DEMONSTRATIO MATHEMATICA
Vol. XXXVIII No 2 2005

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A PROBLEM FOR THE HYPERBOLIC SYSTEM OF DIFFERENTIAL EQUATIONS WITHOUT INITIAL CONDITIONS

Abstract. In the present paper we consider a problem for the system of hyperbolic equations of the first order with periodic conditions and without initial conditions. Some conditions for uniqueness and existence of solution for this problem are given.

Mixed problems for the system of hyperbolic equations of the first order have been considered in many papers. We can mention some papers which refer to the linear and semilinear systems [2–3], [9–17]. In particular in the papers [4–6], [8], some problems without initial conditions under assumption that solutions are bounded have been considered. In this paper we consider uniqueness and existence of solution of the problem without initial conditions for the system of hyperbolic equations without conditions on the solution when $t \to -\infty$.

Let $\Omega = \{x \in R^n;\ 0 < x_i < 2\pi;\ i = 1, \ldots, n\}$, $Q_{t_1,t_2} = \Omega \times (t_1, t_2)$ where $t_1 < t_2$, $t_1, t_2 \in (-\infty, T)$, $T < +\infty$ and $Q_T = Q_{-\infty,T}$.

We shall consider the system of hyperbolic equations of the form

$$A(u) \equiv u_t(x,t) + \sum_{k=1}^{n} A_k(x,t) u_{x_k}(x,t) + C(x,t)u(x,t) + G(t,u) = F(x,t), \tag{1}$$

in the domain Q_T.

For this system we put the following boundary conditions

$$u(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n, t) = u(x_1, \ldots x_{i-1}, 2\pi, x_{i+1}, \ldots x_n, t) \tag{2}$$

for $i = 1, \ldots, n$, where A_k, C are matrices of the order m, $k = 1, \ldots n$, and

$$G = (g_1, \ldots, g_m)^T,\ u = (u_1, \ldots, u_m)^T,$$
$$F = (f_1, \ldots, f_m)^T,\ \ x = (x_1, \ldots, x_n).$$

Now, we fix the notation. Put $L^r_s(D) = \prod_{i=1}^{s} L^r_i(D)$, where D is a domain in $\mathbb{R}^n$. We have

$$L^r_{s,loc}(\overline{Q_T}) = \{v \in L^r_s(Q_{\tau,T}) \forall_{\tau\in(-\infty,T)}\}$$

and $r \in [1, +\infty]$. By $(\cdot,\cdot)$ we denote the scalar product in R^n.

For equation (1) we consider the following conditions:

(A) $A_k(x,t) = A_k^T(x,t)$, $\forall_{(x,t)\in Q_T}$,

$A_k(x_1,\dots x_{i-1},0,x_{i+1},\dots,x_n,t) = A_k(x_1,\dots x_{i-1},2\pi,x_{i+1},\dots,x_n,t)$

for $k = 1,\dots,n$, $i = 1,\dots,n$,

$$(A_{kx_k}(x,t)\xi,\xi) \le a_k|\xi|^2, \quad \text{and} \quad \sum_{i=1}^{m} a_k := a_0,$$

for almost all $(x,t) \in Q_T$ and for every $\xi \in R^m$

$A_k, A_{kx_i} \in L^\infty_{m^2}(Q_T)$, for $k = 1,\dots,n$, $i = 1,\dots,m$;

(C) $C, C_{x_i} \in L^\infty_{m^2}(Q_T)$, for $i = 1,\dots,n$,

C satisfies (2), $(C(x,t),\xi,\xi) \ge c_0|\xi|^2$

where $c_0 = const$, for every $\xi \in R^n$ and almost all $(x,t) \in Q_T$;

(G) the functions $g_j(t,\xi)$ are continous in R^m for almost all $t \in (-\infty,T)$ and measurable with respect to t for every $\xi \in R^m$, $j = 1,\dots,m$ and satisfies the following assumptions:

(i) $(G(x,\xi) - G(x,\mu), \xi - \mu) \ge g_0|\xi - \mu|^p$, *where* $g_0 = const$, $g_0 > 0$,

(ii) $|g_i(t,\xi)| \le g^0 \sum_{j=1}^{n} |\xi_j|^{p-1}$, $g^0 = const$, $i = 1,\dots m$ *for* $p > 2$,

(iii) $(J(t,\nu),\xi,\xi) \ge 0$, $\forall \xi,\nu \in R^m$, where

$$J(t,\nu) = \begin{pmatrix} \frac{\partial g_1(t,\nu)}{\partial u_1} & \dots & \frac{\partial g_1(t,\nu)}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m(t,\nu)}{\partial u_1} & \dots & \frac{\partial g_m(t,\nu)}{\partial u_m} \end{pmatrix}.$$

First, we consider the system (1) in the domain $Q_{0,T}$ with the boundary conditions (2) and with the following initial conditions

$$u(x,0) = \Theta(x), \tag{3}$$

where $\Theta = (\Theta_1,\dots,\Theta_m)^T$.

