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ON THE STRUCTURE OF SOLUTION SET OF AN INTEGRO-DIFFERENTIAL EQUATION IN BANACH SPACES

In this paper we shall present two existence theorems for local solutions of an initial value problem for nonlinear integro-differential equation in a Banach space. Moreover, we also shall prove that the set of all these solutions is an R_δ in the Aronszajn sense [1], i.e. it is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts. Let us mention that in the case of ordinary differential equations this problem was studied by many authors (for example see to [9], [10] and [5], [6], [8]).

1. Consider the following Cauchy problem

$$x'(t) = f(t, x(t), \int_0^t g(t, s, x(s))ds), \tag{1}$$

$$x(0) = 0 \tag{2}$$

in a Banach space E. We assume that $D = [0, a]$, $B = \{x \in E : \|x\| \leq b\}$ and $f : D \times B \times E \to E$, $g : D^2 \times B \to E$ are bounded continuous functions. Let

$$\|f(t, x, z)\| \leq m_1 \text{ for } t \in D,\ x \in B,\ z \in a \cdot \overline{\text{conv}} g(D^2 \times B)$$
$$\|g(t, s, x)\| \leq m_2 \text{ for } t, s \in D,\ x \in B.$$

We choose a positive number d such that $d \leq a$ and

$$m_1 d \leq b \text{ and } m_2 d \leq b. \tag{3}$$

Let $J = [0, d]$. Denote by $C = C(J, E)$ the Banach space of continuous functions $z : J \to E$ with the usual norm $\|z\|_C = \max_{t \in J} \|z(t)\|$.

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Let $W = \{x \in C : \|x\|_C \leq b\}$. For $t \in J$ and $x \in W$ put

$$\tilde{g}(t,x) = \int_0^t g(t,s,x(s))ds.$$

Fix $\tau \in J$ and $x \in W$. As the set $J^2 \times x(J)$ is compact, from the continuity of g it follows that for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|g(t,s,x(s)) - g(\tau,s,x(s))\| < \varepsilon \text{ for } t,s \in J \text{ with } |t-\tau| < \delta.$$

In view of the following inequality

$$\|\tilde{g}(t,x) - \tilde{g}(\tau,x)\| \leq m_2 |t-\tau| + \int_0^\tau \|g(t,s,x(s)) - g(\tau,s,x(s))\| \, ds,$$

this implies the continuity of the function $t \to \tilde{g}(t,x)$.

On the other hand, the Lebesgue dominated convergence theorem proves that for each fixed $t \in J$ the function $x \to \tilde{g}(t,x)$ is continuous on W. Moreover

$$\|\tilde{g}(t,x)\| \leq m_2 t \text{ for } t \in J \text{ and } x \in W. \tag{4}$$

2. Assume that h is a Kamke function, i.e. $(t,r) \to h(t,r)$ is a nonnegative function defined on $D \times R_+$ which is Lebesgue measurable in t for fixed r, and continuous in r for fixed t, and

(i) for every bounded subset Z of $D \times R_+$ there exists a function ψ_z defined on $(0,a]$ such that $h(t,r) \leq \psi_z(t)$ for $(t,r) \in Z$ and ψ_z is Lebesgue integrable on $[c,a]$ for every $c > 0$;

(ii) for each c, $c \in (0,a]$, identically zero function is the only absolutely continuous function on $[0,c]$ which satisfies $z'(t) = h(t,z(t))$ almost everywhere on $[0,c]$ and such that $D_+z(0) = z(0) = 0$.

THEOREM 1. *If for any $t \in D$ and $x,y \in B$, $z \in E$*

$$\|f(t,x,z) - f(t,y,z)\| \leq h(t, \|x-y\|)$$

and the set $g(D^2 \times B)$ is relatively compact in E, then the set S of all solutions of the problem (1)-(2), defined on J, is an R_δ.

Proof. Let us remark that on J the problem (1)-(2) is equivalent to

$$x'(t) = f(t,x(t),\tilde{g}(t,x)), \; x(0) = 0.$$

Let $H = \cup_{0 \leq \lambda \leq d} \lambda \overline{\text{conv}} g(D^2 \times B)$. By (3) and the Mazur Lemma, H is a compact subset of B. Fix $n \in N$ and $v \in W$. Put $t_i = \frac{di}{n}$ for $i = 0,1,\ldots,n$.

