DEMONSTRATIO MATHEMATICA
Vol. XXXVIII No 2 2005

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DEGENERATE SYSTEMS DESCRIBED BY GENERALIZED INVERTIBLE OPERATORS AND CONTROLLABILITY

Abstract. The theory of right invertible operators was started with works of D. Przeworska-Rolewicz and then it has been developed by M. Tasche, H. von Trotha, Z. Binderman and many other mathematicians (see [10]). Nguyen Dinh Quyet (in [5, 7]), has considered the controllability of linear system described by right invertible operators where the resolving operator is invertible. These results were generalized by A. Pogorzelec in the case of one-sized invertible resolving operator (see [9]) and by Nguyen Van Mau for the system described by generalized invertible operator (see [3]). However, for the degenerate systems, the problem has not been investigated. In this paper, we deal with the initial value problem for degenerate system of the form (2.7)-(2.8) and the controllability of this system.

1. Preliminaries

Let X be a linear space over a field $\mathcal{F}$ of scalars ($\mathcal{F} = \mathbb{R}$ or $\mathbb{C}$). Denote by $L(X)$ the space of linear operators defined on linear subspaces of X, taking values in X, and write

$$L_0(X) = \{A \in L(X) : \mathrm{dom} A = X\}.$$

DEFINITION 1.1 ([10]). An operator $D \in L(X)$ is said to be right invertible if there is an operator $R \in L_0(X)$ such that $\mathrm{Im} R \subset \mathrm{dom} D$ and

$$DR = I, \tag{1.1}$$

where I is an identity operator. In this case, R is called a right inverse operator of D.

The set of all right invertible operators belonging to $L(X)$ will be denoted by $R(X)$. If $D \in R(X)$, we denote $\mathcal{R}_D = \{R \in L_0(X) : DR = I\}$.

DEFINITION 1.2 ([3]).

(i) An operator $V \in L(X)$ is said to be generalized invertible if there is an operator $W \in L(X)$ (called a generalized inverse of V) such that

$$\mathrm{Im} V \subset \mathrm{dom} W, \ \mathrm{Im} W \subset \mathrm{dom} V \text{ and } VWV = V \quad \text{on } \mathrm{dom} V.$$

The set of all generalized invertible operators in $L(X)$ will be denoted by $W(X)$. For a given $V \in W(X)$, the set of all generalized inverses of V is denoted by $\mathcal{W}_V$.

(ii) If $V \in W(X), W \in \mathcal{W}_V$ and $WVW = W$ on dom$W$, then W is called an almost inverse of V. The set of all almost inverse operators of V will be denoted by $\mathcal{W}_V^1$.

PROPOSITION 1.1 ([3]). *Suppose that $V \in W(X)$ and $W \in \mathcal{W}_V$. Then*

$$\text{dom}V = WV(\text{dom}V) \oplus \ker V. \tag{1.2}$$

DEFINITION 1.3 ([3]). An operator $F^{(r)} \in L(X)$ is said to be a right initial operator for $V \in W(X)$ corresponding to $W \in \mathcal{W}_V^1$ if

(i) $(F^{(r)})^2 = F^{(r)}$, Im$F^{(r)} = \ker V$, dom$F^{(r)} =$ dom$V$,

(ii) $F^{(r)}W = 0$ on dom$W$.

The set of all right initial operators for $V \in W(X)$ will be denoted by $\mathcal{F}_V^{(r)}$.

THEOREM 1.1 ([3]). *An operator $F^{(r)} \in L(X)$ is a right initial operator for $V \in W(X)$ corresponding to $W \in \mathcal{W}_V^1$ if and only if*

$$F^{(r)} = I - WV \quad \text{on dom}V. \tag{1.3}$$

Other properties of generalized invertible operators can be found in [2, 3], the theory of right invertible operators and their applications can be seen in [10].

2. The initial value problem for degenerate system

Assume that $V \in W(X)$, with dim(ker$V$) $\neq 0$. Denote by $F^{(r)} \in \mathcal{F}_V^{(r)}$ a right initial operator for V corresponding to $W \in \mathcal{W}_V^1$.

