcal{R}(x);$$

(3.2) $$\forall r, s \in \mathcal{R}(x) \quad \exists D_x(r, s),\ \exists D_x(s, r) \text{ and } D_x(r, s) = D_x(s, r);$$

(3.3) $$r, s \in \mathcal{R}(x),\ D_x(r, s) = 0 \Longrightarrow r = s;$$

(3.4) $$\exists \delta > 0\ \forall y \in X : 0 < d(x, y) < \delta \Longrightarrow \mathcal{R}(x) \bigcap \mathcal{R}(y) \neq \emptyset.$$

We set $\mathcal{R} = \bigcup_{x \in X} \mathcal{R}(x)$.

For $r \in \mathcal{R}(x) \bigcap \mathcal{R}(y)$, we write $r = r_{xy}$, but observe that, in general, $\mathcal{R}(x) \bigcap \mathcal{R}(y)$ can contain more than one element. From (2.3) and (2.13) we get that for every $r, s, t \in \mathcal{R}(x)$ the inequality

$$|D_x(r,t) - D_x(t,s)| \leq 2D_x(r,s) \tag{3.5}$$

holds. Moreover, we have the following "continuity" property of the function D_x in $\mathcal{R}(x)$.

PROPOSITION 3.2. *Let $(r_n), (s_n)$ be sequences in $\mathcal{R}(x)$ and $r, s \in \mathcal{R}(x)$. If $\lim_n D_x(r_n, r) = \lim_n D_x(s_n, s) = 0$ then*

$$\lim_n D_x(r_n, s_n) = D_x(r, s).$$

Proof. From (3.5), we have:

$$\begin{aligned} |D_x(r_n, s_n) - D_x(r,s)| &\leq |D_x(r_n, s_n) - D_x(s_n, r)| + |D_x(s_n, r) - D_x(r,s)| \\ &\leq 2[D_x(r_n, r) + D_x(s_n, s)], \end{aligned}$$

and the thesis follows. ∎

We remark explicitly that in a metric space (X, d) we can define different "line elements". Moreover, if d_1 is another metric on X, in general $(X, d_1, \mathcal{R})$ is not a line-structured metric space, even if d_1 is equivalent to d.

If $(X, d, \mathcal{R})$ is a L-SMS, we may give the following definition:

DEFINITION 3.3. Let A be a nonempty, compact (or locally compact) subset of X and x_0 an accumulation point of A. We say that r is *tangent to A at x_0* if the following conditions hold:

$$r \in \mathcal{R}(x_0); \tag{3.6}$$

$$\overline{D}_{x_0}(r, A) = 0; \tag{3.7}$$

$$\forall s \in \mathcal{R}(x_0),\ s \neq r \Longrightarrow \underline{D}_{x_0}(s, A) > 0. \tag{3.8}$$

In particular, let $(X, d, \mathcal{R})$ be a L-SMS and $\gamma :]0,1[\to X$ a continuous curve in X; set $\mathrm{stg}\gamma := \{\gamma(t) | t \in]0,1[\}$, and consider $x_0 = \gamma(t_0) \in \mathrm{stg}\gamma$. If we specialize the above definition to $A = \mathrm{stg}\gamma$, we can say that r is tangent to γ at x_0 if r is tangent to $\mathrm{stg}\gamma$ at x_0, according to Definition 3.3.

From Proposition 2.5, we can state conditions equivalent to (3.7).

Moreover, for given a L-SMS $(X, d, \mathcal{R})$, we can give the following definitions:

DEFINITION 3.4. Let A be a nonempty, compact (or locally compact) subset of X and x_0 an accumulation point of A.

(i) We say that r is *normal (or orthogonal) to A at x_0* if $r \in \mathcal{R}(x_0)$ and $\underline{D}_{x_0}(r, A) = 1$.
(ii) We say that r is *weakly normal to A at x_0* if $r \in \mathcal{R}(x_0)$ and for every $s \in \mathcal{R}(x_0)$, if $s \neq r$ then $\overline{D}_{x_0}(s, A) < \underline{D}_{x_0}(r, A)$.