We define a mapping $u_n(v) : J \to B$ by

$$u_n(v)(0) = 0,$$

$$u_n(v)(t) = u_n(v)(t_i) + \int_{t_i}^{t} f(s, u_n(v)(t_i), \tilde{g}(s,v))ds$$

for $t \in [t_i, t_{i+1}]$, $i = 0, 1, \ldots, n-1$. Notice that $\tilde{g}(s,v) \in H$.

Similarly as in [5], [6], [7] it can be shown that for every $t, \tau \in J$ and $v \in W$,

$$\|u_n(v)(t) - u_n(v)(\tau)\| \le m_1 |t - \tau| ; \tag{5}$$

$$u_n(v)(t) \in V_n \text{ for } t \in J \text{ and } v \in W, \tag{6}$$

where V_n is a compact subset of B defined by

$V_0 = \{0\}$, $V_{k+1} = \cup_{0 \le \lambda \le d} \lambda \overline{\mathrm{conv}} f\,(J \times V_k \times \tilde{g}(J, W))$ for $k = 0, 1, \ldots, n-1$;

$$D_+ \|u_n(v)(t) - u_m(v)(t)\| \le \min\left(\mu(t), h(t, \|u_n(v)(t) - u_m(v)(t)\|) + 2\varepsilon\,(t, q_n)\right) \tag{7}$$

for $m \ge n, t \in J$ and $v \in W$, where $\varepsilon(t,p) = \sup_{0 \le r \le p} h(t,r)$,

$\mu(t) = \sup\{\|f(t,x,z) - f(t,y,z)\| :$

$\|x\| \le m_1 t, \|y\| \le m_1 t, z \in a \cdot \overline{\mathrm{conv}} g(D^2 \times B)\}$ and $q_n = \dfrac{m_1 d}{n}$;

$$\left\| u_n(v)(t) - \int_0^t f(s, u_n(v)(s), \tilde{g}(s,v))ds \right\| \le m(q_n) \tag{8}$$

for $t \in J$ and $v \in W$, where $m(p) = \int_0^d \min\left(\mu(t), \varepsilon(t,p)\right) dt$ for $p \ge 0$; and $\lim_{p \to 0_+} m(p) = 0$; for any $\varepsilon > 0$ and $v_0 \in W$ there exists $\delta > 0$ such that

$$\int_0^t \|\tilde{g}(s,v) - \tilde{g}(s,v_0)\|\, ds \le \varepsilon \text{ for } t \in J, v \in W, \|v - v_0\|_C < \delta, \tag{9}$$

and consequently $\|u_n(v)(t) - u_n(v_0)(t)\| \le m_n(\varepsilon)$, where $m_0(p) = 0$, $m_{k+1}(p) = m(m_k(p)) + p$ for $k = 0, 1, \ldots$ and $p \ge 0$; obviously $\lim_{p \to 0_+} m_n(p) = 0$.

Let u_n denote the mapping $v \to u_n(v)$ for $v \in W$. From (5) and (6) it follows that $u_n(W)$ is a relatively compact set in C. Since, by (9), u_n is continuous, it is completely continuous mapping $W \to W$. Furthermore, analogously as in [4], the inequality (7) becomes

$$\|u_n(v)(t) - u_m(v)(t)\| \le w_n(t) \text{ for } m \ge n,\ t \in J,\ v \in W, \tag{10}$$

where w_n is the maximal solution of $z'(t) = \min(\mu(t), h(t, z(t)) + 2\varepsilon(t, q_n))$ issuing from $(0,0)$. Since w_n uniformly converges to 0 as $n \to \infty$, from (10) we conclude that the sequence (u_n) converges uniformly on W to a limit u. By passing to the limit in (8), we have

$$u(v)(t) = \int_0^t f(s, u(v)(s), \tilde{g}(s, v))ds \quad \text{for } t \in J,\ v \in W. \tag{11}$$

Since u is the uniform limit of the sequence of completely continuous mappings u_n, u is a completely continuous mapping $W \to W$. Note that

$$u(v)'(t) = f(t, u(v)(t), \tilde{g}(t, v)) \text{ for } t \in J \text{ and } v \in W.$$

Now we shall show that for each $\varepsilon > 0$

$$v|_{[0,\varepsilon]} = z|_{[0,\varepsilon]} \Longrightarrow u(v)|_{[0,\varepsilon]} = u(z)|_{[0,\varepsilon]} \quad (v, z \in W). \tag{12}$$