In this paragraph, we deal with a linear system of the form:

$$Vx = y, \quad y \in \text{Im}V, \tag{2.1}$$

$$F^{(r)}x = x_0, \quad x_0 \in \ker V. \tag{2.2}$$

THEOREM 2.1. *Suppose that $V \in W(X)$, and $F^{(r)} \in \mathcal{F}_V^{(r)}$ be a right initial operator for V corresponding to $W \in \mathcal{W}_V^1$. Then the initial value problem (2.1)–(2.2) have a unique solution of the form*

$$x = Wy + x_0. \tag{2.3}$$

Proof. By the assumption of the problem $y \in$ ImV, there exists $x_1 \in$dom$V$ such that $y = Vx_1$, and since $V = VWV$, the equation (2.1) can be written $V(x - WVx_1) = 0$. This implies that $x - WVx_1 = z$, where $z \in \ker V$, so that

$$x = Wy + z, \quad z \in \ker V. \tag{2.4}$$

By the condition (2.2) and $z \in \ker V$, there follows

$$x_0 = F^{(r)}x = F^{(r)}Wy + F^{(r)}z = z\ .$$

The theorem is proved. ■

DEFINITION 2.1. Suppose that $A, B \in L_0(X)$, where $A \neq 0$ is non-invertible and $V \in W(X)$, with $\dim(\ker V) \neq 0$. The linear system

$$AVx = Bx + y, \quad y \in X, \tag{2.5}$$

is called a degenerate system.

PROPOSITION 2.1. *Suppose that* $V \in W(X), \dim(\ker V) \neq 0$, $F^{(r)} \in \mathcal{F}_V^{(r)}$ *is a right initial operator for* V *corresponding to* $W \in \mathcal{W}_V^1$ *and* $A, B \in L_0(X)$. *Then the following identity holds on* $\operatorname{dom} V$

$$AV - B = (A - BW)V - BF^{(r)}. \tag{2.6}$$

Proof. Formula (1.3) and $\operatorname{Im} F^{(r)} = \ker V$, on $\operatorname{dom} V$ imply

$$\begin{aligned} AV - B &= (AV - B)I = (AV - B)(F^{(r)} + WV) \\ &= AVF^{(r)} + AVWV - BF^{(r)} - BWV \\ &= AV - BWV - BF^{(r)} \\ &= (A - BW)V - BF^{(r)}. \ ■ \end{aligned}$$

Now consider the initial value problem for the degenerate system:

$$AVx = Bx + y, \quad y \in X, \tag{2.7}$$

$$F^{(r)}x = x_0, \quad x_0 \in \ker V. \tag{2.8}$$

THEOREM 2.2. *Suppose that all assumptions of Proposition* 2.1 *are satisfied, moreover,*

$$y + Bx_0 \in (A - BW)(\operatorname{Im} V).$$

(i) *If* $A - BW \in R(X)$ *and* $R_{AW} \in \mathcal{R}_{A-BW}$, *then all solutions of the problem* (2.7)–(2.8) *are given by*

$$x = W[R_{AW}(y + Bx_0) + z] + x_0, \quad z \in \ker(A - BW), \tag{2.9}$$

(ii) *If* $A - BW \in \Lambda(X)$ *and* $L_{AW} \in \mathcal{L}_{A-BW}$, *then all solutions of the problem* (2.7)–(2.8) *are given by*

$$x = WL_{AW}(y + Bx_0) + x_0, \tag{2.10}$$

(iii) *If* $A - BW$ *is invertible then the unique solution of problem* (2.7)–(2.8) *is given by*

$$x = W(A - BW)^{-1}(y + Bx_0) + x_0, \tag{2.11}$$

(iv) *If* $A - BW \in W(X)$ *and* $W_{AW} \in \mathcal{W}_{A-BW}$, *then all solutions of the problem* (2.7)–(2.8) *are given by*

$$x = W[W_{AW}(y + Bx_0) + z] + x_0, \quad z \in \ker(A - BW). \tag{2.12}$$

Proof. It is known that the one-sided invertible and invertible operators are generalized invertible, it is sufficient to consider the case $A-BW$ generalized invertible.

