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UNIQUE SOLVABILITY OF CERTAIN NONLINEAR BOUNDARY VALUE PROBLEMS VIA A GLOBAL INVERSION THEOREM OF HADAMARD–LÉVY TYPE

Abstract. The paper deals with the nonlinear even-order boundary value problem

$$\begin{aligned} &u^{(2n)}(x) = f(x, u(x)), \qquad && x \in [0,1], \\ &u^{(2k)}(0) = u^{(2k)}(1) = 0, \qquad && 0 \le k \le n-1, \end{aligned}$$

where $f : [0,1] \times \mathbf{R} \to \mathbf{R}$ is continuous, while its partial derivative with respect to the second argument, denoted by f'_u, exists and is continuous on $[0,1] \times \mathbf{R}$. It is proved that if there exists a continuous nondecreasing function $\eta : \mathbf{R}_+ \to]-\infty, \pi^{2n}[$, such that

$$(-1)^n f'_u(x,u) \le \eta(|u|) < \pi^{2n} \qquad \text{for all } (x,u) \in [0,1] \times \mathbf{R}$$

and

$$\int_0^\infty \left(\pi^{2n} - \eta(x)\right) dx = \infty,$$

then the above problem has a unique solution.

1. Introduction

M. Lees [7, Theorem 1] established an existence-uniqueness theorem, concerning the two-point boundary value problem

$$\text{(1.1)} \qquad \begin{aligned} &u''(x) = f(x, u(x)), \qquad && x \in [0,1], \\ &u(0) = u(1) = 0. \end{aligned}$$

Namely, he proved the following

THEOREM 1.1. *Suppose that $f : [0,1] \times \mathbf{R} \to \mathbf{R}$ is a continuous function, whose partial derivative with respect to the second argument, denoted by f'_u, exists and is continuous on $[0,1] \times \mathbf{R}$. If there exists a real constant η such*

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that

$$f'_u(x,u) \geq -\eta > -\pi^2 \qquad \text{for all } (x,u) \in [0,1] \times \mathbf{R},$$

then the boundary value problem (1.1) *has a unique solution.*

Lees' result was refined by M. Rădulescu and S. Rădulescu [12, Theorem 6] as follows:

THEOREM 1.2. *Let* $f : [0,1] \times \mathbf{R} \to \mathbf{R}$ *be a continuous function, whose partial derivative with respect to the second argument, denoted by* f'_u, *exists and is continuous on* $[0,1] \times \mathbf{R}$. *If there exists a continuous nondecreasing function* $\eta : \mathbf{R}_+ \to \,]-\infty, \pi^2[$, *such that*

$$f'_u(x,u) \geq -\eta(|u|) > -\pi^2 \qquad \text{for all } (x,u) \in [0,1] \times \mathbf{R}$$

and

$$\int_0^\infty \left(\pi^2 - \eta(x)\right) dx = \infty,$$

then the boundary value problem (1.1) *has a unique solution.*

Another extention of Theorem 1.1 to a more general problem than (1.1) has been established by J. Tippett [13].

Now we turn our attention to the fourth-order boundary value problem

$$\begin{aligned} &u^{(4)}(x) = f(x, u(x)), \qquad x \in [0,1], \\ &u(0) = u(1) = u''(0) = u''(1) = 0. \end{aligned} \tag{1.2}$$

It describes the deformations of an elastic beam in equilibrium state, whose two ends are simply supported. Due to the fact that beams are used in structures such as aircraft, buildings, ships, and bridges, to find conditions ensuring the existence of at least one solution of (1.2) was the main purpose of numerous investigations (see, for instance, [2], [3], [8], [9], [11], [16]). We recall here only a few existence-uniqueness results concerning the problem (1.2). Thus, R. A. Usmani [15] and Y. Yang [16] dealt with a special case of (1.2). More precisely, they proved the following

THEOREM 1.3. *Let* $f : [0,1] \to \mathbf{R}$ *be a continuous function, such that*

$$f(x) < \pi^4 \qquad \text{for all } x \in [0,1].$$

Then for every continuous function $e : [0,1] \to \mathbf{R}$ *the boundary value problem*

$$\begin{aligned} &u^{(4)}(x) = f(x)u(x) + e(x), \qquad x \in [0,1], \\ &u(0) = u(1) = u''(0) = u''(1) = 0 \end{aligned}$$

has a unique solution.