Indeed, if $v, z \in W$ and $v(t) = z(t)$ for $t \in [0, \varepsilon]$, then $\tilde{g}(t, v) = \tilde{g}(t, z)$ for $t \in [0, \varepsilon]$, and hence

$$\begin{aligned} D_+ \|u(v)(t) - u(z)(t)\| &\le \|u(v)'(t) - u(z)'(t)\| \\ &= \|f(t, u(v)(t), \tilde{g}(t, v)) - f(t, u(z)(t), \tilde{g}(t, z))\| \\ &\le \min(\mu(t), h(t, \|u(v)(t) - u(z)(t)\|)) \text{ for } t \in [0, \varepsilon]. \end{aligned}$$

From the fact that $u(v)(0) = u(z)(0) = 0$ and h is a Kamke function, by Olech's Lemma ([4], Lemma 1) this implies $\|u(v)(t) - u(z)(t)\| = 0$ for $t \in [0, \varepsilon]$. This proves (12). We see that the mapping $v \to u(v)$ satisfies all asumptions of a Vidossich theorem ([10], Corollary 1.2). By applying this theorem, we conclude that the set Fix u is an R_δ. From (11) it is clear that Fix $u \subset S$. Conversely, let $v \in S$. Since f satisfies the Kamke condition, the Cauchy problem

$$z'(t) = f(t, z(t), \tilde{g}(t, v)),\ z(0) = 0 \tag{13}$$

has a unique solution $z = u(v)$. As v satisfies (13), we get $v = u(v)$, so that $v \in$Fix u. Thus $S =$Fix u which ends the proof.

3. Let α be the Kuratowski measure of noncompactness [2] in E.

THEOREM 2. *If there exist Lebesgue integrable functions $h : D \to R_+$ and $k : D^2 \to R_+$ such that*

$$\alpha(f(t, X \times Y)) \le h(t) \cdot \alpha(X) + \alpha(Y) \quad \textit{and} \tag{14}$$

$$\alpha(g(t, s, X)) \le k(t, s) \cdot \alpha(X) \tag{15}$$

for $t, s \in D$ and for each sets $X \subset B$ and $Y \subset E$, then the set S of all solutions of the problem (1)-(2), defined on J, is an R_δ.

Proof. Put

$$r(x) = \begin{cases} x & \text{for } x \in B, \\ \dfrac{bx}{\|x\|} & \text{for } x \in E \backslash B. \end{cases}$$

Then r is a continuous function $E \to B$ and

$$r(X) \subset \cup_{0 \leq \lambda \leq 1} \lambda X \quad \text{for} \quad X \subset E,$$

so that $\alpha(r(X)) \leq \alpha(X)$ for each bounded subset X of E. Consequently, putting

$$\bar{f}(t,x,z) = f(t, r(x), z),$$
$$\bar{g}(t,s,x) = g(t,s,r(x)) \quad (t,s \in D, z \in E),$$

we obtain bounded continuous functions $\bar{f}: D \times E^2 \to E$ and $\bar{g}: D^2 \times E \to E$, satisfying (14)-(15) for bounded subsets X of E, such that on J the problem (1)-(2) is equivalent to

$$x'(t) = \bar{f}(t, x(t), \tilde{g}(t,x)), \quad x(0) = 0.$$

Define a mapping F by

$$F(x)(t) = \int_0^t \bar{f}(s, x(s), \tilde{g}(s,x))ds \quad (t \in J, x \in C).$$

By Lebesgue dominated convergence theorem, from the considerations of Section 1 we deduce that F is a contiuous mapping $C \to C$. Moreover, $F(C)$ is an equiuniformly continuous subset of C, $F(x)(0) = 0$ for $x \in C$ and for each $\varepsilon > 0$, $x, y \in C$

$$x|_{[0,\varepsilon]} = y|_{[0,\varepsilon]} \Rightarrow F(x)|_{[0,\varepsilon]} = F(y)_{[0,\varepsilon]}.$$

From (3) and (4) it follows that

$$\|F(x)(t)\| \leq b \quad \text{for } x \in C.$$

Then a function $x \in C$ is a solution of (1)-(2) iff $x = F(x)$. Now we shall show that

(16) each sequence (x_n) in C such that $\lim\limits_{n \to \infty} \|x_n - F(x_n)\|_C = 0$

has a limit point.

Let (x_n) be a sequence in C such that

$$\lim_{n \to \infty} \|x_n - F(x_n)\|_C = 0. \tag{17}$$

Put $V = \{x_n : n \in N\}$ and $V(t) = \{x_n(t) : n \in N\}$. As $V \subset \{x_n - F(x_n) : n \in N\} + F(V)$, from (16) it follows that V is an equicontinuous set.