Proposition 2.1 shows that equality (2.7) is equivalent to $(A-BW)Vx - BF^{(r)}x = y$, and then by the condition (2.8), we have

$$(A - BW)Vx = y + Bx_0. \tag{2.13}$$

With assumptions $A - BW \in W(X), W_{AW} \in \mathcal{W}_{A-BW}$ and $y + Bx_0 \in (A - BW)(\mathrm{Im}V)$, it is apparent that the equation (2.13) has solution. By the same way as the proving of Theorem 2.1, from (2.13), we have $Vx = W_{AW}(y + Bx_0) + z$, where $z \in \ker(A - BW)$. Therefore, the problem (2.7)-(2.8) is equivalent to

$$Vx = W_{AW}(y + Bx_0) + z\,, \quad z \in \ker(A - BW), \tag{2.14}$$

$$F^{(r)}x = x_0, \qquad x_0 \in \ker V. \tag{2.15}$$

By virtue of Theorem 2.1, all solutions of problem (2.14)-(2.15) are given by

$$x = W[W_{AW}(y + Bx_0) + z] + x_0. \ \blacksquare$$

EXAMPLE 2.1. Suppose that X is the space (s) of all real sequences $\{x_n\}$, $n = 0, 1, 2, \ldots$ with addition and multiplication by scalars defined as following:

If $x = \{x_n\}$, $y = \{y_n\}$, $\lambda \in \mathbb{R}$ then $x + y = \{x_n + y_n\}$, $\lambda x = \{\lambda x_n\}$, $n = 0, 1, 2, \ldots$

Let $V, A, B \in L_0(X)$ be defined by:

$V\{x_n\} = \{v_n\}$, $v_n = x_n$, $n = 0, 1, 2$ and $v_n = 0, n \geqslant 3$;
$A\{x_n\} = \{a_n\}$, $a_0 = x_0 + x_1, a_1 = x_1 + x_2, a_2 = x_2, a_3 = x_3, a_n = 0, n \geqslant 4$;
$B\{x_n\} = \{x_{n+1}\}$, $n = 0, 1, 2, \ldots$

It is proved that $\ker V = \{\{0, 0, 0, x_3, x_4, x_5, \ldots\} : x_n \in \mathbb{R}, n = 3, 4, 5, \ldots\}$ $\neq \{0\}$, and $VX \neq X$, $VVV\{x_n\} = V\{x_n\}$, i.e. V is generalized invertible operator and its almost inverse $W = V \in \mathcal{W}_V^1$. Hence, a right initial operator $F^{(r)}$ for V corresponding to the almost inverse W is given by

$$F^{(r)}\{x_n\} = (I - WV)\{x_n\} = \{f_n\},$$

where $f_n = 0$, $n = 0, 1, 2$ and $f_n = x_n$, $n \geqslant 3$.

In addition, the operator $A \neq 0$ is non-invertible. Indeed, we have

$$AX = \{A\{x_n\} = \{x_0 + x_1, x_1 + x_2, x_2, x_3, 0, 0, 0, ...\} : \{x_n\} \in X\} \neq X$$

and

$$\ker A = \{\{0, 0, 0, 0, x_4, x_5, x_6, ...\} : x_n \in \mathbb{R}, n = 4, 5, 6, ...\} \neq \{0\}.$$

Let $y = \{y_n^0\} \in X$ and $\bar{x}_0 = \{0, 0, 0, x_3^0, x_4^0, x_5^0, ...\} \in \ker V$, where $x_n^0 \in \mathbb{R}$, $n = 3, 4, 5, ...$ and $x_n^0 = -y_{n-1}^0$ if $n \geqslant 4$.

