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ON THE EQUATION ASSOCIATED WITH BOUNDED PARACONCAVE ENTROPIES

Abstract. A paraconcave entropy function [2] is represented by a pair of two real functions of a real variable satisfying certain natural conditions. The subject of this paper is the functional equation, $L(\sum_j f(p_j)) = \sum_j g(p_j)$, that describes equivalence between two representations of a paraconcave entropy function with concave functions f and g satisfying the condition for a bounded entropy. With the use of E-transforms of the functions f and g we reduce the problem of solvability of the equation to the problem of injectivity of a certain nonlinear operator defined on the set of concave homeomorphisms of the interval $[0,1]$ onto itself. Additionally, we prove some facts about concavity of the E-transform $\hat{f}$.

1. Introduction

Let us denote by Λ_N the set of components of a a complete probability distribution:

$$\Lambda_N = \{p = (p_1, p_2, \ldots, p_N) \mid 0 \le p_j \le 1 \text{ and } \sum_j p_j = 1\}. \tag{1.1}$$

Next, denote by Λ the respective union:

$$\Lambda = \bigcup_{N=1}^{\infty} \Lambda_N. \tag{1.2}$$

Given $p = (p_1, p_2, \ldots, p_N) \in \Delta$, denote by $|p|$ the sum of its components, that is

$$|p| = |(p_j)| = \sum_j p_j. \tag{1.3}$$

Let Ω be a subset of the real space $C_{\mathbb{R}}[0,1]$ of continuous functions on the interval $[0,1]$, defined by

$$\Omega = \{f \in C_{\mathbb{R}}[0,1] \mid f(0) = f(1) = 0 \text{ and } f \text{ is concave on } [0,1]\}. \tag{1.4}$$

Next, let Γ denote the set of all homeomorphisms of the interval $[0,\infty)$ onto itself. It is clear that Γ is a group with respect to the composition of maps and that for every $L \in \Gamma$, we have $L(0) = 0$.

Given maps $f \in \Omega$ and $F \in \Gamma$, define a map $H : \Lambda \to [0,\infty)$ by

$$(1.5) \qquad H(p_1, p_2, \ldots, p_N) = F\Big[\sum_j f(p_j)\Big].$$

The map H given by (1.5) is called a complete paraconcave entropy function [2] with the representatives f and F. Given two representations, (f,F) and (g,G), of the paraconcave entropy H, we have the identity

$$(1.6) \qquad F\Big[\sum_j f(p_j)\Big] = G\Big[\sum_j g(p_j)\Big] \text{ for } p = (p_1, p_2, \ldots, p_N) \in \Lambda.$$

In all of that follows we are assuming that the entropy functions under consideration are nontrivial and that, consequently, their representations consist of nonzero functions. The terms entropy and entropy function will be interchanged.

The subject of this paper is the equation

$$(1.7) \qquad L\Big[\sum_j f(p_j)\Big] = \sum_j g(p_j), \quad \text{for } 0 \le p_j \le 1, \text{ with } \sum_j (p_j) = 1,$$

where $f, g \in \Omega$ and $L \in \Gamma$. The equation (1.7) is obtained from (1.6) by setting

$$(1.8) \qquad L = G^{-1} \circ F.$$

The equations (1.6) and (1.7) are referred to as sum form equations and have quite a large literature, cf. [3], [5] or [6]. We would like to emphasize here that the number N that indicates how many terms appear on both sides of the equation (1.7) is an arbitrary natural number. This assumption may seem too strong. In this paper, however, we consider a sort of boundary case, the case of bounded entropies, described below, where a certain limit process is performed. To do so we need the number of terms in (1.7) to be infinite. Thus, the assumption about the range of N remains valid through all of that follows.