DEFINITION 1. A function u is called a solution of problem (1)-(3) if

$$u \in L^p_m(Q_{0,T});\ u_{x_i} \in L^2_m(Q_{0,T});\ \ i = 1,\dots,n,$$
$$u_t \in L^2_m(Q_{0,T}) + L^q_m(Q_{0,T})$$

and u satisfies (1)–(3) for almost all $(x,t) \in Q_{0,T}$, where $\frac{1}{p} + \frac{1}{q} = 1$.

THEOREM 1. *If the conditions* (**A**), (**C**), (**G**) *hold,* $F, F_{x_i} \in L^2_m(Q_{0,T})$, $\Theta \in H^1_m(\Omega)$, $i = 1, \dots, n$ *and* F, Θ *satysfying* (2), *then there exists a solution* $u = (u_1, \dots, u_m)$ *of the problem* (1)–(3).

Proof. Denote by $H^1_{per}(\Omega)$ the space of the functions which belong to $H^1(\Omega)$ and satysfying (2). Analogously to [7] we can consider the special basis $\{\phi_k\}$ in $H^1_{per}(\Omega)$ which is composed with eigenfunctions of the following problem

$$\Delta u = \lambda u, \tag{4}$$

$$\frac{\partial^j u(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)}{\partial x_i^j} = \frac{\partial^j u(x_1, \dots, x_{i-1}, 2\pi, x_{i+1}, \dots, x_n)}{\partial x_i^j} \tag{5}$$

$j = 0, 1$; $i = 1, \dots, n$.

Let us consider a sequence of functions of the form

$$u^N(x,t) = \sum_{k=1}^{N} C_k^N(t)\phi_k(x), \ N = 1, 2, \dots, \tag{6}$$

where $\{C_1^N, \dots, C_N^N\}$ are solutions of the following Cauchy problems:

$$\begin{aligned} \int_\Omega \Big[&(u_t^N(x,t), \phi_k(x)) + \sum_{i=1}^{n} (A_i(x,t) u_{x_i}^N(x,t), \phi_k(x)) \\ &+ (C(x,t)u^N(x,t), \phi_k(x)) + (G(t, u^N), \phi_k(x)) \\ &- (F(x,t), \phi_k(x)) \Big] dx = 0, \ k = 1, \dots, N, \end{aligned} \tag{7}$$

$$C_k^N(0) = \Theta_k^N, \tag{8}$$

$$\Theta^N(x) = \sum_{k=1}^{N} \Theta_k^N(x)\phi_k(x), \tag{9}$$

where $\Theta^N(x) \to \Theta(x)$ in $H^1_{per}(\Omega)$.

From Carathéodory's Theorem we can observe that this problem has a solution. Multiplying (5) by the functions C_k^N respectively, then summing them up by k from 1 to N and integrating with respect to $t \in [0, \tau]$, $\tau < T$ we obtain

$$\begin{aligned} \int_{Q_{0,\tau}} \Big[&(u_t^N(x,t), u^N(x,t)) + \sum_{k=1}^{n} (A_k(x,t) u_{x_k}^N(x,t), u^N(x,t)) \\ &+ (C(x,t)u^N(x,t), u^N(x,t)) + (G(t, u^N), u^N(x,t)) \\ &- (F(x,t), u^N(x,t)) \Big] dxdt = 0. \end{aligned} \tag{10}$$

If we consider the respective components of the last equality we have

$$I_1 = \int\limits_{Q_{0,\tau}} (u_t^N(x,t), u^N(x,t))dxdt = \frac{1}{2}\int\limits_{\Omega_\tau} |u^N(x,\tau)|^2 dx - \frac{1}{2}\int\limits_{\Omega_0} |\Theta^N(x)|^2 dx$$

and

$$\begin{aligned} I_2 &= \int\limits_{Q_{0,\tau}} \sum_{k=1}^{n} (A_k(x,t)u_{x_k}^N(x,t), u^N(x,t))dxdt \\ &= \frac{1}{2}\int\limits_{Q_{0,\tau}} \sum_{k=1}^{n} (A_k(x,t)u^N(x,t), u^N(x,t))_{x_k} dxdt \\ &\quad - \frac{1}{2}\int\limits_{Q_{0,\tau}} \sum_{k=1}^{n} (A_{kx_k}(x,t)u^N(x,t), u^N(x,t))dxdt. \end{aligned}$$