This theorem was generalized by C. P. Gupta [4, Theorem 3.1] as follows:

THEOREM 1.4. *Let $f : [0,1] \times \mathbf{R} \to \mathbf{R}$ be a continuous function, whose partial derivative with respect to the second argument, denoted by f'_u, exists and is continuous on $[0,1] \times \mathbf{R}$. If there exists a real constant η such that*

$$f'_u(x,u) \le \eta < \pi^4 \qquad \text{for all } (x,u) \in [0,1] \times \mathbf{R},$$

then the boundary value problem (1.2) *has a unique solution.*

In fact, it should be mentioned that in [4] a much more general result than Theorem 1.4 is presented. On the other hand, for existence-uniqueness results regarding more general problems than (1.2), the reader is referred to [1], [4], [6].

Finally, we point out the following existence-uniqueness theorem established by C. P. Gupta [5, Theorem 3.2] for a special case of (1.2), when the elastic beam is at resonance:

THEOREM 1.5. *Suppose that $g : [0,1] \times \mathbf{R} \to \mathbf{R}$ satisfies Caratheodory's conditions for $L^1[0,1]$, that $g(x,\cdot)$ is strictly increasing on $\mathbf{R}$ for a.e. $x \in [0,1]$, and that $g(x,0) = 0$ for a.e. $x \in [0,1]$. Then for each continuous function $e : [0,1] \to \mathbf{R}$, satisfying $\int_0^1 e(x) \sin \pi x dx = 0$, the boundary value problem*

$$\begin{aligned} &u^{(4)}(x) = \pi^4 u(x) - g(x,u(x)) + e(x), \qquad x \in [0,1], \\ &u(0) = u(1) = u''(0) = u''(1) = 0 \end{aligned} \tag{1.3}$$

has a unique solution.

The main purpose of the present paper is to prove a common generalization of Theorems 1.1–1.4. It refers to the even-order two-point boundary value problem

$$\begin{aligned} &u^{(2n)}(x) = f(x,u(x)), \qquad x \in [0,1], \\ &u^{(2k)}(0) = u^{(2k)}(1) = 0, \qquad 0 \le k \le n-1, \end{aligned} \tag{1.4}$$

generalizing (1.1), (1.2), and (1.3).

2. Unique solvability of the boundary value problem (1.4)

In the proof of our main result we will use the following global inversion theorem of Hadamard-Lévy type established by M. Rădulescu and S. Rădulescu (cf. [12, Theorem 2]):

THEOREM 2.1. *Let $(Y, \|\cdot\|_0)$ be a Banach space and let $L : D(L) \to Y$ be a linear operator with closed graph, where $D(L)$ is a linear subspace of Y. Then $D(L)$ is a Banach space with respect to the norm defined by*

$$\|u\|_1 := \|u\|_0 + \|L(u)\|_0, \qquad u \in D(L).$$

Further, let X be a linear subspace of D(L) which is closed in the norm $\|\cdot\|_1$, *let* $N : (X, \|\cdot\|_1) \to (Y, \|\cdot\|_0)$ *be a* C^1 *mapping, and* $S : (X, \|\cdot\|_1) \to (Y, \|\cdot\|_0)$ *be the mapping defined by* $S := L - N$. *If S is a local diffeomorphism and there exists a continuous function* $c : \mathbf{R}_+ \to \,]0, \infty[$, *such that*

$$\|S'(u)(h)\|_0 \geq c(\|u\|_0)\|h\|_0, \qquad \text{for all } u, h \in X$$

and

$$\int_0^\infty c(x)dx = \infty,$$

then S is a global diffeomorphism.

