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INITIAL VALUE PROBLEMS FOR SECOND-ORDER INTEGRO-DIFFERENTIAL EQUATIONS ON UNBOUNDED DOMAINS IN A BANACH SPACE

Abstract. In this paper, the Mönch fixed point theorem is used to investigate the existence of solutions of initial value problem(IVP, for short) for second order nonlinear integro-differential equations on infinite intervals in a Banach space. At the same time, the uniqueness of solution for IVP is obtained also.

1. Introduction

Nonlinear integro-differential equations arise from many nonlinear problems in science (see [1]). In [2, Section 3.3], Dajun Guo had discussed the initial value problems (IVP, for short) for first order integro-differential equations of Volterra type on infinite interval $J = [0, +\infty)$ in a real Banach space by using the Banach contraction principle. He assumed that the nonlinear term satisfies Lipschitz condition. In the recent papers [3], [4], D. Guo has investigated the IVP for second-order integro-differential equations in ordered Banach space by using the upper and lower solutions and monotone iterative technique on finite and infinite interval, respectively. It can be mentioned use such method, a Lipschitz condition or one-sided Lipschitz condition and upper-lower solutions are needed. Unfortunately, to find upper-lower solutions is as difficult as well as to find a solution in most cases.

In present paper, we investigate the existence of IVP for second-order such equations similar to [3], [4] by means of completely different method. We propose an use of H. Mönch fixed point theorem, where the Lipschitz condition and upper-lower solutions are not needed. In addition, the uniqueness result is obtained.

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Consider the IVP for second-order integro-differential equations of mixed type in Banach space E:

$$(1)\qquad \begin{cases} x''(t) = f(t, x, x', Tx, Sx), \forall t \in J; \\ x(0) = x_0, \quad x'(0) = x_1, \end{cases}$$

where $f \in C[J \times E \times E \times E \times E, E], J = [0, +\infty)$, $x_0, x_1 \in E$, and

$$(2)\quad (Tx)(t) = \int_0^t k(t,s)x(s)ds, \qquad (Sx)(t) = \int_0^{+\infty} h(t,s)x(s)ds, \quad \forall t \in J,$$

$k \in C[D, R]$, $D = \{(t, s) \in J \times J : t \geq s\}$; $h \in C[J \times J, R]$, $R = (-\infty, +\infty)$.

Let

$$FC[J, E] = \left\{ x \in C[J, E] : \sup_{t \in J} \frac{\|x(t)\|}{1+t} < +\infty \right\}$$

and

$$DC^1[J, E] = \left\{ x \in C^1[J, E] : \sup_{t \in J} \frac{\|x(t)\|}{1+t} < +\infty \quad \text{and} \quad \sup_{t \in J} \|x'(t)\| < \infty \right\}.$$

Evidently, $C^1[J, E] \subset C[J, E]$, $DC^1[J, E] \subset FC[J, E]$. It is easy to see that $FC[J, E]$ is a Banach space with norm

$$(3)\qquad \|x\|_F = \sup_{t \in J} \frac{\|x(t)\|}{1+t}$$

and $DC^1[J, E]$ is Banach space with norm

$$(4)\qquad \|x\|_D = \max\{\|x\|_F, \|x'\|_C\},$$

where $\|x\|_F$ is defined by (3) and

$$\|x'\|_C = \sup_{t \in J} \|x'(t)\|.$$

The basic space using in this paper is $DC^1[J, E]$. A mapping $x \in C^2[J, E]$ is called a solution of IVP(1) if it satisfies Eq. (1).

For a bounded subset V of Banach space E let $\alpha(V)$ be the Kuratowski noncompactness measure of V(for detail, please see [2][5][6]). In this paper, the Kuratowski measures of noncompactness of bounded set in E, $VC[J, E]$, $FC[J, E]$, and $DC^1[J, E]$ are denoted by $\alpha(\cdot)$, $\alpha_C(\cdot)$, $\alpha_F(\cdot)$, and $\alpha_D(\cdot)$, respectively.

At the end of this section we state some lemmas which will be used in Section 2.

LEMMA 1.1 [2] *If $H \subset C[I, E]$ is bounded and equicontinuous, then $\alpha(H(t))$ is continuous on I and*

$$\alpha_C(H) = \max_{t \in I} \alpha(H(t)) \qquad \alpha\left(\left\{ \int_I x(t)dt : x \in H \right\}\right) \leq \int_I \alpha(H(t))dt$$

where $I = [a, b]$, $H(t) = \{x(t) : x \in H\}$, $t \in I$.

LEMMA 1.2 [2] *Let H be a countable set of strongly measurable function $x : I \to E$ such that there exists a function $M \in L[I, R^+]$ such that $\|x(t)\| \le M(t)$ a.e. $t \in I$ for all $x \in H$. Then $\alpha(H(t)) \in L[I, R^+]$ and*

$$\alpha\Big(\Big\{\int_I x(t)dt : x \in H\Big\}\Big) \le 2\int_I \alpha(H(t))dt,$$

where $I = [a, b]$.