Let

$$I_2^1 = \frac{1}{2}\int\limits_{Q_{0,\tau}} \sum_{k=1}^{n} (A_k(x,t)u^N(x,t), u^N(x,t))_{x_k} dxdt,$$

$$I_2^2 = -\frac{1}{2}\int\limits_{Q_{0,\tau}} \sum_{k=1}^{n} (A_{kx_k}(x,t)u^N(x,t), u^N(x,t))dxdt.$$

Using the assumption of Theorem 1 and (**A**) we obtain

$$\begin{aligned} I_2^1 &= \frac{1}{2}\int_0^\tau \sum_{k=1}^{n} (A_k(x,t)u^N(x,t), u^N(x,t))|_{x_k=2\pi} dt \\ &\quad - \frac{1}{2}\int_0^\tau \sum_{k=1}^{n} (A_k(x,t)u^N(x,t), u^N(x,t))|_{x_k=0} dt = 0. \end{aligned}$$

Moreover $I_2^2 \geq \frac{-a_0}{2} \int\limits_{Q_{0,\tau}} |u^N|^2 dxdt$.

From (**C**), (**G**) and the assumption of Theorem 1 we have

$$I_3 = \int\limits_{Q_{0,\tau}} (C(x,t)u^N(x,t), u^N(x,t))dxdt \geq c_0 \int\limits_{Q_{0,\tau}} |u^N(x,t)|^2 dxdt,$$

$$I_4 = \int\limits_{Q_{0,\tau}} (G(t,u^N)u^N(x,t), u^N(x,t))dxdt \geq \int\limits_{Q_{0,\tau}} g_0|u^N|^p dxdt$$

and

$$\begin{aligned} I_5 &= \int\limits_{Q_{0,\tau}} (F(x,t), u^N(x,t))dxdt \\ &\leq \frac{1}{2\delta_0}\int\limits_{Q_{0,\tau}} |F(x,t)|^2 dxdt + \frac{\delta_0}{2}\int\limits_{Q_{0,\tau}} |u^N(x,t)|^2 dxdt, \text{ for } \delta_0 > 0. \end{aligned}$$

Thus (10) will be of the form

$$(11)\qquad \int_{\Omega_\tau} |u^N(x,\tau)|^2dx + 2\int_{Q_{0,\tau}} g_0|u^N(x,t)|^p dxdt$$
$$\leq \int_{Q_{0,\tau}} (a_0-2c_0+\delta_0)|u^N(x,t)|^2dxdt + \frac{1}{\delta_0}\int_{Q_{0,\tau}} |F(x,t)|^2dxdt + \int_{\Omega_0} |\Theta^N(x)|^2dx.$$

Hence from Gronwall-Bellman Lemma we have

$$(12)\qquad \int_{\Omega_\tau} |u^N(x,t)|^2dx \leq \mu_0\Big(\int_{\Omega_0} |\Theta|^2dx + \int_{Q_{0,\tau}} |F(x,t)|^2dxdt\Big) = \mu_0F_0$$

and

$$(13)\qquad \int_{Q_{0,\tau}} |u^N|^p dxdt \leq \mu_1F_0.$$

Using in (7) the fact, that

$$\Delta\phi_k(x) = \lambda_k\phi_k(x),\ k = 1,2,\ldots$$

and multiplying (7) by the functions $-\lambda_k C_k^N$ respectively, then summing them up by i from 1 to N and integrating with respect to t from 0 to τ we obtain

$$(14)\qquad \sum_{s=1}^{n}\int_{Q_{0,\tau}} \Big[(u_t^N(x,t), u_{x_sx_s}^N(x,t)) + \sum_{i=1}^{m}(A_i(x,t)u_{x_i}^N(x,t), u_{x_sx_s}^N(x,t))$$
$$+ (C(x,t)u^N(x,t), u_{x_sx_s}^N(x,t))$$
$$+ (G(t,u^N), u_{x_sx_s}^N(x,t)) - (F(x,t), u_{x_sx_s}^N(x,t))\Big]dxdt = 0.$$