Besides, we need several inequalities, contained in the following lemma.

LEMMA 2.2. *If* $u : [0,1] \to \mathbf{R}$ *is a continuously differentiable function such that* $u(0) = u(1) = 0$, *then*

$$\|u'\|_2 \geq \pi\|u\|_2, \tag{2.1}$$

$$\|u'\|_2 \geq 2\|u\|_0. \tag{2.2}$$

In the above lemma (and throughout the rest of the paper)

$$\|u\|_0 := \sup\{\, |u(x)| \mid x \in [0,1]\,\}$$

denotes the usual sup-norm of an arbitrary function $u \in C([0,1], \mathbf{R})$, while

$$\|u\|_2 := \left(\int_0^1 u^2(x)dx\right)^{1/2}$$

denotes the L^2-norm of an arbitrary function $u \in L^2([0,1], \mathbf{R})$. Inequality (2.1) is known as the Wirtinger inequality, while inequality (2.2) is known as the Lees inequality.

LEMMA 2.3. *If n is a positive integer and* $u : [0,1] \to \mathbf{R}$ *is a 2n-times continuously differentiable function satisfying*

$$u^{(2k)}(0) = u^{(2k)}(1) = 0 \qquad \text{for all } 0 \leq k \leq n-1, \tag{2.3}$$

then for each $k \in \{1, 2, \ldots, 2n\}$ *it holds that*

$$\|u^{(k)}\|_2 \geq \pi^{k-1}\|u'\|_2. \tag{2.4}$$

Proof. For $k = 1$ inequality (2.4) holds with equality. Let $m \leq 2n$ be an arbitrary positive integer. Assuming that (2.4) holds true for all positive integers $k < m$, let us prove that it holds also for $k = m$. Depending on m, we have two possible cases.

Case I. m is even: $m = 2p$.

Integrating by parts p times and taking into account (2.3), we get

$$\int_0^1 u(x)u^{(m)}(x)dx = (-1)^p \int_0^1 \left[u^{(p)}(x)\right]^2 dx = (-1)^p \|u^{(p)}\|_2^2.$$

Using (2.1), the Cauchy-Schwarz inequality, as well as the induction hypothesis, we deduce that

$$\|u\|_2^2 \|u^{(m)}\|_2^2 \geq \Big(\int_0^1 u(x)u^{(m)}(x)dx\Big)^2 = \|u^{(p)}\|_2^4 \geq \pi^{4p-4} \left\|u'\right\|_2^4$$
$$\geq \pi^{4p-2} \left\|u'\right\|_2^2 \|u\|_2^2,$$

whence $\left\|u^{(m)}\right\|_2 \geq \pi^{2p-1} \|u'\|_2 = \pi^{m-1} \|u'\|_2$.

Case II. m is odd: $m = 2p-1$.

Integrating by parts $p-1$ times and taking into account (2.3), we get

$$\int_0^1 u'(x)u^{(m)}(x)dx = (-1)^{p-1} \int_0^1 \left[u^{(p)}(x)\right]^2 dx = (-1)^{p-1} \|u^{(p)}\|_2^2.$$

Using again the Cauchy-Schwarz inequality and the induction hypothesis, we find that

$$\|u'\|_2^2 \|u^{(m)}\|_2^2 \geq \Big(\int_0^1 u'(x)u^{(m)}(x)dx\Big)^2 = \|u^{(p)}\|_2^4 \geq \pi^{4p-4} \left\|u'\right\|_2^4,$$

whence $\left\|u^{(m)}\right\|_2 \geq \pi^{2p-2} \|u'\|_2 = \pi^{m-1} \|u'\|_2$. ∎

LEMMA 2.4. *Let n be a positive integer, let*

$$X := \{\, h \in C^{2n}([0,1], \mathbf{R}) \mid h^{(2k)}(0) = h^{(2k)}(1) = 0,\ k = 0, 1, \dots, n-1 \,\},$$

and let $v : [0,1] \to \mathbf{R}$ be a continuous function for which there exists a real constant b such that