So the function $t \to v(t) = \alpha(V(t))$ is continuous on J. By (14)–(15) and Heinz's theorem ([3], Th. 2.1) we obtain

$$\begin{aligned}
\alpha\Big(\Big\{\tilde{g}(\tau, x_n) : n \in N\Big\}\Big) &= \alpha\Big(\Big\{\int_0^\tau g\big(\tau, s, r\big(x_n(s)\big)\big)ds : n \in N\Big\}\Big) \\
&\le 2\int_0^\tau \alpha\Big(\Big(g\{\tau, s, r\big(x_n(s)\}\Big)ds : n \in N\Big)\Big)ds \\
&= 2\int_0^\tau \alpha\Big(g\Big(\tau, s, r\Big(V\Big(s\Big)\Big)\Big)\Big)ds \\
&\le 2\int_0^\tau k(\tau, s) \cdot \alpha\Big(r\Big(V(s)\Big)\Big)ds \\
&\le 2\int_0^\tau k(\tau, s)\alpha\Big(V(s)\Big)ds \\
&= 2\int_0^\tau k(\tau, s) \cdot v(s)ds, \quad \text{for } \tau \in J, x \in B
\end{aligned}$$

and further

$$\begin{aligned}
\alpha\Big(\Big\{\int_0^t \tilde{g}(\tau, x_n)d\tau : n \in N\Big\}\Big) &\le 2\int_0^t \alpha\Big(\Big\{\tilde{g}(\tau, x_n) : n \in N\Big\}\Big)d\tau \\
&\le 4\int_0^t \Big(\int_0^\tau k(\tau, s)v(s)ds\Big)d\tau \text{ for } t \in J.
\end{aligned}$$

Therefore

$$\begin{aligned}
\alpha\Big(FV\Big(t\Big)\Big) &\le \alpha\Big(\Big\{\int_0^t \bar{f}(s, x_n\Big(s\Big), \tilde{g}(s, x_n))ds : n \in N\Big\}\Big) \\
&\le 2\int_0^t \alpha\Big(\Big\{\bar{f}(s, x_n\Big(s\Big), \tilde{g}(s, x_n)) : n \in N\Big\}\Big)ds \\
&\le 2\int_0^t \alpha\Big(\Big\{f(s, r\Big(V\Big(s\Big)\Big), \tilde{g}\Big(s, V\Big)\Big\}\Big)ds \\
&\le 2\int_0^t h(s)\alpha\Big(V\Big(s\Big)\Big)ds + \alpha\Big(\Big\{\int_0^t \tilde{g}(\tau, x_n)d\tau : n \in N\Big\}\Big) \\
&\le 2\int_0^t h(s)v\Big(s\Big)ds + 4\int_0^t \Big(\int_0^\tau k(\tau, s)v(s)ds\Big)d\tau \ \text{ for } t \in J.
\end{aligned}$$

Since $V(t) \subset \{x_n(t) - F(x_n)(t) : n \in N\} + F(V)(t)$ and

$\alpha(\{x_n(t) - F(x_n)(t) : n \in N\}) = 0$, we get $\alpha(V(t)) \leq \alpha(F(V)(t))$. This implies that

$$v(t) \leq \alpha(F(V)(t)) \leq 2\int_0^t h(s)v(s)ds + 4\int_0^t \Big(\int_s^t k(\tau, s)d\tau\Big)v(s)ds.$$

Then

$$v(t) \leq 2\int_0^t h(s)v(s)ds + 4\int_0^t q(t,s)v(s)ds \quad \text{for } t \in J, \tag{18}$$

where

$$q(t,s) = \int_s^t k(\tau, s)d\tau \quad \text{for } 0 \leq s \leq t \leq a.$$

The function $t \to q(t,s)$ is continuous and the function $s \to q(t,s)$ is integrable, because

$$\int_0^t q(t,s)ds = \int_0^t \Big(\int_s^t k(\tau,s)d\tau\Big)ds = \int_0^t \Big(\int_0^\tau k(\tau,s)ds\Big)d\tau.$$

As the function v is contiuous, from (17) we see that $\alpha(V(t)) = v(t) = 0$ for $t \in J$. Therefore the set $V(t)$ is relatively compact in E. Then by Ascoli's theorem, V is a relatively compact subset of C. This proves (14). Using now Th. 5 of [8], we find that the set S =Fix F is an R_δ. This completes the proof.

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