LEMMA 1.3 [2] (H. Mönch fixed point theorem). *Let D be a closed and convex subset of E and $u \in D$. Assume that the continuous operator $A : D \to D$ has the following property:*

$C \subset D$ countable, $C \subset \overline{co}(\{u\} \cup A(C)) \to C$ is relatively compact.
Then A has a fixed point in D.

2. Main result

For convenience, let us list some conditions.

H_1) There exist nonnegative functions $a, b, c, d, e \in C[J, J]$ such that

$$\|f(t, x, y, z, w)\| \le a(t)\|x\| + b(t)\|y\| + c(t)\|z\| + d(t)\|w\| + e(t) \quad \forall t \in J, x, y, z, w \in E$$

and

$$\int_0^{+\infty} [(1 + t)a(t) + b(t) + k^*(t)c(t) + h^*(t)d(t)]dt < 1, \qquad \int_0^{+\infty} e(t)dt < +\infty,$$

where

$$k^*(t) = \int_0^t |k(t, s)|(1+s)ds, \qquad h^*(t) = \int_0^{+\infty} |h(t, s)|(1+s)ds < +\infty, \quad \forall t \in J.$$

H_2) There exist nonnegative functions l_1, l_2, l_3, $l_4 \in L[0, +\infty)$ such that

$$\alpha(f(t, D_1, D_2, D_3, D_4)) \le l_1(t)\alpha(D_1) + l_2(t)\alpha(D_2) + l_3(t)\alpha(D_3) + l_4(t)\alpha(D_4),$$
$$\forall t \in J, \quad \text{bounded subsets} \quad D_1, D_2, D_3, D_4 \subset E$$

and

$$l =: \int_0^{+\infty} [(1 + t)l_1(t) + l_2(t) + 2k^*(t)l_3(t) + 2l_4(t)h^*(t)]dt < \frac{1}{2}.$$

LEMMA 2.1. *If condition H_1) is satisfied, then $x \in DC^1[J, E] \cap C^2[J, E]$ is a solution of IVP(1) if and only if $x \in DC^1[J, E]$ is a solution of the following integral equation*

$$x(t) = x_0 + tx_1 + \int_0^t (t - s)f(s, x(s), x'(s), (Tx)(s), (Sx)(s))ds. \tag{5}$$

Proof. First, we show that the abstract infinite integral

$$\int_0^{+\infty} f(s, x(s), x'(s), (Tx)(s), (Sx)(s))ds$$

is convergent for $x \in DC^1[J, E]$.

In fact, from condition H_1), we know

$$\begin{aligned}
(6)\qquad & \int_0^{+\infty} \|f(t, x(t), x'(t), (Tx)(t), (Sx)(t))\|dt \\
\le & \int_0^{+\infty} \Big[a(t)\|x(t)\| + b(t)\|x'(t)\| + c(t)\Big\|\int_0^t k(t,s)x(s)ds\Big\| \\
& + d(t)\Big\|\int_0^{+\infty} h(t,s)x(s)ds\Big\| + e(t)\Big]dt \\
\le & \int_0^{+\infty} \Big[(1+t)a(t)\cdot\frac{\|x(t)\|}{1+t} + b(t)\|x'(t)\| + c(t)\Big\|\int_0^t k(t,s)(1+s)\cdot\frac{x(s)}{1+s}ds\Big\| \\
& + d(t)\Big\|\int_0^{+\infty} h(t,s)(1+s)\cdot\frac{x(s)}{1+s}ds\Big\| + e(t)\Big]dt \\
\le & \int_0^{+\infty} \Big[(1+t)a(t)\cdot\|x\|_F + b(t)\|x'\|_c + c(t)\int_0^t |k(t,s)|(1+s)ds\cdot\|x\|_F \\
& + d(t)\int_0^{+\infty} |h(t,s)|(1+s)ds\cdot\|x\|_F + e(t)\Big]dt \\
\le & \int_0^{+\infty} [(1+t)a(t) + b(t) + k^*(t)c(t) + h^*(t)d(t)]dt\cdot\|x\|_D \\
& + \int_0^{+\infty} e(t)dt < +\infty.
\end{aligned}$$

Thus, $\int_0^{+\infty} f(s, x(s), x'(s), (Tx)(s), (Sx)(s))ds$ is convergent.

For $x \in C^2[J, E]$, we have

$$(7)\qquad x'(t) = x'(0) + \int_0^t x''(s)ds, \quad \forall t \in J$$

and

$$(8)\qquad x(t) = x(0) + \int_0^t x'(s)ds \quad \forall t \in J.$$

Substituting Eq. (7) into (8), it is easy to get the following formula:

$$(9)\qquad x(t) = x(0) + tx'(0) + \int_0^t (t-s)x''(s)ds, \quad \forall t \in J, \quad x \in C^2[J,E].$$

Now, if $x \in C^2[J,E]$ is a solution of IVP(1), then, substituting Eq. (1) into Eq. (9), we see that $x(t)$ satisfies Eq. (5).