(2.5) $$(-1)^n v(x) \leq b < \pi^{2n} \qquad \textit{for all } x \in [0,1].$$

If $A : X \to C([0,1], \mathbf{R})$ is the operator defined by

$$A(h)(x) := h^{(2n)}(x) - v(x)h(x), \qquad h \in X,\ x \in [0,1],$$

then it holds that

(2.6) $$\|A(h)\|_0 \geq \frac{2\left(\pi^{2n} - b\right)}{\pi} \|h\|_0 \qquad \textit{for all } h \in X.$$

Proof. Let $h \in X$ be arbitrarily chosen. Integrating by parts n times, we get

$$\int_0^1 h(x)A(h)(x)dx = (-1)^n \|h^{(n)}\|_2^2 - \int_0^1 v(x)h^2(x)dx.$$

Using now (2.1), (2.4), (2.5), and the Cauchy-Schwarz inequality, we deduce that

$$\begin{aligned}\|h\|_2\|A(h)\|_2 &\geq (-1)^n \int_0^1 h(x)A(h)(x)dx \\ &= \|h^{(n)}\|_2^2 - \int_0^1 (-1)^n v(x)h^2(x)dx \\ &\geq \pi^{2n-2} \|h'\|_2^2 - b\int_0^1 h^2(x)dx \\ &\geq \pi^{2n-1}\|h\|_2 \|h'\|_2 - b\|h\|_2^2,\end{aligned}$$

whence

$$\|A(h)\|_2 \geq \pi^{2n-1} \|h'\|_2 - b\|h\|_2.$$

Using again (2.1) we have

$$\|A(h)\|_2 \geq \pi^{2n-1} \|h'\|_2 - \frac{b}{\pi} \|h'\|_2 = \frac{\pi^{2n} - b}{\pi} \|h'\|_2. \tag{2.7}$$

Taking into account that $\|A(h)\|_0 \geq \|A(h)\|_2$, from (2.2) and (2.7) it follows that (2.6) holds true. ∎

Now we are ready to state and prove the main result of the paper, concerning the unique solvability of the boundary value problem (1.4).

THEOREM 2.5. *Let n be a positive integer and let $f : [0,1] \times \mathbf{R} \to \mathbf{R}$ be a continuous function, whose partial derivative with respect to the second argument, denoted by f'_u, exists and is continuous on $[0,1] \times \mathbf{R}$. If there exists a continuous nondecreasing function $\eta : \mathbf{R}_+ \to \ ]-\infty, \pi^{2n}[$, such that*

$$(-1)^n f'_u(x,u) \leq \eta(|u|) < \pi^{2n} \qquad \text{for all } (x,u) \in [0,1] \times \mathbf{R} \tag{2.8}$$

and

$$\int_0^\infty \left(\pi^{2n} - \eta(x)\right) dx = \infty, \tag{2.9}$$

then the boundary value problem (1.4) *has a unique solution.*

Proof. Let $Y := C([0,1], \mathbf{R})$ be endowed with the usual sup-norm $\|\cdot\|_0$ and let $L : D(L) \to Y$ be the linear operator defined by

$$L(u)(x) := u^{(2n)}(x), \qquad u \in D(L),\ x \in [0,1],$$

where $D(L) := C^{2n}([0,1], \mathbf{R})$. Since L has closed graph (see Lemma 2.6 below), by Theorem 2.1 we conclude that $D(L)$ is a Banach space with respect to the norm defined by

$$\|u\|_1 := \|u\|_0 + \|u^{(2n)}\|_0, \qquad u \in D(L).$$

Further, let

$$X := \{ h \in C^{2n}([0,1], \mathbf{R}) \mid h^{(2k)}(0) = h^{(2k)}(1) = 0, \ k = 0, 1, \dots, n-1 \}.$$