Given $f \in \Omega$, and a positive integer n, denote by $\beta_n(f)$ and $\beta_\infty(f)$, the following two numbers:

$$(1.9) \qquad \beta_n(f) = n\, f\left(\frac{1}{n}\right), \quad \text{and}$$

$$(1.10) \qquad \beta_\infty(f) = \sup_n n\, f\left(\frac{1}{n}\right) = f'(0+) \text{ (possibly extended).}$$

The sequence $\{\beta_n(f)\}_{n=1}^{\infty}$ given by (1.9) is nondecreasing [2]. If its limit is ∞, that is, if $\beta_\infty(f) = \infty$, we will say that the entropy given by a representative (f,F) is unbounded; otherwise it is called bounded. The range of sums

$\sum_j f(p_j)$ in (1.5) equals the interval $[0, \infty)$, if the entropy is unbounded; and otherwise, if the entropy is bounded, then this range equals the closed interval $[0, \beta_\infty(f)]$ [4]. A particular case of bounded entropies is obtained by functions being linear around the origin.

DEFINITION 1.1 [4]. A real function f defined in a right neighborhood of $0 \in \mathbb{R}$ is called germinally linear if there is a unique constant $c(f) \in \mathbb{R}$ and a $\delta > 0$ such that the interval $[0, \delta]$ is included in the neighborhood and

$$f(x) = c(f)x \text{ for all } x \in [0, \delta]. \tag{1.11}$$

An immediate consequence of the above definition is that for every function $f \in \Omega$ which is germinally linear, the following identities hold:

$$nf\left(\frac{1}{n}\right) = c(f) \text{ for sufficiently large } n, \text{ and} \tag{1.12}$$

$$\beta_\infty(f) = c(f) < \infty. \tag{1.13}$$

The following lemma will be used frequently in the sequel.

LEMMA 1.1. *If φ is a concave function on an open interval (a, b) and if x, y, x', y' are points of (a, b) with $x \le x' < y'$ and $x < y \le y'$, then*

$$\frac{\varphi(y) - \varphi(x)}{y - x} \ge \frac{\varphi(y') - \varphi(x')}{y' - x'}. \tag{1.15}$$

Proof. It follows directly from Lemma 15 in [5] formulated there for convex functions. ■

The inequality (1.15) means that the chord over (x, y) has larger slope than the chord over (x', y'). If the inequality (1.15) is strict for all possible choices, the function φ is strictly concave.

Given functions f, $g \in \Omega$ and the sums $\sum_j f(p_j)$ and $\sum_j g(p_j)$ of their values at certain points $p_1, p_2, \dots, p_n$, we will adopt the following convention to denote these sums:

$$f_\Sigma(p) = \sum_j f(p_j), \text{ and} \tag{1.16}$$

$$g_\Sigma(p) = \sum_j g(p_j). \tag{1.17}$$

When $f \in \Omega$ is fixed and $p = (p_1, p_2, \dots, p_n) \in \Lambda$ is regarded as a variable, the formula (1.16) defines a nonnegative functional on a subset of Λ. The functional f_Σ will be called the summation functional associated with the function f. It is well defined on the entire set Λ, if only $\beta_\infty(f) < \infty$ [4].

2. Algebraization of the entropy equation

Let us consider the entropy equation (1.7), with unknown functions $f, g \in \Omega$, and a homeomorphism $L \in \Gamma$. Since the entropy is bounded, the numbers $\beta_\infty(f)$ and $\beta_\infty(g)$, given by (1.10), are finite. Since the ranges of the summation functionals f_Σ and g_Σ are then the finite intervals $[0, \beta_\infty(f)]$ and $[0, \beta_\infty(g)]$, respectively, we will require only that the homeomorhism L of (1.7) maps the interval $[0, \beta_\infty(f)]$ onto the interval $[0, \beta_\infty(g)]$, with $L(0) = 0$ and, consequently, $L(\beta_\infty(f)) = \beta_\infty(g)$. Such homeomorphisms will be called ordered homeomorphisms, and the set they form will be again denoted by Γ. Respectively, denote by Ω_{bnd}, the set of all $f \in \Omega$, satisfying $\beta_\infty(f) < \infty$.

