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REMARKS ON FIXED POINTS FOR INVOLUTIONS OF ORDER $n = 3$ IN BANACH SPACES

Abstract. In this paper we study the problem of existence of fixed points of k-Lipschitzian and uniformly k-Lipschitzian mappings ($k > 1$) defined on nonempty closed convex subset of Banach space. Using very simple method we extend Kirk and Linhart's result [5, 8] in the case of involution of order $n = 3$.

1. Introduction

Let C be a nonempty closed convex subset of Banach space E. A mapping $T: C \to C$ is called *k-Lipschitzian* if

$$\exists k > 0 \, \forall x, y \in C \; \|Tx - Ty\| \leq k\|x - y\|.$$

It is called *nonexpansive* if the same condition with $k = 1$ holds.

In 1970, K. Goebel [1] showed that involutions (mappings for which $T^2 = I$) always have fixed point if they are k-Lipschitzian for $k < 2$.

The Goebel's result was next extended in 1971 by W. A. Kirk [5] who showed that a mapping $T: C \to C$ for which $T^n = I$ ($n > 1$) has a fixed point if

$$\|T^i x - T^i y\| \leq k\|x - y\|, \;\; x, y \in C, \; i = 1, 2, \ldots, n-1,$$

where k satisfies

$$(n-1)(n-2)k^2 + 2(n-1)k < n^2.$$

Hence, for $n = 3$ we have the following estimate for k:

$$k < \frac{1}{2}[\sqrt{22} - 2] \approx 1.345.$$

In 1973, J. Linhart [8] showed that a k-Lipschitzian mapping $T: C \to C$ for which $T^n = I$ ($n > 1$) has a fixed point if

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$$\frac{1}{n}\sum_{j=n-1}^{2n-3} k^j < 1.$$

Hence, for $n = 3$ we have the following estimate for k:

$$k < \sqrt[3]{\frac{79}{54}+\sqrt{\frac{6237}{2916}}}+\sqrt[3]{\frac{79}{54}-\sqrt{\frac{6237}{2916}}}-\frac{1}{3}\approx 1.174.$$

In 1991, J. Górnicki [2] showed that a k-Lipschitzian mapping $T: C \to C$ with $T^3 = I$ has a fixed point if $k < 1.2208$.

Recently M. Koter - Mórgowska [7] showed that a k-Lipschitzian mapping $T: C \to C$ in a Hilbert space with $T^3 = I$ has a fixed point if $k < \gamma_3^H(0)$, where $\gamma_3^H(0) \geq 1.366$.

In the present paper we extend previous results in the general case of Banach space for involutions of order $n = 3$ (mappings T for which $T^3 = I$).

2. Lipschitzian mappings

We will start with the following lemma:

LEMMA 1 ([2, 3]). *Let C be a nonempty closed subset of a Banach space E and $T: C \to C$ be a k-Lipschitzian. Given $A, B \in \mathbb{R}$ with $0 \leq A < 1$ and $0 < B$. If for arbitrary $x \in C$ there exists $u \in C$ such that*

$$\|Tu - u\| \leq A \cdot \|Tx - x\|$$

and

$$\|u - x\| \leq B \cdot \|Tx - x\|$$

then T has a fixed point in C.

THEOREM 1. *Let C be a nonempty closed convex subset of a Banach space E and $T: C \to C$ be a k-Lipschitzian such that $T^3 = I$. If*

$$k < k_0 = \sup_{\alpha\in(0,1)} \left\{s > 1 : \alpha^2 s^3 + \left[\alpha^2(1-\alpha) + (1-\alpha)^3\right] s^2 - 1 = 0\right\},$$

then T has a fixed point in C.

Note, that for $\alpha = 0.345$ we obtain the evaluation $k_0 \geq 1.3822$, which is better than that obtained in [7] even for a Hilbert space.

Proof. We consider a sequence generated by first steps of Halpern's iteration procedure [4] as follows: let x be an arbitrary point in C, i.e.

$$\begin{aligned} x_0 &= x \in C, \\ x_1 &= \alpha x_0 + (1-\alpha)Tx_0, \\ x_2 &= \alpha x_0 + (1-\alpha)Tx_1, \end{aligned}$$

where $\alpha \in (0,1)$. Take $z = x_2$, and observe that

$$\begin{aligned}\|Tz - z\| &= \|\alpha(Tz - x_0) + (1-\alpha)(Tz - Tx_1)\| \\ &\le \alpha k\|z - T^2x_0\| + (1-\alpha)k\|z - x_1\| \\ &= \alpha k\|\alpha(x_0 - T^2x_0) + (1-\alpha)(Tx_1 - T^2x_0)\| \\ &\quad + (1-\alpha)k\|(1-\alpha)(Tx_1 - Tx_0)\| \\ &\le \alpha^2k^3\|Tx_0 - x_0\| + \alpha(1-\alpha)k^2\|x_1 - Tx_0\| + (1-\alpha)^2k^2\|x_1 - x_0\| \\ &= \left\{\alpha^2k^3 + \alpha^2(1-\alpha)k^2 + (1-\alpha)^3k^2\right\}\|Tx_0 - x_0\|.\end{aligned}$$