If we consider the respective components of the equality (14) we will have the following estimates

$$I_6 = -\int_{Q_{0,\tau}} (u_t^N(x,t), u_{x_sx_s}^N(x,t))dxdt$$
$$= -\int_{Q_{0,\tau}} (u_t^N(x,t), u_{x_s}^N(x,t))_{x_s}dxdt + \int_{Q_{0,\tau}} (u_{tx_s}^N(x,t), u_{x_s}^N(x,t))dxdt$$
$$= \frac{1}{2}\int_{Q_{0,\tau}} (u_{x_s}^N(x,t))_t^2dxdt = \frac{1}{2}\int_{\Omega_\tau} (u_{x_s}^N(x,\tau))^2dx - \frac{1}{2}\int_{\Omega_0} (\Theta_{x_s}^N(x))^2dx$$

and

$$I_7 = \int_{Q_{0,\tau}} (A_k(x,t)u_{x_k}^N(x,t), u_{x_sx_s}^N(x,t))dxdt$$
$$= \int_{Q_{0,\tau}} (A_k(x,t)u_{x_k}^N(x,t), u_{x_s}^N(x,t))_{x_s}dxdt$$

$$
\begin{aligned}
&- \int\limits_{Q_{0,\tau}} (A_{kx_s}(x,t)u^N_{x_k}(x,t), u^N_{x_s}(x,t))dxdt \\
&- \int\limits_{Q_{0,\tau}} (A_k(x,t)u^N_{x_kx_s}(x,t), u^N_{x_s}(x,t))dxdt \\
&= I_7^1 - I_7^2 - I_7^3.
\end{aligned}
$$

From (**A**) and (5) we get

$$
\begin{aligned}
I_7^1 &= \int\limits_0^\tau\int\limits_\Omega (A_k(x,t)u^N_{x_k}(x,t), u^N_{x_s}(x,t))_{x_s}dxdt = 0, \\
I_7^2 &= \int\limits_0^\tau\int\limits_\Omega (A_{kx_s}(x,t)u^N_{x_k}(x,t), u^N_{x_s}(x,t))dxdt \\
&\le \int\limits_0^\tau\int\limits_\Omega \|A_{kx_s}(x,t)\||u^N_{x_k}(x,t)||u^N_{x_s}(x,t)|dxdt \\
&\le \frac{1}{2}\sup_{Q_T}\|A_{kx_s}(x,t)\| \int\limits_0^\tau\int\limits_\Omega (|u^N_{x_k}(x,t)|^2 + |u^N_{x_s}(x,t)|^2)dxdt
\end{aligned}
$$

and

$$
\begin{aligned}
I_7^3 &= \int\limits_\Omega\int\limits_0^\tau (A_k(x,t)u^N_{x_kx_s}(x,t), u^N_{x_s}(x,t))dxdt \\
&= \frac{1}{2}\int\limits_0^\tau\int\limits_\Omega (A_k(x,t)u^N_{x_s}(x,t), u^N_{x_s}(x,t))_{x_k}dxdt - \frac{1}{2}\int\limits_0^\tau\int\limits_\Omega (A_{kx_k}u^N_{x_s}, u^N_{x_s})dxdt.
\end{aligned}
$$

Hence $I_7^3 \le \int\limits_{Q_{0,\tau}} a_k|u^N_{x_s}(x,t)|^2dxdt$. Thus

$$
\begin{aligned}
I_7 \le \left(a_0 + \frac{1}{2}\sup_{Q_T}\|A_{kx_s}(x,t)\|\right) &\int\limits_{Q_{0,\tau}} \sum_{i=1}^m |u^N_{ix_s}(x,t)|^2dxdt \\
+ \frac{1}{2}\sup_{Q_T}\|A_{kx_s}(x,t)\| &\int\limits_{Q_{0,\tau}} \sum_{i=1}^m |u^N_{ix_k}(x,t)|^2dxdt.
\end{aligned}
$$

From (**G**) we get

$$
\begin{aligned}
I_8 &= -\int\limits_{Q_{0,\tau}} (G(t,u^N), u^N_{x_sx_s}(x,t))dxdt \\
&= -\int\limits_{Q_{0,\tau}} (G(t,u^N), u^N_{x_s}(x,t))_{x_s}dxdt + \int\limits_{Q_{0,\tau}} (G_{x_s}(t,u^N), u^N_{x_s}(x,t))dxdt \\
&= \int\limits_{Q_{0,\tau}} (J(t,u^N)u^N_{x_s}(x,t), u^N_{x_s}(x,t))dxdt
\end{aligned}
$$