Consider the initial value problem

$$AVx = Bx + y, \tag{2.16}$$

$$F^{(r)}x = \bar{x}_0. \tag{2.17}$$

We have $\ker(A - BW) = \{\{0, 0, 0, 0, x_4, x_5, x_6, ...\} : x_n \in \mathbb{R},\ n = 4, 5, 6, ...\} \neq \{0\}$, and the operator $A - BW$ is generalized invertible, where $W_{AW} = I \in \mathcal{W}_{A-BW}$. Indeed,

$$\begin{aligned}(A - BW)I(A - BW)\{x_n\} &= \{x_0, x_1, x_2, x_3, 0, 0, 0, ...\} \\ &= (A - BW)\{x_n\}.\end{aligned}$$

Moreover, it is possible to verify that $y + B\bar{x}_0 \in (A - BW)\mathrm{Im}V$. According to Theorem 2.2, the solution of (2.16)-(2.17) is given by

$$\begin{aligned}x &= W[W_{AW}(y + B\bar{x}_0) + z] + \bar{x}_0, \\ z &= \{0, 0, 0, 0, z_4, z_5, z_6, ...\} \in \ker(A - BW) = \{x_n\},\end{aligned}$$

where $x_0 = y_0^0$, $x_1 = y_1^0$, $x_2 = y_2^0 + x_3^0$, $x_3 = x_3^0$ and $x_n = -y_{n-1}^0$, $n \geqslant 4$.

3. Controllability of degenerate systems

Let X and U be linear spaces over the same field $\mathcal{F}$ of scalars ($\mathcal{F} = \mathbb{R}$ or $\mathbb{C}$). Suppose that $V \in W(X)$, $\dim(\ker V) \neq 0$. Let $F^{(r)} \in \mathcal{F}_V^{(r)}$ be a right initial operator for V corresponding to $W \in \mathcal{W}_V^1$; operators $A, B \in L_0(X)$, $A \neq 0$ is non-invertible, and $C \in L_0(U, X)$.

We consider the degenerate system $(DS)_0$ of the form:

$$AVx = Bx + Cu, \quad CU \oplus \{Bx_0\} \subset (A - BW)(\mathrm{Im}V), \tag{3.1}$$

$$F^{(r)}x = x_0, \quad x_0 \in \ker V. \tag{3.2}$$

The spaces X and U are called the spaces of states and the spaces of controls, respectively. Elements $x \in X$ and $u \in U$ are called states and controls, respectively. The element $x_0 \in \ker V$ is said to be an initial state. A pair $(x_0, u) \in (\ker V) \times U$ is called an input. If (3.1)-(3.2) has solution $x = \Phi(x_0, u)$ then this solution is called output correspondent to input (x_0, u).

By a similar proof as in Theorem 2.2, the problem (3.1)-(3.2) is equivalent to

$$(A - BW)Vx = Cu + Bx_0 . \tag{3.3}$$

Hence, the inclusion $CU \oplus \{Bx_0\} \subset (A - BW)(\mathrm{Im}V)$ is necessary and sufficient condition for the problem (3.1)-(3.2) to have solutions for every $u \in U$.

Note that the properties of degenerate systems depend on the properties of the resolving operator $A - BW$. There are four cases to deal with:

(i) $A - BW \in R(X)$ ($A - BW$ is right invertible),
(ii) $A - BW \in \Lambda(X)$ ($A - BW$ is left invertible),
(iii) $A - BW$ is invertible,
(iv) $A - BW \in W(X)$ ($A - BW$ is generalized invertible).

Since both one-sided invertible and invertible operators are generalized invertible, it is enough to consider the case when $A - BW$ is generalized invertible. In this case, we get

$$\Phi(x_0, u) = \{x = W[T(Cu + Bx_0) + z] + x_0 : T \in \mathcal{W}_{A-BW}, \quad z \in \ker(A - BW)\}. \tag{3.4}$$

Clearly, $\Phi(x_0, u)$ is the set of all solutions of the problem (3.1)–(3.2) and for every fixed input (x_0, u) there corresponds an output $x = \Phi(x_0, u)$. Write

$$\mathrm{Rang}_{U,x_0}\Phi = \bigcup_{u \in U} \Phi(x_0, u), \quad x_0 \in \ker V. \tag{3.5}$$

The set $\mathrm{Rang}_{U,x_0}\Phi$ is called *reachable* from the initial x_0 by means of controls $u \in U$.