Conversely, if $x \in DC^1[J,E]$ is a solution of Eq. (5), then, differentiation of Eq. (5) gives

$$x'(t) = x_1 + \int_0^t f(s, x(s), x'(s), (Tx)(s), (Sx)(s))ds, \quad \forall t \in J$$

and

$$x'' = f(t, x(t), x'(t), (Tx)(t), (Sx)(t)), \quad \forall t \in J,$$

hence $x \in C^2[J,E]$ and $x(t)$ satisfies Eq. (1). ∎

For $x \in DC^1[J,E]$, we define an operator A by

$$(10)\qquad (Ax)(t) =: x_0 + tx_1 + \int_0^t (t-s)f(s, x(s), x'(s), (Tx)(s), (Sx)(s))ds.$$

Then, by Lemma 2.1, we know that the existence of solution for IVP(1) in $DC^1[J,E]$ is equivalent to the existence of a fixed point of the operator A in $DC^1[J,E]$. Therefore, we need to investigate only the existence of fixed a point of A in $DC^1[J,E]$.

LEMMA 2.2. *Suppose H_1) is satisfied. Then A: $DC^1[J,E] \to DC^1[J,E]$ is bounded.*

Proof. Firstly, we claim that $Ax \in DC^1[J,E]$ for any $x \in DC^1[J,E]$.

In fact, for $x \in DC^1[J,E]$, by (6) (10) we know that $Ax \in C^1[J,E]$ and

$$(11)\qquad \left\| \frac{(Ax)(t)}{1+t} \right\| \le \frac{\|x_0 + tx_1\|}{1+t} + \int_0^t \frac{t-s}{1+t} \|f(s, x(s), x'(s), (Tx)(s), (Sx)(s))\| ds$$

$$\le \|x_0\| + \|x_1\| + \int_0^{+\infty} \|f(s, x(s), x'(s), (Tx)(s), (Sx)(s))\| ds$$

$$\le \|x_0\| + \|x_1\| + \int_0^{+\infty} [(1+t)a(t) + b(t) + k^*(t)c(t) + h^*(t)d(t)]dt \cdot \|x\|_D$$

$$+ \int_0^{+\infty} e(t)dt.$$

Similarly, we have

$$\|(Ax)'(t)\| = \Big\|x_1 + \int_0^t f(s, x(s), x'(s), (Tx)(s), (Sx)(s))ds\Big\| \tag{12}$$
$$\leq \|x_1\| + \int_0^{+\infty} [(1+t)a(t) + b(t) + k^*(t)c(t) + h^*(t)d(t)]dt \cdot \|x\|_D$$
$$+ \int_0^{+\infty} e(t)dt.$$

Now (11) and (12) guarantee that $Ax \in DC^1[J, E]$ and A is a bounded operator. ∎

LEMMA 2.3. *Let H_1) be satisfied, V is a bounded subset of $DC^1[J, E]$. Then $\frac{(AV)(t)}{1+t}$, $(AV)'(t)$ are equicontinuous on any finite subinterval of J and for any $\varepsilon > 0$, there exists $N > 0$ such that*

$$\left\|\frac{(Ax)(t_1)}{1+t_1} - \frac{(Ax)(t_2)}{1+t_2}\right\| < \varepsilon, \qquad \|(Ax)'(t_1) - (Ax)'(t_2)\| < \varepsilon$$

uniformly with respect to $x \in V$ as $t_1, t_2 \geq N$.

Proof. For $x \in V$, $t_2 > t_1$, by using (10), we get

$$\left\|\frac{(Ax)(t_1)}{1+t_1} - \frac{(Ax)(t_2)}{1+t_2}\right\| \leq \left\|\frac{x_0 + t_1 x_1}{1+t_1} - \frac{x_0 + t_2 x_1}{1+t_2}\right\| \tag{13}$$
$$+ \Big\|\int_0^{t_1} \frac{t_1 - s}{1+t_1} f(s, x(s), x'(s), (Tx)(s), (Sx)(s))ds$$
$$- \int_0^{t_2} \frac{t_2 - s}{1+t_2} f(s, x(s), x'(s), (Tx)(s), (Sx)(s))ds\Big\|$$
$$\leq (\|x_0\| + \|x_1\|)|t_2 - t_1| + \int_{t_1}^{t_2} \frac{t_2 - s}{1+t_2} \|f(s, x(s), x'(s), (Tx)(s), (Sx)(s))\|ds$$
$$+ \int_0^{t_1} \left|\frac{t_1 - s}{1+t_1} - \frac{t_2 - s}{1+t_2}\right| \cdot \|f(s, x(s), x'(s), (Tx)(s), (Sx)(s))\|ds.$$

Then by using H_1) together with (13), it is easy to see that $\{\frac{(Ax)(t)}{1+t} : x \in V\}$ is equicontinuous on any finite subinterval of J.