and hence $I_8 \geq 0$. From **(C)** we have

$$
\begin{aligned}
I_9 = & \int_{Q_{0,\tau}} (C(x,t)u^N(x,t), u^N_{x_s x_s}(x,t))dxdt \\
= & \int_{Q_{0,\tau}} (C(x,t)u^N(x,t), u^N_{x_s}(x,t))_{x_s} dxdt \\
& - \int_{Q_{0,\tau}} (C_{x_s}(x,t)u^N(x,t), u^N_{x_s}(x,t))dxdt \\
\leq & \int_{Q_{0,\tau}} \|C_{x_s}(x,t)\| |u^N(x,t)| |u^N_{x_s}(x,t)| dxdt \\
\leq & \frac{c_2}{2\delta_1} \int_{Q_{0,\tau}} |u^N(x,t)|^2 dxdt + \frac{\delta_1 c_2}{2} \int_{Q_{0,\tau}} |u^N_{x_s}(x,t)|^2 dxdt, \ \delta_1 > 0,
\end{aligned}
$$

where c_2 depends on the matrix C_{x_s}, $s = 1, 2, \ldots, n$.

$$
\begin{aligned}
I_{10} = & \int_{Q_{0,\tau}} (F(x,t), u^N_{x_s x_s}(x,t))dxdt \\
= & \int_{Q_{0,\tau}} (F(x,t), u^N_{x_s}(x,t))_{x_s} dxdt - \int_{Q_{0,\tau}} (F_{x_s}(x,t), u^N_{x_s}(x,t))dxdt \\
= & I^1_{10} - I^2_{10}.
\end{aligned}
$$

From the assumption $I^1_{10} = 0$ we thus obtain

$$
I_{10} = -I^2_{10} \leq \frac{1}{2\delta_3} \int_{Q_{0,\tau}} |F_{x_s}(x,t)|^2 dxdt + \frac{\delta_3}{2} \int_{Q_{0,\tau}} |u^N_{x_s}(x,t)|^2 dxdt.
$$

Using (14) we obtain the following inequality

$$
\begin{aligned}
(15) \qquad & \int_{\Omega_\tau} \sum_{s=1}^{n} |u^N_{x_s}(x,t)|^2 dx - l_1 \int_{Q_{0,\tau}} \sum_{s=1}^{n} |u^N_{x_s}(x,t)|^2 dxdt \\
\leq & \frac{1}{\delta_3} \int_{Q_{0,\tau}} \sum_{s=1}^{n} |f_{x_s}(x,t)|^2 dxdt + \int_{\Omega_0} \sum_{s=1}^{n} |\Theta^N_{x_s}(x)|^2 dx \\
& + \frac{c_2}{\delta_1} \int_{Q_{0,\tau}} \sum_{s=1}^{n} |u^N(x,t)|^2 dxdt - \sup_{Q_{0,T}} \|A_{kx_s}(x,t)\| \int_{Q_{0,\tau}} \sum_{s=1}^{n} |u^N_{x_k}(x,t)|^2 dxdt,
\end{aligned}
$$

where $l_1 = 2a_0 - \delta_1 c_2 + \frac{\delta_3}{2}$. Therefore

$$\int_{\Omega_\tau} \sum_{s=1}^{n} |u_{x_s}^N(x,t)|^2 dx - l_1 \int_{Q_{0,\tau}} \sum_{s=1}^{n} |u_{x_s}^N(x,t)|^2 dxdt \tag{16}$$

$$\leq \mu_0 F_0 + \|F_{x_s}\|^2_{L^2_m(Q_{0,T})} + \|\Theta^N_{x_s}\|^2_{H^1_m(\Omega)} + \frac{c_2}{\delta_1} \int_{Q_{0,\tau}} \sum_{s=1}^{n} |u_i^N(x,t)|^2 dxdt$$

$$- \sup_{Q_{0,T}} \|A_{kx_s}(x,t)\| \int_{Q_{0,\tau}} \sum_{s=1}^{n} |u_{x_k}^N(x,t)|^2 dxdt,$$

and thus

$$\|u_{x_s}^N(x,t)\|_{L^2_m(Q_{0,T})} + \|u^N(x,t)\|_{L^2_m(Q_{0,T})} \tag{17}$$

$$\leq \|F\|_{L^2_m(Q_{0,T})} + \|\Theta\|_{H^1_{m,per}(\Omega)} + \|F_{x_s}\|_{L^2_m(Q_{0,T})} + \|\Theta_{x_s}\|_{H^1_{m,per}(\Omega)}.$$

Hence, there exist subsequences $\{u_i^l\}$, $\{u_{ix_s}^l\}$ of sequences $\{u_i^N(x,t)\}$, $\{u_{ix_s}^N(x,t)\}$ such that

$$u_i^l \to u_i \quad \text{weakly in} \quad L^2(Q_{0,T}),$$
$$u_{ix_s}^l \to u_{ix_s} \quad \text{weakly in} \quad L^2(Q_{0,T}),$$
$$u_i^l \to u_i \quad \text{weakly in} \quad L^p(Q_{0,T}).$$