(iii) We say that r is *strongly normal to A at x_0* if r is normal to A at x_0 and for every $s \in \mathcal{R}(x_0)$, if $s \neq r$ then $\overline{D}_{x_0}(s, A) < 1$.

We have the following general result:

PROPOSITION 3.5. *Let A be compact (or locally compact) in a L-SMS X, x_0 an accumulation point of A, and $r \in \mathcal{R}(x_0)$. If for every pair of sequences $(x_n) \subset r \setminus \{x_0\}$ and $(a_n) \subset A$ such that $d(x_n, A) = d(x_n, a_n)$ for all $n \in \mathbb{N}$ one has:*

$$\lim_n d(x_n, x_0) = 0 \implies \lim_n \frac{d(a_n, x_0)}{d(x_n, x_0)} = 0,$$

then $D_{x_0}(r, A) = 1$.

Proof. It is sufficient to observe that the following inequality is true:

$$\left|\frac{d(x_n, A)}{d(x_n, x_0)} - 1\right| = \left|\frac{d(x_n, a_n) - d(x_n, x_0)}{d(x_n, x_0)}\right| \leq \frac{d(a_n, x_0)}{d(x_n, x_0)}. \blacksquare$$

When $A = \text{stg}\gamma$ for a continuous curve γ in X, as for the tangent, we say that r is normal (weakly normal, strong normal) to γ at x_0 if it is so to $\text{stg}\gamma$.

In a natural way, the functions $\underline{D}_{x_0}$, $\overline{D}_{x_0}$ lead to a definition of curvature of a continuous curve.

DEFINITION 3.6. Let $(X, d, \mathcal{R})$ be a L-SMS such that for every pair of points $x \neq y$ in X there is only one element in $\mathcal{R}(x) \bigcap \mathcal{R}(y)$, which we denote by r_{xy}. Let $\gamma :]0, 1[\to X$ be a continuous curve and $x_0 = \gamma(t_0)$. We say that γ has *strong curvature to* x_0, denoted by $C^*(x_0)$, if

$$C^*(x_0) = \lim_{\text{stg}\gamma \ni x_i \to x_0} \frac{D_{x_2}(r_{12}, r_{23})}{d(x_1, x_3)}. \tag{3.9}$$

We say that γ has *weak curvature to* x_0, denoted by $C(x_0)$, if

$$C(x_0) = \lim_{\text{stg}\gamma \ni x_i \to x_0} \frac{D_{x_0}(r_{10}, r_{02})}{d(x_1, x_2)}. \tag{3.10}$$

It is evident that if $r \in \mathcal{R}(x)$ is a continuous curve then $C^*(x_0) = C(x_0) = 0$ at every point $x_0 \in r$.

Henceforth, we assume the following further property of $\mathcal{R}$:

ASSUMPTION 3.7. *For every $x \in X$ and for every sequence $(r_n) \subset \mathcal{R}(x)$ there is a subsequence (r_{n_k}) and $\overline{r} \in \mathcal{R}(x)$ such that* $\lim_n D_x(r_{n_k}, \overline{r}) = 0$.

The following proposition is easy to prove:

PROPOSITION 3.8. *Let X be a L-SMS, and $\gamma :]0, 1[\to X$ a continuous curve. If γ has a finite weak curvature $C(x_0)$ at $x_0 = \gamma(t_0)$ then there is only one*

$r^* \in \mathcal{R}(x_0)$ such that

$$\lim_{\mathrm{stg}\gamma \ni x \to x_0} D_{x_0}(r^*, r_{xx_0}) = 0 \tag{3.11}$$

and

$$C(x_0) = \lim_{\mathrm{stg}\gamma \ni x \to x_0} \frac{D_{x_0}(r^*, r_{xx_0})}{d(x, x_0)}. \tag{3.12}$$

Proof. Given $\epsilon > 0$, let I be a neighbourhood of x_0 such that:

$$(C(x_0) - \epsilon) d(x,y) < D_{x_0}(r_{xx_0}, r_{yx_0}) < (C(x_0) + \epsilon) d(x,y) \quad \forall x, y \in I. \tag{3.13}$$