Now, let $f \in \Omega_{bnd}$, t be any number in the interval $[0,1]$, and n be any positive integer. Then, by (1.7), we get the following identity:

$$L\left[nf\left(\frac{t}{n}\right) + f(1-t)\right] = ng\left(\frac{t}{n}\right) + g(1-t). \tag{2.1}$$

Since $\lim_{n\to\infty} nf(\frac{t}{n}) = t$ [4], if we let $n \to \infty$ in (2.1) then we obtain the identity:

$$L[t\beta_\infty(f) + f(1-t)] = t\beta_\infty(g) + g(1-t), \quad 0 \le t \le 1. \tag{2.2}$$

Let us rewrite the identity (2.1) to a form with two variables s and t:

$$\begin{aligned} L\left[nf\left(\frac{s}{2n}\right) + nf\left(\frac{t}{2n}\right) + f\left(\frac{1-s}{2}\right) + f\left(\frac{1-t}{2}\right)\right] \\ = ng\left(\frac{s}{2n}\right) + ng\left(\frac{t}{2n}\right) + g\left(\frac{1-s}{2}\right) + g\left(\frac{1-t}{2}\right), \end{aligned} \tag{2.3}$$

where $0 \le s, t \le 1$, $s + t = 1$, and n is a positive integer. If we let $n \to \infty$ in (2.3) then we get the following identity:

$$\begin{aligned} L\left[\frac{s+t}{2}\beta_\infty(f) + f\left(\frac{1-s}{2}\right) + f\left(\frac{1-t}{2}\right)\right] \\ = \frac{s+t}{2}\beta_\infty(g) + g\left(\frac{1-s}{2}\right) + g\left(\frac{1-t}{2}\right). \end{aligned} \tag{2.4}$$

Increasing the number of variables in (2.3) to an arbitrary $m \ge 1$ we get a general case

$$\begin{aligned} L\left[\frac{t_1 + t_2 + \ldots t_m}{m}\beta_\infty(f) + \sum_{j=1}^{m} f\left(\frac{1-t_j}{m}\right)\right] \\ = \frac{t_1 + t_2 + \ldots t_m}{m}\beta_\infty(g) + \sum_{j=1}^{m} g\left(\frac{1-t_j}{m}\right), \end{aligned} \tag{2.5}$$

where $0 \le t_1, t_2, \ldots, t_m \le 1$ and $t_1 + t_2 + \ldots t_m = 1$.

Obviously, if L were linear, or even additive, then following any of the identities (2.2), (2.3), or (2.3), we would have that $L(x) = cx$ with $c = \beta_\infty(g)/\beta_\infty(f)$.

DEFINITION 2.1. Given function $f \in \Omega_{bnd}$, the map $\hat{f}$ defined below,

$$\hat{f}(t) = t\beta_\infty(f) + f(1-t), \quad \text{for } 0 \le t \le 1, \tag{2.6}$$

will be called an *E-transform of the function* f, and the map that assigns to every function $f \in \Omega_{bnd}$ its E-transform $\hat{f}$ will be called *E-transformation*, or just *E-transform* as well.

COROLLARY 2.1. *If the maps* L, f, *and* g *are solutions of the equation* (1.7), *then the maps* $L, \hat{f}$, *and* $\hat{g}$ *satisfy the equation*

$$L \circ \hat{f} = \hat{g}. \ \blacksquare \tag{2.7}$$

E-transforms are characterized by the following theorem.

THEOREM 2.1. *Assume that* (i) $f \in \Omega_{bnd}$, *and* (ii) f *is not germinally linear. Then the map* $\hat{f}$ *is a concave homeomorhism of the intervals* $[0,1]$ *and* $[0, \beta_\infty(f)]$.

Proof. Of course $\hat{f}$ is continuous, $\hat{f}(0) = 0$, $\hat{f}(1) = \beta_\infty(f)$, so the interval $[0, \beta_\infty(f)]$ is included in the range of the map $\hat{f}$. Next, for $0 < t < 1$, we have the following

$$\begin{aligned} 0 < \hat{f}(t) = \beta_\infty(f)t + f(1-t) &= (1-t)\{\beta_\infty(f)\frac{t}{1-t} + \frac{f(1-t_2)}{1-t_2}\} \\ &< (1-t)\{\beta_\infty(f)\frac{t}{1-t} + \beta_\infty(f)\} = \beta_\infty(f). \end{aligned} \tag{2.8}$$

Therefore, the range of the map $\hat{f}$ equals the interval $[0, \beta_\infty(f)]$.