Since $|g_j(t,u_1^N,\dots,u_n^N)| \leq c\sum_{i=1}^{n} |u_i^N|^{p-1}$, $j = 1,2\dots,m$ then from (13)

$$\int |u_i|^p dx \leq \mu_1 F_0, \tag{18}$$

and Hölder-Young inequality we obtain

$$I_{11} = \int_{Q_{0,\tau}} (|u_i(x,t)|^{p-1}) dxdt \leq \Big(\int_{Q_{0,\tau}} (|u_i(x,t)|^{p-1})^q dxdt\Big)^{1/q} \Big(\int_{Q_{0,\tau}} dxdt\Big)^{1/p}$$

Hence $I_{11} \leq \frac{1}{q}\int_{Q_{0,\tau}} |u_i(x,t)|^p dxdt + \frac{1}{p}\int_{Q_{0,\tau}} dxdt$, and $\int_{Q_{0,\tau}} |G(t,u^k)| dxdt \leq \mu_2$, where μ_2 is a constant independent on u_i. So $G(\cdot,u_1^k,\dots,u_n^k) \to \omega$ weakly in $L^q_m(Q_T)$.

From (**G**), analogously to [7], it is easy to prove that $\omega = G(t,u)$.

Using (7) and the Galerkin's system it is easy to prove, that for all $v \in C^\infty_{0,m}(Q_{0,\tau})$ the following equality is satisfied

$$- \int_{Q_{0,T}} (u(x,t), v_t(x,t)) + \int_{Q_{0,T}} \Big[\sum_{k=1}^{n} (A_k(x,t) u_{x_k}^N(x,t), v(x,t))$$

$$+ (C(x,t)u^N(x,t), v(x,t)) + (G(t,u^N), v(x,t)) - (F(x,t), v(x,t))\Big] dxdt = 0.$$

Hence $u_t = Z$, where

$$Z(x,t) = -\sum_{i=1}^{n} A_i(x,t)u_{x_i} - C(x,t)u - G(t,u) + F(x,t).$$

Thus $u_t \in L^2_m(Q_{0,T}) + L^q_m(Q_{0,T})$, which completes the proof of Theorem 1.

Now, we shall consider the problem (1), (2) in the domain Q_T without initial conditions.

DEFINITION 2. A function u is called a weak solution of problem (1),(3) if

$$u \in L^p_{loc}(Q_T) \cap C((-\infty,T];L^2(\Omega)),$$

and u satisfies the following integral equality

$$\begin{aligned}(19)\quad &\int_\Omega \Big[(u(x,t_2),v(x,t_2)) - (u(x,t_1),v(x,t_1))\Big]dx \\ &+ \int_{Q_{t_1,t_2}} \Big[-(u(x,t),v_t(x,t)) - \sum_{k=1}^{n}(A_k(x)u(x,t),v_{x_k}(x,t)) \\ &- \sum_{k=1}^{n}(u(x,t),A_{x_k}(x)v(x,t)) + (C(x,t)u(x,t),v(x,t)) \\ &+ (G(t,u),v(x,t)) - (F(x,t),v(x,t))\Big]dxdt = 0\end{aligned}$$

for all $t_1,t_2 \in (-\infty,T]$, $t_1 < t_2$, where v satisfies (3) and $v \in C^1(\overline{Q}_T)$.

Let $(C(x,t) - \sum_{j=1}^{n} A_{j,x_i}(x,t)\xi,\xi) \geq h_0|\xi|^2$ for almost all $(x,t) \in Q_T$ and all $\xi \in R^m$.

THEOREM 2. *If the conditions* (**A**), (**C**), (**G**) *hold,* $h_0 > 0$, $F, F_{x_i} \in L^2_m(\overline{Q}_T)$, $i = 1,\dots n$ *and* F *satisfies* (2), *then the problem* (1), (2) *has at most one weak solution.*

Proof. Let $Q_{-l,T} = \Omega \times (-l,T)$ for all $l \in N$ and

$$F^{(l)}(x,t) = \begin{cases} F(x,t) & \text{for } (x,t) \in Q_{-l,T}, \\ 0, & \text{for } (x,t) \in Q_{-\infty,-l}. \end{cases}$$

From Theorem 1 we see that for every positive integer l the problem (1),(3) has a solution $u^l(x,t)$ of the class $L^p_m(Q_{-l,T}) \cap H^1_m(Q_{-l,T})$ with the initial condition

$$u(x,-l) = 0.$$

We extend the function $u^l(x,t)$ on $\Omega_{-\infty,-l}$ assuming that $u^l(x,t) = 0$ on this domain. Let t_1, t_2 be such that $t_1, t_2 \in R, t_1 < t_2 < T$.