DEFINITION 3.1. Let a degenerate system $(DS)_0$ of the form (3.1)–(3.2) be given. Suppose that $F_1^{(r)} \in \mathcal{F}_V^{(r)}$ is arbitrary right initial operator for V.

(i) A state $x_1 \in \ker V$ is said to be $F_1^{(r)}$-reachable from an initial state $x_0 \in \ker V$ if there exists a control $u_1 \in U$ such that $x_1 \in F_1^{(r)}\Phi(x_0, u_1)$. The state x_1 is called a final state.

(ii) The system $(DS)_0$ is said to be $F_1^{(r)}$-controllable if for every initial state $x_0 \in \ker V$, we have

$$F_1^{(r)}(\mathrm{Rang}_{U,x_0}\Phi) = \ker V. \tag{3.6}$$

(iii) The system $(DS)_0$ is said to be $F_1^{(r)}$-controllable to $x_1 \in \ker V$ if

$$x_1 \in F_1^{(r)}(\mathrm{Rang}_{U,x_0}\Phi) \tag{3.7}$$

for every initial state $x_0 \in \ker V$.

LEMMA 3.1. *Suppose that the system $(DS)_0$ is $F_1^{(r)}$-controllable to zero and that for every $x_1' \in \ker V$, there exists $x_2' \in \ker V$ and $z_1' \in \ker(A - BW)$ such that*

$$F_1^{(r)}[W(TBx_2' + z_1') + x_2'] = x_1'. \tag{3.8}$$

Then every final state $x_1 \in \ker V$ is $F_1^{(r)}$-reachable from zero.

Proof. Since the system $(DS)_0$ is $F_1^{(r)}$-controllable to zero, we conclude that

$$0 \in F_1^{(r)}(\mathrm{Rang}_{U,x_0}\Phi) \quad \text{for every initial state} \quad x_0 \in \ker V.$$

Therefore, there exists a control $u_0 \in U$ and $z_0 \in \ker(A - BW)$ such that

$$F_1^{(r)}\{W[T(Cu_0 + Bx_0) + z_0] + x_0\} = 0 \tag{3.9}$$

or equivalently,

$$F_1^{(r)}W(TCu_0 + z_0) = -F_1^{(r)}(WTBx_0 + x_0). \tag{3.10}$$

The condition (3.8) means that, for every $x_1 \in \ker V$, there exists $z_1 \in \ker(A - BW)$ and $x_2 \in \ker V$ such that

$$F_1^{(r)}[W(TBx_2 + z_1) + x_2] = x_1. \tag{3.11}$$

Moreover, by formula (3.10), for the element $x_2 \in \ker V$, there exist $u_0' \in U$ and $z_0' \in \ker(A - BW)$ such that

$$F_1^{(r)}W(TCu_0' + z_0' + z_1) = F_1^{(r)}[W(TBx_2 + z_1) + x_2]. \tag{3.12}$$

So (3.11) and (3.12) yield $F_1^{(r)}W(TCu_0' + z_1') = x_1$, with $z_1' = z_0' + z_1 \in \ker(A - BW)$. This proves that every final state $x_1 \in \ker V$ is $F_1^{(r)}$-reachable from zero. ∎

THEOREM 3.1. *Suppose that all assumptions of Lemma 3.1 are satisfied. Then the degenerate system $(DS)_0$ is $F_1^{(r)}$-controllable.*

Proof. Assume that the system $(DS)_0$ is $F_1^{(r)}$-controllable to zero, i.e. there exists $u_0 \in U$ and $z_0 \in \ker(A - BW)$ such that

$$F_1^{(r)}\{W[T(Cu_0 + Bx_0) + z_0] + x_0\} = 0. \tag{3.13}$$

By Lemma 3.1, for every $x_1 \in \ker V$ there exists $u_1 \in U$ and $z_1 \in \ker(A - BW)$ such that

$$F_1^{(r)}W(TCu_1 + z_1) = x_1. \tag{3.14}$$

By adding (3.13) and (3.14) we obtain

$$F_1^{(r)}\{W[T(C(u_0 + u_1) + Bx_0) + (z_0 + z_1)] + x_0\} = x_1,$$

or

$$F_1^{(r)}\{W[T(Cu_0' + Bx_0) + z_0'] + x_0\} = x_1,$$

where $u_0' = u_0 + u_1 \in U$ and $z_0' = z_0 + z_1 \in \ker(A - BW)$. It means that x_1 is $F_1^{(r)}$-reachable from the initial state x_0. The arbitrariness of $x_0, x_1 \in \ker V$ gives $F_1^{(r)}(\mathrm{Rang}_{U,x_0}\Phi) = \ker V$, for every $x_0 \in \ker V$. The theorem is proved. ∎