Similarly, from the same reason, one can prove that $\{(Ax)'(t) : x \in V\}$ is equicontinuous on any finite subinterval of J.

In the following, we shall prove that for any $\varepsilon > 0$, there exists sufficiently large $N > 0$, which satisfies

$$\left\|\frac{(Ax)(t_1)}{1+t_1}-\frac{(Ax)(t_2)}{1+t_2}\right\|<\varepsilon, \qquad \|(Ax)'(t_1)-(Ax)'(t_2)\|<\varepsilon$$

holds for all $x \in V$ and $t_1, t_2 \geq N$.

By (2.9), we need to show only that for any $\varepsilon > 0$, there exists sufficiently large $N > 0$ such that

$$\left\|\int_0^{t_1}\frac{t_1-s}{1+t_1}f(s,x(s),x'(s),(Tx)(s),(Sx)(s))ds - \int_0^{t_2}\frac{t_2-s}{1+t_2}f(s,x(s),x'(s),(Tx)(s),(Sx)(s))ds\right\|<\varepsilon, \quad \forall t_1,t_2\geq N. \tag{14}$$

Since V is bounded, then, by using (6), there exists $M > 0$ which satisfies

$$\int_0^{+\infty}\|f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\|ds\leq M, \qquad \forall x\in V. \tag{15}$$

Therefore, there exists $L > 0$ such that

$$\int_L^{+\infty}\|f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\|ds < \frac{\varepsilon}{3} \text{ uniformly with respect to } x\in V. \tag{16}$$

Choose $N > L > 0$ such that

$$\left|\frac{t_1-s}{1+t_1}-\frac{t_2-s}{1+t_2}\right|<\frac{\varepsilon}{3M}, \quad \forall t_1,t_2>N, \quad \forall s\in[0,L]. \tag{17}$$

Then (15)–(17) yield that

$$\begin{aligned}
&\left\|\int_0^{t_1}\frac{t_1-s}{1+t_1}f(s,x(s),x'(s),(Tx)(s),(Sx)(s))ds - \int_0^{t_2}\frac{t_2-s}{1+t_2}f(s,x(s),x'(s),(Tx)(s),(Sx)(s))ds\right\| \\
&\leq 2\left\|\int_L^{+\infty}f(s,x(s),x'(s),(Tx)(s),(Sx)(s))ds\right\| \\
&\quad+\left\|\int_0^{L}\left|\frac{t_1-s}{1+t_1}-\frac{t_2-s}{1+t_2}\right|f(s,x(s),x'(s),(Tx)(s),(Sx)(s))ds\right\| \\
&\leq\frac{2\varepsilon}{3}+\frac{\varepsilon}{3M}\cdot M=\varepsilon,
\end{aligned}$$

what means that (14) holds.

Thus, our conclusion follows. ∎

LEMMA 2.4. *Let H_1) be satisfied, V is a bounded subset of $DC^1[J,E]$. Then,*

$$\alpha_D(AV) = \max\left\{ \sup_{t\in J} \alpha\left(\frac{(AV)(t)}{1+t}\right), \quad \sup_{t\in J} \alpha((AV)'(t)\right\}.$$

Using Lemma 1.1 and Lemma 2.3, the proof of Lemma 2.4 is very similar to that of [7, Lemma 2.4], so we omit it.

LEMMA 2.5 *Let H_1) be satisfied. Then A: $DC^1[J,E] \to DC^1[J,E]$ is continuous.*

Proof. Let $\{x_n\}, \{x\} \subset DC^1[J,E]$ and $\|x_n - x\|_D \to 0 \quad (n \to +\infty)$. Hence $\{x_n : n \geq 1\}$ is a bounded subset of $DC^1[J,E]$. Thus, there exists $M > 0$ such that $\|x_n\|_D \leq M$. At the same time, $\|x\|_D \leq M$.

In the following, we first show that $\{Ax_n\}$ is relatively compact. Since

$$\left\|\frac{(Ax_n)(t)}{1+t} - \frac{(Ax)(t)}{1+t}\right\| \leq \int_0^t \frac{t-s}{1+t} \|f(s,x_n(s),x_n'(s),(Tx_n)(s),(Sx_n)(s)) - f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\| ds \tag{18}$$

and

$$\|(Ax_n)'(t) - (Ax)'(t)\| \leq \int_0^t \|f(s,x_n(s),x_n'(s),(Tx_n)(s),(Sx_n)(s)) - f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\| ds.$$

By H_1) (6)(11)(12)(18)(19) together with the Lebesgue dominated convergence theorem, one can get

$$\begin{aligned} &(Tx_n)(t) \to (Tx)(t) \quad (n \to +\infty), \\ &(Sx_n)(t) \to (Sx)(t) \quad (n \to +\infty), \end{aligned} \quad \forall t \in J.$$