Now, we are going to show that the map $\hat{f}$ is a bijection. So, let $0 < t_1 < t_2 < 1$. If $\hat{f}(t_1) = \hat{f}(t_2)$, then

$$t_1\beta_\infty(f) + f(1-t_1) = t_2\beta_\infty(f) + f(1-t_2).$$

Therefore,

$$f(1-t_1) - f(1-t_2) = \beta_\infty(f)(t_2 - t_1),$$

and consequently,

$$\beta_\infty(f) = \frac{f(t_o + \Delta t) - f(t_o)}{\Delta t}, \tag{2.9}$$

where $0 < t_o = 1 - t_2 < 1$, and $0 < \Delta t = (1-t_1) - (1-t_2) = t_2 - t_1$.

Since f is concave, but not germinally linear, we have

$$\beta_\infty(f) = \frac{f(t_o + \Delta t) - f(t_o)}{\Delta t} < \frac{f(t_o) - f(0)}{t_o} < \beta_\infty(f). \tag{2.10}$$

The contradiction (2.11) proves that $\hat{f}$ must be a bijection. Concerning concavity of $\hat{f}$, for $0 < t_1 < t_2 < 1$ and $0 \le \lambda \le 1$, we get the following.

$$\begin{aligned}(2.11)\quad &\hat{f}(\lambda t_1 + (1-\lambda)t_2)\\ &= \beta_\infty(f)(\lambda t_1 + (1-\lambda)t_2) + f(1 - (\lambda t_1 + (1-\lambda)t_2))\\ &= \lambda\beta_\infty(f)t_1 + (1-\lambda)\beta_\infty(f)t_2 + f(\lambda(1-t_1) + (1-\lambda)(1-t_2))\\ &\ge \lambda[\beta_\infty(f)t_1 + f(1-t_1)] + (1-\lambda)[\beta_\infty(f)t_2 + f(1-t_2)]\\ &= \lambda\hat{f}(t_1) + (1-\lambda)\hat{f}(t_2).\end{aligned}$$

Thus the map $\hat{f}$ is a concave, continuous bijection of the interval $[0,1]$ onto the interval $[0, \beta_\infty(f)]$; obviously its inverse is also continuous. Hence, the map $\hat{f}$ is the required homeomorphism. ∎

COROLLARY 2.2. *For any f satisfying the assumptions of Theorem 2.1 we have*

$$(2.12)\qquad (\hat{f})'(1-) = \beta_\infty(f) - f'(0+) = 0. \;∎$$

COROLLARY 2.3. *The E-transformation $f \mapsto \hat{f}$ is a 1-1 correspondence between the sets:*

$$(2.13D)\qquad \Omega_{bnd}^{(ng)} = \{f \in \Omega_{bnd} : f \textit{ is not germinally linear}\}, \textit{ and}$$

$$(2.13R)\qquad \Gamma_{concave}^{(ng)} = \{\varphi \in \Gamma : \varphi \textit{ is concave and } \varphi'(1-) = 0\}. \;∎$$

COROLLARY 2.4. *The inverse map to the E-transformation, when defined on (2.13D), is the map $\Gamma_{concave}^{(ng)} \ni \varphi \mapsto \varphi^{\curlyvee} \in \Omega_{bnd}^{(ng)}$, given as follows:*

$$(2.14)\qquad \varphi^{\curlyvee}(t) = \varphi(1-t) + t\varphi(1) - \varphi(1), \qquad t \in [0,1]. \;∎$$

COROLLARY 2.5. *If the maps L, f, and g are solutions of the equation (1.7), and the functions f and g are not germinally linear, then*

$$(2.15)\qquad L = \hat{g} \circ (\hat{f})^{-1}. \;∎$$

The right hand side of the formula (2.15) is a composition of the function $\hat{g}$ and the inverse of the function $\hat{f}$.