THEOREM 3.2. *Let a degenerate system $(DS)_0$, a right initial operator $F_1^{(r)} \in \mathcal{F}_V^{(r)}$, and arbitrary $T \in \mathcal{W}_{A-BW}$ be given. Suppose that $V \in L(X, X')$, $C \in L_0(U \to X, X' \to U')$ and $A, B, W \in L_0(X, X')$, then the system $(DS)_0$ is $F_1^{(r)}$-controllable if and only if*

$$\ker C^*T^*W^*(F_1^{(r)})^* = \{0\}. \tag{3.15}$$

Proof. Note that $F_1^{(r)}WTC$ maps U into $\ker V$. Therefore, fixing $x_0, x_1 \in \ker V$, the condition (3.15) is equivalent to

$$F_1^{(r)}WTCU = \ker V. \tag{3.16}$$

The assumption $CU \oplus \{Bx_0\} \subset (A - BW)(\mathrm{Im}V)$ and Proposition 1.1 imply

$$\begin{aligned}
F_1^{(r)}WTCU &= F_1^{(r)}WT(CU \oplus \{Bx_0\}) - \{F_1^{(r)}WTBx_0\} \\
&\subset F_1^{(r)}WT(A - BW)(\mathrm{Im}V) - \{F_1^{(r)}WTBx_0\} \\
&\subset F_1^{(r)}W[T(A - BW)(\mathrm{Im}V) \oplus \ker(A - BW)] \\
&\qquad -\{F_1^{(r)}WTBx_0\} - F_1^{(r)}W(\ker(A - BW)) \\
&\subset F_1^{(r)}W(\mathrm{Im}V) - \{F_1^{(r)}WTBx_0\} - F_1^{(r)}W(\ker(A - BW)) \\
&\subset F_1^{(r)}(WV(\mathrm{dom}V) \oplus \{x_0\}) - \{F_1^{(r)}WTBx_0\} \\
&\qquad -F_1^{(r)}W(\ker(A - BW)) - \{F_1^{(r)}x_0\} \\
&\subset F_1^{(r)}(WV(\mathrm{dom}V) \oplus \ker V) - \{F_1^{(r)}WTBx_0\} \\
&\qquad -F_1^{(r)}W(\ker(A - BW)) - \{F_1^{(r)}x_0\} \\
&\subset F_1^{(r)}(\mathrm{dom}V) - \{F_1^{(r)}WTBx_0\} \\
&\qquad -F_1^{(r)}W(\ker(A - BW)) - \{F_1^{(r)}x_0\} \subset \ker V.
\end{aligned}$$

By (3.16), we have

$$\begin{aligned}
F_1^{(r)}WTCU &= F_1^{(r)}(\mathrm{dom}V) - \{F_1^{(r)}WTBx_0\} \\
&\qquad - F_1^{(r)}W(\ker(A - BW)) - \{F_1^{(r)}x_0\} \\
&= \ker V.
\end{aligned}$$

Thus,

$$F_1^{(r)}WTCU + \{F_1^{(r)}WTBx_0\} + F_1^{(r)}W(\ker(A - BW)) + \{F_1^{(r)}x_0\} = F_1^{(r)}(\mathrm{dom}V) = \ker V.$$

It means that for every $x_1 \in \ker V$, there exist $v \in \mathrm{dom}V$, $u_0 \in U$ and $z_0 \in \ker(A - BW)$ such that

$$x_1 = F_1^{(r)}v = F_1^{(r)}WTCu_0 + F_1^{(r)}WTBx_0 + F_1^{(r)}Wz_0 + F_1^{(r)}x_0 = F_1^{(r)}\{W[T(Cu_0 + Bx_0) + z_0] + x_0\}.$$

Hence, x_1 is $F_1^{(r)}$-reachable from $x_0 \in \ker V$. The arbitrariness of $x_0, x_1 \in \ker V$ shows that $F_1^{(r)}(\mathrm{Rang}_{U,x_0}\Phi) = \ker V$, for every $x_0 \in \ker V$.