Moreover,

$$\begin{aligned} &\frac{(Ax_n)(t)}{1+t} \to \frac{(Ax)(t)}{1+t} \quad (n \to +\infty), \\ &(Ax_n)'(t) \to (Ax)'(t) \quad (n \to +\infty), \end{aligned} \quad \forall t \in J; \tag{20}$$

Therefore,

$$\alpha\left(\left\{\frac{(Ax_n)(t)}{1+t} : n \in N\right\}\right) = \alpha(\{(Ax_n)'(t) : n \in N\}) = 0, \ \forall t \in J.$$

Immediately, Lemma 2.4 guarantees that

$$\alpha_D(\{Ax_n : n \in N\}) = 0,$$

that is, $\{Ax_n\}$ is relatively compact.

Next, we prove $\|Ax_n - Ax\|_D \to 0 \quad (n \to +\infty)$.

In fact, if this is not true, then there exist $\varepsilon_0 > 0$ and $\{x_{n_i}\} \subset \{x_n\}$ such that $\|Ax_{n_i} - Ax\|_D \geq \varepsilon_0 (i = 1,2,3,\cdots)$. Since $\{Ax_n\}$ is relatively compact, there exists a subsequence of $\{Ax_{n_i}\}$ (without loss of generality, we relabel the subsequence still as $\{Ax_{n_i}\}$) and $u \in DC^1[J,E]$ with $Ax_{n_i} \to u$ $(i \to +\infty)$, that is, $\|Ax_{n_i} - u\|_D \to 0 (i \to +\infty)$. Therefore,

$$\left.\begin{aligned} &\left\| \frac{(Ax_{n_i})(t)}{1+t} - \frac{u(t)}{1+t} \right\| \to 0 \quad (n \to +\infty), \\ &\|(Ax_n)'(t) - u'(t)\| \to 0 \quad (n \to +\infty), \end{aligned}\right. \quad \forall t \in J.$$

Combining this with (20), one get that $u = Ax$. This is a contradiction.

Consequently, A is continuous. ∎

Our main results is the following.

THEOREM 2.1. *Let $H_1)H_2)$ be satisfied. Then IVP(1) has at least one solution belonging to $DC^1[J,E] \cap C^2[J,E]$.*

Proof. We need to prove only the existence of fixed point of operator A in $DC^1[J,E]$. Let

$$R > \left(\|x_0\| + \|x_1\| + \int_0^{+\infty} e(t)dt\right)$$

$$\times \left(1 - \int_0^{+\infty} [(1+t)a(t) + b(t) + k^*(t)c(t) + h^*(t)d(t)]dt\right)^{-1},$$

$$B =: B_D(\theta, R) = \{x \in DC^1[J,E] : \|x\|_D \le R\},$$

where θ denotes the zero element in E.

We first show that $A : B \to B$. In fact, for $x \in B$, by (10), (11) and (12) we know

$$\left\| \frac{(Ax)(t)}{1+t} \right\| \le \|x_0\| + \|x_1\| + \int_0^{+\infty} [(1+t)a(t)$$

$$+ b(t) + k^*(t)c(t) + h^*(t)d(t)]dt \cdot \|x\|_D + \int_0^{+\infty} e(t)dt$$

$$\le R, \quad \forall t \in J.$$

Analogously, we can get

$$\|(Ax)'(t)\| \le R, \quad \forall t \in J.$$

Thus, by Lemma 2.5, we know that A is a continuous operator from B into B.

Next, we prove that if $C \subset B$ is countable, $C \subset \overline{co}_D(\{u\} \cup A(C))$, then C is relatively compact.

Indeed, suppose that $C \subset B$ satisfies the above condition. Then we have $\alpha_D(C) \le \alpha_D(AC)$. At the same time, by [8], it follows that

$$C(t) \subset \overline{co}_E(\{u(t)\} \cup (AC)(t)), \quad C'(t) \subset \overline{co}_E(\{u'(t)\} \cup (AC)'(t)), \quad \forall t \in J.$$

Therefore,

$$\alpha\Big(\frac{C(t)}{1+t}\Big) \le \alpha\Big(\frac{(AC)(t)}{1+t}\Big), \quad \alpha(C'(t)) \le \alpha((AC)'(t)). \tag{21}$$

On the other hand, for $t \in J$, $n \in N$, $x \in C$, let

$$(S_n x)(t) =: \int_0^n h(t,s)x(s)ds.$$

By H_1) we know that

$$\|(S_n x)(t) - (Sx)(t)\| \le \int_n^{+\infty} |h(t,s)| \cdot \|x(s)\| ds$$
$$\le \int_n^{+\infty} |h(t,s)|(1+s)ds \cdot \|x\|_D \to 0 \quad (n \to +\infty), \ x \in C.$$