Now, given functions f and g of the entropy equation (1.7) consider their symmetrizations:

$$(2.16a)\qquad f^{\sim}(p) = f(p) + f(1-p), \qquad \text{for } p \in [0,1];$$

$$(2.16b)\qquad g^{\sim}(p) = g(p) + g(1-p), \qquad \text{for } p \in [0,1].$$

In view of the equation (1.7), we get another (algebraic) equation for L:

$$(2.17)\qquad L \circ f^{\sim} = g^{\sim}.$$

In the next section we are going to use equations (2.15) and (2.17) in a more symmetric setting regarding the functions L, f, and g, and solve them for L.

3. Normalization of the entropy equation

Let us consider again the equation (1.7) with the unknown functions L, f, g. We assume that the functions f and g are not germinally linear (in fact, it suffices to assume that one of them is not germinally linear) and that the numbers $\beta_\infty(f)$ and $\beta_\infty(g)$ are finite. Consider the following functions:

$$(3.1.a) \qquad f^{\#}(p) = \frac{f(p)}{\beta_\infty(f)}, \qquad \text{for } p \in [0,1];$$

$$(3.1.b) \qquad g^{\#}(p) = \frac{g(p)}{\beta_\infty(g)}, \qquad \text{for } p \in [0,1];$$

$$(3.1.c) \qquad L^{\#}(u) = \frac{1}{\beta_\infty(g)} L(\beta_\infty(f)u), \qquad \text{for } u \in [0,1].$$

Of course, the reverse formulas hold:

$$(3.2.a) \qquad f(p) = \beta_\infty(f) f^{\#}(p), \qquad \text{for } p \in [0,1];$$

$$(3.2.b) \qquad g(p) = \beta_\infty(g) g^{\#}(p), \qquad \text{for } p \in [0,1];$$

$$(3.2.c) \qquad L(x) = \beta_\infty(g) L^{\#}\left(\frac{x}{\beta_\infty(f)}\right), \qquad \text{for } x \in [0, \beta_\infty(f)].$$

Then all three functions $L^{\#}$, $f^{\#}$, and $g^{\#}$ are defined on the interval $[0,1]$, the range of $L^{\#}$ is $[0,1]$, and the range of the summation functionals corresponding to the functions $f^{\#}$ and $g^{\#}$ is again the interval $[0,1]$ (for both). Therefore the equation (1.7) is equivalent to the following one:

$$(3.3) \quad L^{\#}\Big[\sum_j f^{\#}(p_j)\Big] = \sum_j g^{\#}(p_j) \quad \text{for } 0 \le p_j \le 1, \text{ with } \sum_j (p_j) = 1.$$

The equation (3.3) will be called a normalization of the equation (1.7), and the three functions $L^{\#}$, $f^{\#}$, and $g^{\#}$, the normalized versions of the functions L, f, and g, respectively. It is clear that the operations of symmetrization and normalization are commutative. From now on we will consider only the equation (3.3), where, for the sake of simplicity, we will replace the map $L^{\#}$ with L, and the functions $f^{\#}$ and $g^{\#}$ with f and g, like in the original equation (1.7). This time, however, L denotes an ordered homeomorphism of the interval $[0,1]$ onto itself (so that $L(0) = 0$ and $L(1) = 1$); all such homeomorphisms form a group, which will be again denoted by Γ. Now, both the numbers $\beta_\infty(f)$ and $\beta_\infty(g)$ are equal 1 and the ranges of the summation functionals f_Σ and g_Σ are also equal the interval $[0,1]$. That is, the equation (3.3) again becomes

$$L\Big[\sum_j f(p_j)\Big] = \sum_j g(p_j) \text{ for } 0 \le p_j \le 1, \text{ with } \sum_j (p_j) = 1.$$

with the functions L, f, and g satisfying the above conditions.