Since, $h_\alpha(k) = \alpha^2k^3 + \left[\alpha^2(1-\alpha) + (1-\alpha)^3\right]k^2 < 1$ for all $\alpha \in (0,1)$ and $k < k_0$, and

$$\begin{aligned}\|z - x_0\| &= \|(1-\alpha)(Tx_1 - x_0)\| \le (1-\alpha)k\|x_1 - T^2x_0\| \\ &= (1-\alpha)k\|\alpha(x_0 - T^2x_0) + (1-\alpha)(Tx_0 - T^2x_0)\| \\ &\le \left\{(1-\alpha)\alpha k^3 + (1-\alpha)^2k^2\right\}\|Tx_0 - x_0\|,\end{aligned}$$

hence Lemma 1 implies the existence of fixed points of T in C. The sequence $\{z_n\}$ defined by

$$\begin{aligned}x_0 &= x \in C, \\ x_1 &= \alpha x_0 + (1-\alpha)Tx_0, \\ z_1 &= \alpha x_0 + (1-\alpha)Tx_1,\end{aligned}$$

and

$$z_{n+1} = \alpha z_n + (1-\alpha)T(\alpha z_n + (1-\alpha)Tz_n), \quad n = 1, 2, \ldots,$$

converges strongly to a fixed point of T. ∎

3. Uniformly Lipschitzian mappings

Recall, that a mapping $T: C \to C$ is called *uniformly k-Lipschitzian* if for all $n \in \mathbb{N}$ and $x, y \in C$,

$$\|T^nx - T^ny\| \le k\|x - y\|.$$

THEOREM 2. *Let C be a nonempty closed convex subset of a Banach space E. Let $T: C \to C$ be a mapping such that*

(1) $\|T^ix - T^iy\| \le k\|x - y\|$ *for* $i = 1, 2,\quad x, y \in C$,

(2) $T^3 = I$.

If $k < k_0 = \sqrt{\frac{14\sqrt{7}+88}{59}} \approx 1.455792162$ *then T has a fixed point in C.*

Proof. As in the proof of Theorem 1 we consider the following iteration procedure:

$$\begin{aligned}x_0 &= x \in C, \text{ (arbitrary point in } C) \\ x_1 &= \alpha x_0 + (1-\alpha)Tx_0,\end{aligned}$$

$$x_2 = \alpha x_0 + (1-\alpha)Tx_1,$$

where $\alpha \in (0,1)$. For $z = x_2$, we have

$$\begin{aligned}
\|Tz - z\| &= \|\alpha(Tz - x_0) + (1-\alpha)(Tz - Tx_1)\| \\
&\le \alpha k\|z - T^2x_0\| + (1-\alpha)k\|z - x_1\| \\
&= \alpha k\|\alpha(x_0 - T^2x_0) + (1-\alpha)(Tx_1 - T^2x_0)\| \\
&\quad + (1-\alpha)k\|(1-\alpha)(Tx_1 - Tx_0)\| \\
&\le \alpha^2k^2\|Tx_0 - x_0\| + \alpha(1-\alpha)k^2\|x_1 - Tx_0\| + (1-\alpha)^2k^2\|x_1 - x_0\| \\
&= \left[\alpha^2 + \alpha^2(1-\alpha) + (1-\alpha)^3\right]k^2\|Tx_0 - x_0\|.
\end{aligned}$$

Since $\left[\alpha^2 + \alpha^2(1-\alpha) + (1-\alpha)^3\right]k^2 = (-2\alpha^3 + 5\alpha^2 - 3\alpha + 1)k^2 < 1$ for all $\alpha \in (0,1)$ and $k < k_0 = \sqrt{\frac{14\sqrt{7}+88}{59}}$, and

$$\begin{aligned}
\|z - x_0\| &= \|(1-\alpha)(Tx_1 - x_0)\| \le (1-\alpha)k\|x_1 - T^2x_0\| \\
&= (1-\alpha)k\|\alpha(x_0 - T^2x_0) + (1-\alpha)(Tx_0 - T^2x_0)\| \\
&\le \left[(1-\alpha)\alpha + (1-\alpha)^2\right]k^2\|Tx_0 - x_0\|,
\end{aligned}$$

hence Lemma 1 implies the existence of fixed points of T. ∎

4. Nonexpansive iterate

In this section, using Theorems 1 and 2, we extend the result of W. A. Kirk [5] in a special setting. We obtain conditions sufficient to guarantee the existence of fixed points for mappings T such that T^3 is nonexpansive.

THEOREM 3. *Let E be a reflexive Banach space which has strictly convex norm and suppose that C is a nonempty bounded closed convex subset of E with normal structure. Suppose that the mapping $T: C \to C$ has one of the following properties*

(A) *T^3 is nonexpansive and there is a constant $k < k_0$ (see Theorem 1) such that $\|Tx - Ty\| \le k\|x - y\|$ for all $x, y \in C$,*

or

(B) *T^3 is nonexpansive and there is a constant $k < k_0$ (see Theorem 2) such that $\|T^ix - T^iy\| \le k\|x - y\|$ for all $x, y \in C$, $i = 1, 2$.*

Then, T has a fixed point in C.

Proof. By the result of Browder–Göhde–Kirk [6] the set

$$C^* = \{x \in C : T^3x = x\} \neq \emptyset.$$

Because of the strict convexity of E, C^* has to be convex. Cleary C^* is closed, $T: C^* \to C^*$ and T^3 is the identity on C^*. Thus the assumptions of

Theorems 1 and 2, respectively, are satisfied for T on C^*. Consequently T has a fixed point in C. ∎

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