Fix y and consider a sequence (x_n) in $I \bigcap \mathrm{stg}\gamma$, converging to x_0. Define $r_n := r_{x_n x_0}$ and assume, without loss of generality because of Assumption 3.7, that

$$\lim_n D_{x_0}(r^*, r_n) = 0 \tag{3.14}$$

with $r^* \in \mathcal{R}(x_0)$. From Proposition 3.2 and (3.13), we have

$$\begin{aligned} (C(x_0) - \epsilon) d(x_0, y) &\le D_{x_0}(r^*, r_{yx_0}) \\ &= \lim_n D_{x_0}(r_n, r_{yx_0}) \le (C(x_0) + \epsilon) d(x_0, y). \end{aligned} \tag{3.15}$$

From the previous inequality, (3.11) follows. Uniqueness of r^* is an easy consequence of (3.5) and, finally, condition (3.12) follows from (3.15). ∎

Let us show an estimate concerning the curvature C^*.

PROPOSITION 3.9. *Let $\gamma :]0,1[\to X$ be a continuous curve having in all its points x finite strong curvature $C^*(x)$. Assume that C^* is a continuous function on* stgγ*; then for any $\epsilon > 0$ there is $\delta > 0$ such that for all $x_1 \in \mathrm{stg}\gamma$ and for every pair $x_2, x_3 \in \mathrm{stg}\gamma$, if $d(x_1, x_i) < \delta$ for $i = 2, 3$ then*

$$\left| C^*(x_1) - \frac{D_{x_1}(r_{12}, r_{13})}{d(x_2, x_3)} \right| < \epsilon.$$

Proof. Assume that the thesis is not true; then there exist $\epsilon > 0$, three sequences (x_n), (y_n) and (z_n) with elements in stgγ and there exists $\overline{x} \in \mathrm{stg}\gamma$ such that:

$$\lim_n d(x_n, \overline{x}) = \lim_n d(y_n, \overline{x}) = \lim_n d(z_n, \overline{x}) = 0$$

and

$$\left| C^*(x_n) - \frac{D_{x_n}(r_{x_n y_n}, r_{x_n z_n})}{d(y_n, z_n)} \right| > \epsilon \qquad \forall n \in \mathbb{N}.$$

By the continuity of C^*, we can assume that $|C^*(\overline{x}) - C^*(\xi)| < \epsilon$ for $\xi = x_n, y_n, z_n$ and n large enough, and a contradiction follows. ∎

4. Results on curvature

In this section we give some considerations on curvature; we prove that, with simple conditions on the family $\mathcal{R}$, a continuous rectificable curve of X with zero strong curvature at every point is a part of some element of $\mathcal{R}$.

Let $(X, d, \mathcal{R})$ be a L-SMS such that for every pair of points $x \neq y$ in X there is only one element in $\mathcal{R}(x) \bigcap \mathcal{R}(y)$, as in Definition 3.6 and that Assumption 3.7 holds.

Henceforth, we further require the following conditions:

(i) for all $x, y \in X$ with $x \neq y$, there exists $\pi_{x,y} : \mathcal{R}(x) \to \mathcal{R}(y)$ such that

$$\pi_{x,y}(r_{xy}) = r_{xy}; \tag{4.1}$$

(ii) there exists $\alpha_{xy} \in]0, 1]$ such that for all $r, s \in \mathcal{R}(x)$

$$D_x(r, s) \leq \alpha_{xy} D_y(\pi_{x,y}(r), \pi_{x,y}(s)); \tag{4.2}$$

(iii) for all $r, s \in \mathcal{R}(x)$, and for every $\epsilon > 0$, $x_0 = x, x_1, x_2, \ldots, x_n$ with $x_i \neq x_j$ $(i \neq j)$, there is $\pi_{i,i+1} = \pi_{x_i,x_{i+1}}$ such that:

$$|D_x(\pi_{n,0} \circ \pi_{n-1,n} \circ \ldots \circ \pi_{0,1} r, s) - D_x(r, s)| \leq \epsilon; \tag{4.3}$$

(iv) for every $r \in \mathcal{R}(x)$, $y \neq x$, $\pi_{x,y}$, $\epsilon > 0$, $x_0 = x, x_1 = y, x_2, \ldots, x_{n-1}, x_n$ with $x_i \neq x_j$ $(i \neq j)$, there exists $\pi_{i,i+1} = \pi_{x_i,x_{i+1}}$ $(i \neq 0)$ verifying (4.1), (4.2) and

$$|D_x(\pi_{n,0} \circ \pi_{n-1,n} \circ \ldots \circ \pi_{0,1} r, r)| \leq \epsilon. \tag{4.4}$$

Let us set $\Pi_{x,y} = \bigcup_{x,y \in X} \pi_{x,y}$. The next proposition is easy to prove.