This implies that

$$d_H((S_nC)(t), (SC)(t)) \to 0 \quad (n \to +\infty), t \in J,$$

where $d_H(\cdot,\cdot)$ denotes the Housdorff distance. Thus, by the property of the measure of noncompactness, we get

$$\alpha((S_nC)(t)) \to \alpha((SC)(t)), \quad t \in J. \tag{22}$$

Moreover, by Lemma 1.2 we know

$$\alpha((S_nC)(t)) = \alpha\Big(\Big\{\int_0^n h(t,s)x(s)ds : x \in C\Big\}\Big)$$
$$\le 2\int_0^n |h(t,s)|(1+s)\cdot\alpha\Big(\frac{C(s)}{1+s}\Big)ds \le 2\int_0^n |h(t,s)|(1+s)ds \cdot \alpha_D(C).$$

So, (22) guarantees that

$$\alpha\Big(\Big\{\int_0^{+\infty} h(t,s)x(s)ds : x \in C\Big\}\Big) \le 2\int_0^{+\infty} |h(t,s)|(1+s)ds \cdot \alpha_D(C). \tag{23}$$

Now, by H_2), Lemma 1.2 and (10)(21)(23), we obtain

$$\alpha\Big(\frac{(AC)(t)}{1+t}\Big) \le 2\int_0^t \frac{t-s}{1+t}\alpha(\{f(s,x(s),x'(s),(Tx)(s),(Sx)(s)) : x \in C\})ds$$
$$\le 2\int_0^t [l_1(s)\alpha(C(s)) + l_2(s)\alpha(C'(s)) + l_3(s)\alpha\Big(\Big\{\int_0^s k(s,\tau)x(\tau)d\tau : x \in C\Big\}\Big)$$
$$+ l_4(s)\alpha\Big(\Big\{\int_0^{+\infty} h(s,\tau)x(\tau)d\tau : x \in C\Big\}\Big)]ds$$

$$\le 2\int_0^t [l_1(s)(1+s)\alpha\left(\frac{C(s)}{1+s}\right) + l_2(s)\alpha(C'(s))$$

$$+ 2l_3(s)\int_0^s |k(s,\tau)|(1+\tau)\alpha\left(\frac{C(\tau)}{1+\tau}\right)d\tau$$

$$+ 2l_4(s)\int_0^{+\infty} |h(s,\tau)|(1+\tau)d\tau \cdot \alpha_D(C)]ds$$

$$\le 2\int_0^{+\infty} [(1+s)l_1(s) + l_2(s) + 2l_3(s)\int_0^s |k(s,\tau)|(1+\tau)d\tau$$

$$+ 2l_4(s)\int_0^{+\infty} |h(s,\tau)|(1+\tau)d\tau]ds \cdot \alpha_D(C)$$

$$\le 2l\alpha_D(AC).$$

Since t is arbitrary, it follows that

$$\sup_{t\in J}\alpha\left(\frac{(AC)(t)}{1+t}\right) \le 2l\alpha_D(AC). \tag{24}$$

Very similarly, one can get

$$\sup_{t\in J}\alpha((AC)'(t)) \le 2l\alpha_D(AC). \tag{25}$$

Immediately, by (24), (25) and Lemma 2.4, we obtain $\alpha_D(AC) = 0$. Furthermore, $\alpha_D(C) = 0$. This implies that C is a relatively compact subset of $DC^1[J,E]$. It follows from Lemma 1.3 that A has a fixed point in B, that is, IVP(1) has at least one solution in $DC^1[J,E]$. ∎

REMARK 1. In Theorem 2.1, f needs not to be uniformly continuous.

REMARK 2. If $f(t,x,x',Tx,Sx) \equiv f(t,x,Tx,Sx)$ in IVP(1), we may use similar method to study IVP(1) in basic space $FC[J,E]$ to obtain the same result as Theorem 2.1 under $H_1) - H_2)$ (here $b(t) \equiv 0$, $l_2(s) \equiv 0$). Moreover, the proof may be simpler since we do not need to estimate the derivative term.

REMARK 3. If $f(t,x,x',Tx,Sx) \equiv f(t,x,x',Tx)$ in IVP(1), that is, the term Sx does not emerge in f, the condition "$l < \frac{1}{2}$" in $H_2)$ may be removed.