The symmetrization of the functions f and g leads to the functions $f^{\sim}$ and $g^{\sim}$, which are nondecreasing on the interval $[0, \frac{1}{2}]$ and nonincreasing on the interval $[\frac{1}{2}, 0]$. If the functions f and g are strictly concave, then the functions $f^{\sim}$ and $g^{\sim}$ are increasing on the interval $[0, \frac{1}{2}]$ and decreasing on the interval $[\frac{1}{2}, 0]$. Thus we get the following

PROPOSITION 3.1. *If the functions L, f, and g are solutions of the equation of* (3.4) *and the functions f, g are not germinally linear, then*

$$L = \hat{g} \circ (\hat{f})^{-1}, \text{ where } L, \hat{g}, \hat{f} \in \Gamma; \tag{3.5}$$

$$L(x) = g^{\sim} \circ (f^{\sim})^{-1}(x), \text{ for } x \in [0, 2f(\frac{1}{2})]. \tag{3.6}$$

The equation (3.5) is an (algebraic) equation in the group Γ of ordered homeomorhisms of the interval $[0, 1]$. The equation (3.6) gives L only on a subinterval via symmetrizations $f^{\sim}$ and $g^{\sim}$. ∎

In a normalized case and with the function f not being germinally linear, the E-transform of the function f is an ordered and concave homeomorphism of the interval $[0, 1]$ onto itself. For such maps the following proposition holds.

PROPOSITION 3.2. *Let φ be an ordered and concave homeomorphism of the interval $[0, 1]$ onto itself different from the identity map id. Then*

$$\lim_{n\to\infty} \varphi^n(x) = 1, \quad \text{for all } x \in (0, 1). \tag{3.7}$$

Proof. Let φ be an ordered homeomorphism of $[0, 1]$ onto itself that is concave and $x \in (0, 1)$. Since $\varphi(0) = 0$ and $\varphi(1) = 1$, we have $0 < \varphi(x) < 1$. Next, by concavity of φ, we have $\frac{\varphi(x)}{x} > \frac{\varphi(1)}{1} = 1$, so that $\varphi(x) > x$. Therefore, $1 \ge \varphi^n(x) = \varphi(\varphi^{n-1}(x)) > \varphi^{n-1}(x)$ for all $n > 1$. Thus the sequence $\{\varphi^n(x)\}$ converges to a certain $y \in (x, 1]$, for which we have $\varphi(y) = \varphi(\lim_{n\to\infty} \varphi^n(x)) = \lim_{n\to\infty} \varphi^{n+1}(x) = y$. It follows that y must be equal 1. Otherwise we would get a contradiction

$$1 = \frac{\varphi(y)}{y} > \frac{\varphi(1)}{1} = 1. \ ∎$$

Let us now rewrite formula (2.17) to the normalized case

$$L \circ f^{\sim} = g^{\sim}. \tag{3.7}$$

Combining formulas (3.5) and (3.7) we obtain

$$\hat{g} \circ (\hat{f})^{-1} \circ f^{\sim} = g^{\sim}. \tag{3.8}$$

Separating terms that depend only on one of the functions, f or g, we arrive at

$$(\hat{f})^{-1} \circ f^{\sim} = (\hat{g})^{-1} \circ g^{\sim}. \tag{3.9}$$

Now, we are in a position to obtain an operator equation involving the E-transforms $\hat{f}$ and $\hat{g}$ only.

PROPOSITION 3.3. *If the functions L, f, and g are solutions of the equation* (3.4) *and the functions f, g are not germinally linear, then*

$$(\hat{f})^{-1}(\hat{f}(t) + \hat{f}(1-t) - 1) = (\hat{g})^{-1}(\hat{g}(t) + \hat{g}(1-t) - 1), \quad t \in [0,1]. \tag{3.10}$$