PROPOSITION 4.1. *Let γ be a continuous curve in X and x_1, x_2, x_3 be distinct points in* stgγ. *Setting $D_i = D_{x_i}, r_{i,j} = r_{x_i x_j}$, for any $r \in \mathcal{R}(x_1)$ and for any $s \in \mathcal{R}(x_2)$ we have:*

$$|D_1(r, r_{1,2}) - D_2(s, r_{1,2})| \leq 2|D_1(r, r_{1,3}) + D_3(r_{1,3}, r_{3,2}) + D_2(r_{3,2}, s)|. \tag{4.5}$$

Proof. Fix x_i, $i = 1, 2, 3$, in stgγ, $r, s \in \mathcal{R}$, $\epsilon > 0$ and $\pi_{1,3}, \pi_{3,2}$ for which condition (4.3) holds. From (3.5) the following inequality follows:

$$D_1(r, r_{1,2}) \leq 2[D_1(r, r_{1,3}) + D_3(r_{1,3}, r_{2,3}) + D_2(r_{2,3}, s)] + D_2(s, r_{1,2}) + \epsilon. \tag{4.6}$$

In fact we have

$$\begin{aligned}
D_1(r, r_{1,2}) &\leq 2D_1(r, r_{1,3}) + D_1(r_{1,3}, r_{1,2}) \\
&\leq 2D_1(r, r_{1,3}) + \alpha_{13} D_3(r_{1,3}, \pi_{1,3}(r_{1,2})) \\
&\leq 2D_1(r, r_{1,3}) + D_3(r_{1,3}, \pi_{1,3}(r_{1,2})) \\
&\leq 2[D_1(r, r_{1,3}) + D_3(r_{1,3}, r_{2,3})] + \alpha_{23} D_2(r_{2,3}, \pi_{3,2}(\pi_{1,3}(r_{1,2}))) \\
&\leq 2[D_1(r, r_{1,3}) + D_3(r_{1,3}, r_{2,3}) + D_2(r_{2,3}, s)] \\
&\quad + D_2(s, \pi_{3,2}(\pi_{1,3}(r_{1,2}))) \\
&\leq 2[D_1(r, r_{1,3}) + D_3(r_{1,3}, r_{2,3}) + D_2(r_{2,3}, s)] + D_2(s, r_{1,2}) + \epsilon.
\end{aligned}$$

Hence, changing the role of x_1, x_2 and r, s, (4.6) holds and the thesis follows from the arbitrariness of ϵ. ∎

Now we can prove the following proposition:

PROPOSITION 4.2. *Let $(X, d, \mathcal{R})$ be a L-SMS verifying all the preceding conditions and, for simplicity, (3.4) with $\delta = \infty$. Let $\gamma : [0,1] \to X$ be a continuous, rectifiable curve such that $C^*(x) = 0$ for all $x \in$ stgγ. Then, for every pair x, y of distinct points of* stgγ *we have:*

$$D_x(r_x^*, r_{xy}) = D_y(r_y^*, r_{xy}) \tag{4.7}$$

and, for every $\pi_{y,x} \in \Pi_{y,x}$,

$$D_x(r_x^*, \pi_{y,x}(r_y^*)) = 0, \tag{4.8}$$

where, for all $x \in$ stgγ, r_x^* is the element of $\mathcal{R}(x)$ given in Proposition 3.8.