Conversely, suppose that $F_1^{(r)}(\mathrm{Rang}_{U,x_0}\Phi) = \ker V$. Choosing $x_0 = 0$, $z_0 = 0$, we obtain $F_1^{(r)}WTCU = \ker V$, thus $\ker C^*T^*W^*(F_1^{(r)})^* = \{0\}$. The proof is completed. ∎

THEOREM 3.3. *Suppose that the system $(DS)_0$ is $F_1^{(r)}$-controllable. Then this system is $F_2^{(r)}$-controllable for an arbitrary right initial operator $F_2^{(r)} \in \mathcal{F}_V^{(r)}$.*

Proof. Let $F_1^{(r)}$ be a right initial operator for V corresponding to $W_1 \in \mathcal{W}_V^1$, we get $F_1^{(r)}W_1 = 0$ on $\mathrm{dom}W_1$. Moreover, for every $x_1 \in \ker V$ and $w \in \mathrm{dom}W_1$, there exists $x_2 \in \ker V$ such that $x_1 = x_2 + F_2^{(r)}W_1w$. Since the system $(DS)_0$ is $F_1^{(r)}$-controllable, for every $x_0, x_2 \in \ker V$, there exists a control $u \in U$ and $z \in \ker(A - BW)$ such that

$$F_1^{(r)}\{W[T(Cu + Bx_0) + z] + x_0\} = x_2,$$

or equivalently

$$W[T(Cu + Bx_0) + z] + x_0 = x_2 + W_1w, \quad \text{where } w \in \mathrm{dom}W_1 \text{ is arbitrary.}$$

Hence,

$$F_2^{(r)}\{W[T(Cu + Bx_0) + z] + x_0\} = F_2^{(r)}(x_2 + W_1w) = x_2 + F_2^{(r)}W_1w = x_1.$$

Since the arbitrariness of $x_0, x_1 \in \ker V$, the system $(DS)_0$ is $F_2^{(r)}$-controllable. ∎

EXAMPLE 3.1. Suppose that X is the space (s) of all real sequences $\{x_n\}$, $n = 0, 1, 2, \ldots$ Let $V, A, B \in L_0(X)$ be defined by:

$$V\{x_n\} = \{v_n\},\ v_0 = v_1 = 0, v_n = x_n,\ n \geqslant 2;$$

$$A\{x_n\} = \{a_n\},\ a_n = x_n + x_{n+2},\ n = 0,1,2,3,\ a_n = x_{n+2},\ n \geqslant 4;$$
$$B\{x_n\} = \{x_{n+2}\},\ n = 0,1,2,...$$

It is completely checkable that $VX \neq X$, $\ker V = \{\{x_n\} : x_n \in \mathbb{R},\ x_n = 0,\ n \geqslant 2\} \neq \{0\}$ and $VVV\{x_n\} = V\{x_n\}$, thus $V \in W(X)$ and $W = V \in \mathcal{W}_V^1$. Therefore, the right initial operator $F^{(r)}$ for V corresponding to the almost inverse W is defined by

$$F^{(r)}\{x_n\} = (I - WV)\{x_n\} = \{f_n\},\ f_0, f_1 \in \mathbb{R},\ f_n = 0,\ n \geqslant 2.$$

The operator A differs from 0 and is non-invertible, since there exists $\{x_n\} \in X$ such that $A\{x_n\} \neq \{0,0,0,...\}$ and

$$\ker A = \{\{x_0, x_1, -x_0, -x_1, x_0, x_1, 0,0,0,...\} : x_0, x_1 \in \mathbb{R}\} \neq \{0\}.$$

Let

$$U = \{\{u_n\} : u_n \in \mathbb{R}, u_0 = u_1 = 0 \text{ and } u_n = 0, n \geqslant 4\}$$

and

$$C = \alpha I \in L_0(U, X)$$

(where $\alpha \in \mathbb{R}, \alpha \neq 0$ and I is an identity operator).