Proof. Following definition (2.6) of the E-transform and its inverse (2.14), adapted to the normalized case, we obtain

$$f(1-t) = \hat{f}(t) - t, \qquad t \in [0,1], \tag{3.11}$$

$$f(t) = \hat{f}(1-t) + t - 1, \qquad t \in [0,1], \quad \text{and} \tag{3.12}$$

$$f^{\sim}(t) = \hat{f}(t) + \hat{f}(1-t) - 1, \qquad t \in [0,1]. \tag{3.13}$$

Therefore, the equation (3.9) becomes (3.10). ∎

Now, denote by $\Gamma_{concave}$ the set of all concave ordered homeomorphisms of the interval $[0,1]$ onto itself, and define an operator $\Psi : \Gamma_{concave} \to C_R[0,1]$, by setting

$$\Psi(F)(t) = F^{-1}(F(t)+F(1-t)-1), \quad \text{for } t \in [0,1], \text{and } F \in \Gamma_{concave}. \tag{3.14}$$

If the operator Ψ of (3.14) is 1-1, then the map L of (3.5), and consequently of (1.7), must be linear. It is clear that if F is linear then its image $\Psi(F)$ is 0. The map Ψ is nonlinear. The function $\Psi(F)$ may be viewed as a sort of measure of nonlinearity (concavity) of F.

4. The case of germinally linear entropies

Let us now consider the case when the function f in the equation (1.7) is germinally linear. The function g must be germinally linear either as is evident from the following proposition.

PROPOSITION 4.1. *Let the functions L, f, and g satisfy the equation* (1.7) *with the function f being germinally linear. Then the function g is germinally linear as well.*

Proof. Assume that there is a $0 < \delta < 1$ and $c > 0$ such that for each $0 \le x \le \delta$, $f(x) = cx$. Then for every selection of $0 \le p_1, p_2, \ldots, p_n \le \delta$ with

$\sum_j p_j = 1$, we have $\sum_j g(p_j) = L[c] = const$. Thus, the sequence $\{ng(\frac{1}{n})\}$ is also constant for $n \geq 1/\delta$, and consequently $g(x) = L(c)x$, for $0 \leq x \leq \delta$. ∎

PROPOSITION 4.1. *Let the function $f \in \Omega$ be germinally linear. Then its E-transform $f\hat{f}$ is constant for t close to* 1.

Proof. Let $0 < d < 1$ denote the length of a maximal interval on which $f(x)$ is linear,

$$d = \max\{0 < \delta \mid f(x) = \beta_\infty(f)x \text{ for all } 0 \leq x \leq \delta < 1\}. \tag{4.1}$$

Then for all $t \in (1 - d, 1]$ we have

$$\hat{f}(t) = t\beta_\infty(f) + f(1-t) = \beta_\infty(f)(t + (1-t)) = \beta_\infty(f). \tag{4.2}$$ ∎

Therefore, in this case, the E-transform $\hat{f}$ is not a homeomorphism of the whole interval $[0, 1]$ onto the interval $[0, \beta_\infty(f)]$. However, the map $\hat{f}$, when restricted to the interval $[0, 1-d]$, with d as in (4.1), and still denoted by $\hat{f}$ is a homeorphism (of the intervals $[0, 1-d]$ and $[0, \beta_\infty(f)]$). With this reservation in mind we can write a formula corresponding to (2.7)

$$L = \hat{g} \circ (\hat{f})^{-1}. \tag{4.3}$$

The normalization and symmetrization for germinally linear functions work the same as for non germinally linear, so the other formulas for solutions of the equation (1.7) hold as well. In particular, we have the analogs of formulas (3.7)-(3.10), which express the map L in terms of the functions f and g.

5. Conclusions

The method used in [4] to prove that the map L in the entropy equation (1.7) is linear (the case of complete unbounded entropies) relied strongly on the fact that the range of the summation functional f_Σ is equal to the infinite interval $[0, \infty)$, which followed from the fact that $\lim_{n\to\infty} nf(\frac{1}{n}) = \infty$. This is not true for the bounded entropies and that is why we did algebraization of the equation and ended with the operator (3.14) defined on the set of ordered and concave homeomorphisms of the interval [0,1] onto itself. This operator is highly nonlinear. However, if it is injective, then the map L of (1.7) must be linear. We leave the issue of injectivity of the operator (3.14) to another paper.

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