In fact, from the proof of Theorem 2.1, for $C \subset B$ countable and $C \subset \overline{co}_D(\{u\} \cup A(C))$, we get

$$\alpha\left(\frac{(AC)(t)}{1+t}\right) \le 2\int_0^t [l_1(s)(1+s)\alpha\left(\frac{C(s)}{1+s}\right) + l_2(s)\alpha(C'(s))$$
$$+ 2l_3(s)\int_0^s |k(s,\tau)|(1+\tau)\alpha\left(\frac{C(\tau)}{1+\tau}\right)d\tau]ds.$$

Also we have

$$\alpha((AC)'(t)) \le 2\int_0^t [l_1(s)(1+s)\alpha\left(\frac{C(s)}{1+s}\right) + l_2(s)\alpha(C'(s))$$
$$+ 2l_3(s)\int_0^s |k(s,\tau)|(1+\tau)\alpha\left(\frac{C(\tau)}{1+\tau}\right)d\tau]ds.$$

Let

$$m(t) = \max\left\{\alpha\left(\frac{(AC)(t)}{1+t}\right), \quad \alpha((AC)'(t))\right\}, \quad t \in J.$$

Then

$$m(t) \le 2\int_0^t \{[l_1(s)(1+s) + l_2(s)]m(s)$$
$$+ 2l_3(s)\int_0^s |k(s,\tau)|(1+\tau)m(\tau)d\tau\}ds.$$

This integral inequality yields $m(t) \equiv 0$ for $t \in J$. By Lemma 2.4 it can be obtained that $\alpha_D(AC) = 0$. The rest is the same as in the proof of Theorem 2.1. Thus, the existence of solution for IVP(1) follows.

The following theorem is an uniqueness result for IVP.

THEOREM 2.2. *Assume that*

H_3) *there exist nonnegative functions $a, b, c, d \in C[J, J]$ such that*

$$\|f(t,x_1,y_1,z_1,w_1) - f(t,x_2,y_2,z_2,w_2)\| \le a(t)\|x_1 - x_2\| + b(t)\|y_1 - y_2\|$$
$$+ c(t)\|z_1 - z_2\| + d(t)\|w_1 - w_2\| \quad \forall t \in J, x, y, z, w \in E$$

and

$$L =: \int_0^{+\infty} [(1+t)a(t) + b(t) + k^*(t)c(t) + h^*(t)d(t)]dt < 1,$$
$$\int_0^{+\infty} \|f(t,0,0,0,0)\|dt < +\infty,$$

where $k^*(t)$ and $h^*(t)$ are the same as in H_1).

Then there exists the only solution of IVP(1) belonging to $DC^1[J,E] \cap C^2[J,E]$.

Proof. First, it is easy to see that H_3) implies H_1). So by Lemma 2.5, A: $DC^1[J,E] \to DC^1[J,E]$ is continuous, where A is the same as in (10).

Next, we show A is a contraction. For any given x_1, $x_2 \in DC^1[J,E]$, by (10) and (H_3) we know that

$$\left\|\frac{(Ax_1)(t)-(Ax_2)(t)}{1+t}\right\| \leq \int_0^{+\infty} \|f(t,x_1(t),x_1'(t),(Tx_1)(t),(Sx_1)(t)) - f(t,x_2(t),x_2'(t),(Tx_2)(t),(Sx_2)(t))\|dt$$

$$\leq \int_0^{+\infty} \Big[a(t)\|x_1(t)-x_2(t)\|+b(t)\|x_1'(t)-x_2'(t)\|+c(t)\Big\|\int_0^t k(t,s)(x_1(s)-x_2(s))ds\Big\| + d(t)\Big\|\int_0^{+\infty} h(t,s)(x_1(s)-x_2(s))ds\Big\|\Big]dt$$

$$\leq \int_0^{+\infty} \Big[(1+t)a(t)\cdot\|x_1-x_2\|_F+b(t)\|x_1'-x_2'\|_c+c(t)\int_0^t |k(t,s)|(1+s)ds\cdot\|x_1-x_2\|_F + d(t)\int_0^{+\infty} |h(t,s)|(1+s)ds\cdot\|x_1-x_2\|_F\Big]dt$$

$$\leq \int_0^{+\infty} [(1+t)a(t)+b(t)+k^*(t)c(t)+h^*(t)d(t)]dt\cdot\|x_1-x_2\|_D = L\|x_1-x_2\|_D.$$

Similarly,

$$\|(Ax_1)'(t)-(Ax_2)'(t)\| \leq L\|x_1-x_2\|_D.$$

So it follows that

$$\|Ax_1-Ax_2\|_D \leq L\|x_1-x_2\|_D.$$

Immediately, the Banach contraction principle guarantees our result. ∎

The following example, may be used to illustrate some applications of Theorem 2.1.

EXAMPLE. Consider the IVP of an infinite system for scalar second order differential equations

$$(26)\quad \begin{cases} x_n''=\dfrac{t+x_n}{7(t+1)e^t}+\dfrac{1+\sqrt[2]{|x_{2n}'|}}{4n(9+t^2)}+\dfrac{1}{n^2(1+t)^2} \\ \times\ln\Big(2+\int_0^t e^{-(t+1)s}x_{n+2}(s)ds+\int_0^{+\infty}\dfrac{\sin(t-s)}{(1+s)^3}x_{n+1}(s)ds\Big),\ 0\leq t<+\infty; \\ x_n(0)=x_{0n},\quad x_n'(0)=x_{1n}\quad (n=1,2,3,\cdots), \end{cases}$$

where $\sup_n |x_{0n}| < +\infty$, $\sup_n |x_{1n}| < +\infty$.