Now consider the system $(DS)_0$:

$$AVx = Bx + Cu, \quad u \in U, \tag{3.17}$$
$$F^{(r)}x = \bar{x}_0, \quad \bar{x}_0 = \{x_0^0, x_1^0, 0,0,0,...\} \in \ker V. \tag{3.18}$$

Since

$$(A - BW)I(A - BW)\{x_n\} = \{x_0, x_1, x_2, x_3, 0,0,0,...\} = (A - BW)\{x_n\},$$

the resolving operator $A - BW$ is generalized invertible with $T = I \in \mathcal{W}_{A-BW}$.

In addition, $\ker(A - BW) = \{\{0,0,0,0,x_4,x_5,x_6,...\} : x_n \in \mathbb{R},\ n = 4,5,6,...\}$ and $CU \oplus \{B\bar{x}_0\} \subset (A - BW)(\mathrm{Im}V)$.

By the formula (3.4), the solution of the problem $(DS)_0$ is given by

$$\begin{aligned} \Phi(\bar{x}_0, u) &= W[T(Cu + B\bar{x}_0) + z] + \bar{x}_0,\ z \in \ker(A - BW) \\ &= W[\alpha Iu + B\bar{x}_0 + z] + \bar{x}_0,\ z = \{0,0,0,0,z_4,z_5,z_6,...\} \\ &= \{x_0^0, x_1^0, \alpha u_2, \alpha u_3, z_4, z_5, z_6, ...\}. \end{aligned} \tag{3.19}$$

Write $W_1\{x_n\} = \{y_n\}$, $y_0 = x_3$, $y_1 = x_2$, $y_n = x_n$, $n \geqslant 2$, then $W_1 \in \mathcal{W}_V^1$. Indeed,

$$VW_1V\{x_n\} = V\{x_n\} \quad \text{and} \quad W_1VW_1\{x_n\} = W_1\{x_n\}.$$

The right initial operator $F_1^{(r)}$ for V corresponding to W_1 is defined by

$$F_1^{(r)}\{x_n\} = (I - W_1V)\{x_n\} = \{x_0 - x_3, x_1 - x_2, 0,0,0,...\}.$$

Clearly, for every initial state $\bar{x}_0 \in \ker V$, there exist

$$\bar{u}_0 = \{0, 0, x_1^0/\alpha, x_0^0/\alpha, 0, 0, 0, ...\} \in U$$

and

$$\bar{z}_0 = \{0, 0, 0, 0, z_4^0, z_5^0, z_6^0, ...\} \in \ker(A - BW)$$

such that

$$F_1^{(r)}\Phi(\bar{x}_0, \bar{u}_0) = F_1^{(r)}\{x_0^0, x_1^0, x_1^0, x_0^0, z_4^0, z_5^0, z_6^0, ...\} = \{0, 0, 0, ...\}.$$

This means that the system $(DS)_0$ is $F_1^{(r)}$-controllable to zero.

Moreover, for every $\bar{x}_1 = \{x_0^1, x_1^1, 0, 0, 0, ...\} \in \ker V$, there exists $\bar{x}_2 = \bar{x}_1$ and $\bar{z}_1 = \{0, 0, 0, 0, z_4^1, z_5^1, z_6^1, ...\} \in \ker(A - BW)$ such that

$$F_1^{(r)}[W(TB\bar{x}_2 + \bar{z}_1) + \bar{x}_2] = \bar{x}_1 .$$

By Theorem 3.1, the system $(DS)_0$ is $F_1^{(r)}$-controllable.

Acknowledgement. I would like to express my sincere thanks to Professors Nguyen Dinh Quyet and Nguyen Van Mau for invaluable suggestions and advice that led to improvement of this work. I am greatly indebted to the colleagues for assisting me in collecting proper readings related to the paper.

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Received December 19, 2003.