Define

$$\psi(t) = \begin{cases} (t_2 - t_1)^\alpha & \text{if } t > t_2, \\ (t - t_1)^\alpha & \text{if } t_1 \le t \le t_2, \\ 0 & \text{if } t < t_1 \end{cases}$$

for $\alpha > 1$. Notice that each function u^l satisfies the system $A(u) = F^{(l)}$ in Q_T. Thus

$$\begin{aligned}(20)\quad & \frac{1}{2}\int\limits_{\Omega_\tau} |u^l - u^s|^2 \psi(\tau)dx + \int\limits_{Q_{t_1,\tau}} \Big[(C(u^l - u^s), u^l - u^s)\psi(t) \\ & - \frac{1}{2}\psi(t)\sum_{j=1}^n (A_{jx_j}(u^l - u^s), u^l - u^s) + \frac{1}{2}\psi'(t)(u^l - u^s, u^l - u^s) \\ & + (G(x,u^l) - G(x,u^s), u^l - u^s)\psi(t) - (F^l(x,t) - F^s(x,t), u^l - u^s)\psi(t)\Big] dxdt \\ & - \frac{1}{2}\int\limits_{\Omega_0} \psi(t)|u^l(x,0) - u^s(x,0)|^2 dx = 0\end{aligned}$$

for $-l < t_1$, $-s < t_1$, where $t_1 < \tau < T$.

From the assumption of Theorem 2 we have

$$\begin{aligned} I_{12} = & \int\limits_{Q_{t_1,\tau}} (C(x,t)(u^l(x,t) - u^s(x,t)), u^l(x,t) - u^s(x,t))\psi(t)dxdt \\ \ge & c_0 \int\limits_{Q_{t_1,\tau}} \psi(t)|u^l(x,t) - u^s(x,t)|^2 dxdt, \\ I_{13} = & \int\limits_{Q_{t_1,\tau}} (G(x,u^l) - G(x,u^s), u^l(x,t) - u^s(x,t))\psi(t)dxdt \\ \ge & g_0 \int\limits_{Q_{t_1,\tau}} \psi(t)|u^l(x,t) - u^s(x,t)|^p dxdt \end{aligned}$$

and

$$\begin{aligned} I_{14} = & \int\limits_{Q_{t_1,\tau}} \psi'(t)|u^l(x,t) - u^s(x,t)|^2 dxdt \\ \le & \frac{g_0}{2}\int\limits_{t_1}^{\tau} \psi(t)|u^l(x,t) - u^s(x,t)|^p dt + \mu(p,g_0)\int\limits_{t_1}^{\tau}\left[\frac{|\psi'|}{\psi^{\frac{2}{p}}}\right]^{\frac{2}{p-2}} dt, \end{aligned}$$

where $\mu(p, g_0)$ is a constant.

By assumption we have

$$I_{15} = \int\limits_{Q_{t_1,\tau}} ((F^l(x,t) - F^s(x,t), u^l(x,t) - u^s(x,t))\psi(t)dxdt = 0.$$

From the estimates $I_{12} - I_{15}$, and (20) we obtain

$$\int\limits_{\Omega_\tau} |u^l(x,\tau) - u^s(x,\tau)|^2\psi(\tau)dx + \frac{g_0}{2}\int\limits_{Q_{t_1,\tau}} |u^l(x,t) - u^s(x,t)|^p\psi(t)dxdt \tag{21}$$

$$+ h_0 \int\limits_{Q_{t_1,\tau}} \psi(t)|u^l(x,t) - u^s(x,t)|^2 dxdt \le \mu_4(t_2 - t_1)^{\alpha+1-\frac{p}{p-2}},$$

and from (21) we have

$$\int\limits_{\Omega_\tau} |u^l(x,\tau) - u^s(x,\tau)|^2 dx + \frac{g_0}{2}\int\limits_{Q_{t_1,\tau}} |u^l(x,t) - u^s(x,t)|^p dxdt \tag{22}$$

$$+ h_0 \int\limits_{Q_{t_1,\tau}} |u^l(x,t) - u^s(x,t)|^2 dxdt \le \mu_4(t_2 - t_1)^{1-\frac{p}{p-2}}.$$