Proof. Consider $x \neq y \in$ stgγ and $\epsilon > 0$; since the curve is rectifiable, it is possible to consider a finite number of points x_i, $i = 0, 1, 2, \dots, n$ in the arc of extreme points x, y such that $x_0 = x$, $x_n = y$ and $D_i(r_{i-1,i}, r_{i,i+1}) < \epsilon d(x_{i+1}, x_{i-1})$ for $i = 1, 2, \dots, n-1$, $D_0(r_0^*, r_{0,1}) < \epsilon d(x_0, x_1)$, $D_n(r^*{}_n, r_{n-1,n}) < \epsilon d(x_{n-1}, x_n)$.

Denote by $L(\gamma)$ the (finite) lenght of the curve. An application of Proposition 4.1 extended to a finite number of points gives the following inequalities:

$$\begin{aligned} D_0(r_0^*, r_{0,n}) &\leq 2[D_0(r_0^*, r_{0,1}) + D_1(r_{0,1}, r_{1,2}) + \dots + D_n(r_{n-1,n}, r_n^*)] \\ &\quad + D_n(r_n^*, \pi_{n-1,n} \circ \pi_{n-1,n-2} \circ \dots \circ \pi_{0,1}(r_{0,n})) \\ &\leq 2[d(x_0, x_1) + d(x_2, x_0) + d(x_1, x_2) + \dots + d(x_{n-1}, x_n)] \\ &\quad + D_n(r_n^*, r_{0,n}) + \epsilon. \end{aligned}$$

Then

$$D_0(r_0^*, r_{0,n}) - D_n(r_n^*, r_{0,n}) \leq 4\epsilon L(\gamma) + \epsilon.$$

Exchanging x and y, we obtain (4.7). Moreover, from (4.6) and (4.4) applied to r_0^* and $\pi_{x,y}(r_y^*)$ with obviuos modification, we prove easily (4.8). ∎

With a very simple auxiliary condition we can obtain a stronger result which can be read as a necessary condition on the possible choices of the family $\mathcal{R}$.

PROPOSITION 4.3. *Let $(X, d, \mathcal{R})$ be a L-SMS verifying all preceding conditions and, for simplicity, $\delta = \infty$ in (3.4); assume also that*

$$\forall x \neq y \in X \quad \exists \pi_{x,y} \in \Pi(x,y) \quad \text{such that} \quad 0 < \alpha_{xy} < 1, \tag{4.9}$$

where a_{xy} is the coefficient in (4.2). Let $\gamma : [0,1] \to X$ be a continuous, rectifiable curve such that $C^*(x) = 0$ for any $x \in$ stgγ. Then there exists $\bar{r} \in \mathcal{R}$ such that stg$\gamma \subseteq \bar{r}$.

Proof. Consider $\pi_{x,y} \in \Pi(x,y)$ as in (4.9). Then, from (4.1), (3.5), (4.8) and (4.7), we have:

$$D_x(r_x^*, r_{xy}) \leq \alpha_{xy} D_y(r_{xy}, \pi_{xy}(r_x^*)) \leq \alpha_{xy} D_y(r_{xy}, r_y^*) = \alpha_{xy} D_x(r_{xy}, r_x^*).$$

From the condition $\alpha_{xy} \in]0,1[$ we deduce that $r_{xy} = r_x^*$; hence, $y \in r_x^*$ and, by the arbitrariness of y, the thesis follows with $\bar{r} = r_x^*$. ∎

EXAMPLE 4.4. Let us consider $X = \mathbb{R}^2$ endowed with the euclidean metric; let further $\mathcal{R}(x)$ be the set of straight lines through the point x and $\pi_{x,y}$ the traslation of the plane along the line r_{xy} (in such situation $\alpha_{x,y} = 1$). Then Proposition 4.3 can be specialized as follows:

> *Every rectificable continuous curve of the real plane, having strong curvature zero in all points, has the following property: for every* $x \neq y \in \mathrm{stg}\gamma$ *the lines* r_x^*, r_y^* *are parallel.*

REMARK 4.5. We remark that most of the results in Sections 3 and 4 do not depend on the definition of the functions $\overline{D}$ and $\underline{D}$, but only on their formal properties summarized in Definition 3.1 and (3.5), which could be taken as a starting point for an axiomatic treatment of structures even more general than L-SMS.

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