CONCLUSION: IVP(26) has at least one solution defined on $[0,+\infty)$.

Proof. Let $E = l^\infty = \{x = (x_1, \cdots x_n, \cdots) : \sup_n |x_n| < +\infty\}$ with norm $\|x\| = \sup_n |x_n|$. Then IVP(26) can be regarded as an IVP of form (1) in E. In this situation, $J = [0,+\infty)$, $x = (x_1, \cdots x_n, \cdots)$, $x_0 = (x_{01}, \cdots x_{0n}, \cdots)$, $x_1 = (x_{11}, \cdots x_{1n}, \cdots) \in E$, $f = (f_1, \cdots f_n, \cdots)$, $f_n = g_n + h_n$, in which

$$g_n = \frac{t + x_n}{7(t+1)e^t},$$

$$h_n = \frac{1 + \sqrt[2]{|x'_{2n}|}}{4n(9+t^2)} + \frac{1}{n^2(1+t)^2}\ln(2 + z_{n+2} + w_{n+1}),$$

where

$$z_n = \int_0^t e^{-(t+1)s} x_n(s)ds, \quad w_n = \int_0^{+\infty} \frac{\sin(t-s)}{(1+s)^3} x_n(s)ds.$$

Evidently, $f \in C[J \times E \times E \times E \times E, E]$. Now we verify that $H_1) - H2)$ hold. First, it is easy to see that

$$\|f(t,x,y,z,w)\| = \sup_n |f_n(t,x,y,z,w)|$$
$$\le a(t)\|x\| + b(t)\|y\| + c(t)\|z\| + d(t)\|w\| + e(t), \quad \forall t \in J, x, y, z, w \in E,$$

where

$$a(t) = \frac{1}{7(1+t)e^t}, \qquad b(t) = \frac{1}{4(9+t^2)},$$
$$c(t) = d(t) = \frac{1}{(1+t)^2},$$
$$e(t) = \frac{t}{7(1+t)e^t} + \frac{1}{2(9+t^2)} + \frac{1}{(1+t)^2}.$$

Now we estimate $k^*(t)$ and $h^*(t)$. After simple calculation, we can get

$$0 \le k^*(t) = \int_0^t k(t,s)(1+s)ds = \int_0^t e^{-(t+1)s}(1+s)ds \le \frac{1}{t+1} + \frac{1}{(1+t)^2}$$

and

$$h^*(t) = \int_0^{+\infty} \frac{\sin(t-s)|}{(1+s)^3}(1+s)ds \le \int_0^{+\infty} \frac{1}{(1+s)^2}ds = \frac{1}{3}.$$

Therefore,

$$\int_0^{+\infty} k^*(t)c(t)dt \le \int_0^{+\infty} \left[\frac{1}{(t+1)^3} + \frac{1}{(1+t)^4}\right]dt = \frac{9}{20},$$

$$\int_0^{+\infty} h^*(t)d(t)dt \le \frac{1}{3}\int_0^{+\infty} \frac{1}{(1+t)^2}dt = \frac{1}{9}.$$

Also we have

$$\int_0^{+\infty}(1+t)a(t)dt = \frac{1}{7}, \qquad \int_0^{+\infty} b(t)dt = \frac{\pi}{24}.$$

Consequently,

$$\int_0^{+\infty}[(1+t)a(t)+b(t)+k^*(t)c(t)+h^*(t)d(t)]dt < 1, \qquad \int_0^{+\infty} e(t)dt < +\infty,$$

that is, H_1) is satisfied.

On the other hand, we obtain

$$0 \le |h_n(t,x,y,z,w)| \le \frac{(2+\|y\|)}{4n(9+t^2)} + \frac{1}{n^2(1+t)^2}\ln(2+\|z\|+\|w\|).$$

As shown in [2, Example 2.1.2], we know $h(t,D_1,D_2,D_3,D_4)$ is relatively compact in l^∞, i.e.

$$\begin{aligned}&\alpha(h(t,D_1,D_2,D_3,D_4)) = 0, \quad \forall t \in J,\\ &\text{and all bounded subsets} \quad D_1,D_2,D_3,D_4 \subset E.\end{aligned} \tag{27}$$

Combining (26) with (27), it follows that

$$\alpha(f(t,D_1,D_2,D_3,D_4)) \le \frac{\alpha(D_1)}{7(1+t)e^t}, \quad \forall t \in J,$$

$$\text{and bounded subsets} \quad D_1,D_2,D_3,D_4 \subset E.$$

This means that $l_1(t) = \frac{1}{7(1+t)e^t}$, $l_2(t) = l_3(t) = l_4(t) \equiv 0$. Since

$$l = \int_0^{+\infty}(1+t)l_1(t)dt = \frac{1}{7} < \frac{1}{2},$$

then, H_2) holds.

By Theorem 2.1, our conclusion follows. ∎

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