Since $1 - \frac{p}{p-2} < 0$ and t_1 is arbitrary then (22) implies, that the sequence $\{u^l(x,t)\}$ satisfies the Cauchy condition in the domain $Q_{t_2,T}$, $T > t_2$. Hence we obtain

$$\forall_{\epsilon>0} \int\limits_{\Omega} |u^l(x,t) - u^s(x,t)|^2 dx < \epsilon \text{ for } t \in (t_2,T) \text{ and } l,s > N_0,$$

$$\forall_{\epsilon>0} \int\limits_{Q_{t_0,\tau}} |u^l(x,t) - u^s(x,t)|^2 dx < \epsilon \text{ for } t \in (t_2,T) \text{ and } l,s > N_0$$

and

$$\forall_{\epsilon>0} \int\limits_{Q_{t_2,T}} |u^l(x,t) - u^s(x,t)|^p dxdt < \epsilon \text{ for } t \in (t_2,T) \text{ and } l,s > N_0.$$

Therefore

$$u^l \to u \ \text{ in } \ C((-\infty,T];L^2(\Omega)),$$

$$u^l \to u \ \text{ in } \ L^2_{m,loc}(\overline{Q}_T)$$

and $u^l \to u$ in $L^p_{m,loc}(\overline{Q}_T)$ when $l \to \infty$.

Hence we have

$$G(x,u^l) \to G(x,u) \ \text{ weakly in } \ L^q_{m,loc}(\overline{Q}_T).$$

Consequently

$$\begin{aligned}(23)\qquad &\int_{\Omega}\Big[(u^l(x,t_2),v(x,t_2))-(u^l(x,t_1),v(x,t_1))\Big]dx\\ &+\int_{Q_{t_1,t_2}}\Big[-(u^l(x,t),v_t(x,t))-\sum_{k=1}^{n}(A_k(x)u^l(x,t),v_{x_k}(x,t))\\ &-\sum_{k=1}^{n}(u^l(x,t),A_{x_k}(x)v(x,t))+(C(x,t)u^l(x,t),v(x,t))\\ &+(G(t,u^l),v(x,t))-(F(x,t),v(x,t))\Big]dxdt=0.\end{aligned}$$

Passing to the limits in (23) with $l\to\infty$ we obtain that there exists solution of the problem (1), (2).

Now we prove that this problem has at most one solution.

By theorem from [1], it is easy to show that the equality (20) holds for $v=u\psi(t)$. If we put $v=u\psi$ in (20) then we obtain

$$\begin{aligned}(24)\qquad &\int_{Q_{t_1,t_2}}\Big[-\frac{1}{2}\int_{\Omega}u^2(x,t)\psi(t)\Big|_{t_1}^{t_2}dx\\ &+\int_{Q_{t_1,t_2}}\Big[-\frac{1}{2}(u(x,t),u(x,t)\psi'(t))-\frac{1}{2}\sum_{k=1}^{n}(u(x,t),A_{x_k}(x)u(x,t)\psi(t))\\ &+(C(x,t)u(x,t),u(x,t)\psi(t))+(G(t,u),u(x,t)\psi(t))\\ &-(F(x,t),u(x,t)\psi(t))\Big]dxdt=0.\end{aligned}$$

To obtain a contradiction, suppose that there exist two solutions u^1, u^2 of the problem (1),(2) such that $u^1\neq u^2$. Denote $u=u^1-u^2$. From (24) it is easy to show that for every $\tau<T$ the following equality is satisfied

$$\begin{aligned}(25)\qquad &\frac{1}{2}\int_{\Omega_\tau}|u^1-u^2|^2\psi(\tau)dx+\int_{Q_{t_1,\tau}}\Big[(C(u^1-u^2),u^1-u^2)\psi(t)\\ &-\frac{1}{2}\psi(t)\Big(\sum_{j=1}^{n}(A_{jx_j}(u^1-u^2),u^1-u^2)\Big)+\frac{1}{2}\psi'(t)(u^1-u^2),u^1-u^2)\\ &+(G(x,u^1)-G(x,u^2),u^1-u^2)\psi(t)\Big]dxdt=0.\end{aligned}$$

Analogously to the estimates of components of equality of (20) we can show that

$$\forall_{\epsilon>0} \int\limits_{Q_{t_1,T}} |u^1(x,t) - u^2(x,t)|^2 dxdt < \epsilon$$

for every ϵ. Hence $u^1 = u^2$ in (t_1, T). Since t_1 is arbitrary, then we have $u^1 = u^2$ in $(-\infty, T)$. This completes the proof of Theorem 2.

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Received January 20, 2004; revised version September 16, 2004.