From the proof of Lemma 2.6 it follows that X is a closed subspace of $D(L)$ in the norm $\| \cdot \|_1$. Consider now the nonlinear operators $N, S : X \to Y$, defined by

$$N(u)(x) := f(x, u(x)), \qquad u \in X, \ x \in [0,1],$$
$$S(u)(x) := u^{(2n)}(x) - f(x, u(x)), \qquad u \in X, \ x \in [0,1].$$

The regularity assumption on f ensures that S is of class C^1. Moreover, we have

$$S'(u)(h)(x) = h^{(2n)}(x) - f'_u(x, u(x))h(x)$$

for all $u, h \in X$ and all $x \in [0,1]$.

We claim that

$$\|S'(u)(h)\|_0 \geq \frac{2\left(\pi^{2n} - \eta(\|u\|_0)\right)}{\pi} \|h\|_0 \qquad \text{for all } u, h \in X. \tag{2.10}$$

To see this, let $u \in X$ be arbitrarily chosen and let $v : [0,1] \to \mathbf{R}$ be the function defined by $v(x) := f'_u(x, u(x))$, $x \in [0,1]$. According to (2.8), we have

$$(-1)^n v(x) \leq \eta(|u(x)|) \leq \eta(\|u\|_0) < \pi^{2n} \qquad \text{for all } x \in [0,1].$$

Applying Lemma 2.4 with $b = \eta(\|u\|_0)$, we conclude that (2.10) holds true.

Next we show that S is a local diffeomorphism. To this end, let us fix $u \in X$ and let $Q : X \to Y$ be the operator defined by

$$Q(h)(x) := v(x)h(x), \qquad h \in X, \ x \in [0,1],$$

where $v : [0,1] \to \mathbf{R}$ is the function defined by $v(x) := f'_u(x, u(x))$, $x \in [0,1]$. By induction on n, it is not difficult to prove that $L : X \to Y$ is invertible and its inverse $L^{-1} : Y \to X$ is given by

$$L^{-1}(z)(x) = \tag{2.11}$$
$$= \int_0^1 \int_0^1 \cdots \int_0^1 G(x, x_1) G(x_1, x_2) \cdots G(x_{n-1}, x_n) z(x_n) dx_1 dx_2 \cdots dx_n$$

for all $z \in Y$ and all $x \in [0,1]$, where $G : [0,1] \times [0,1] \to \mathbf{R}$ denotes the Green function

$$G(x, s) := \begin{cases} x(1-s) & \text{if } 0 \leq x \leq s \leq 1 \\ s(1-x) & \text{if } 0 \leq s \leq x \leq 1. \end{cases}$$

Formula (2.11) together with the Arzelá-Ascoli theorem ensure that the operator $L^{-1} : (Y, \| \cdot \|_0) \to (X, \| \cdot \|_0)$ is compact, whence $K := L^{-1}Q$ is compact, too. Remark that

$$S'(u) = L - Q = L(I - K).$$

From (2.10) it follows that $ker(S'(u)) = \{0\}$, hence $ker(I - K) = \{0\}$, because L is invertible. By the Fredholm alternative, we deduce that $I - K$ is onto. Consequently, $S'(u)$ is bijective. Applying now the local inversion theorem, we conclude that S is a local diffeomorphism around u.

From (2.9) and (2.10) it follows that the function $c : \mathbf{R}_+ \to]0, \infty[$, defined by $c(x) := \frac{2(\pi^{2n} - \eta(x))}{\pi}$, $x \in [0, 1]$, satisfies the hypotheses of Theorem 2.1. Therefore, S is a global diffeomorphism and, consequently, the equation $S(u) = 0$ has a unique solution $u \in X$. This is also the unique solution of the boundary value problem (1.4). ∎

LEMMA 2.6. *The graph of the linear operator L, defined in the proof of Theorem 2.5, is closed in the Banach space $Y \times Y$.*

Proof. We make use of the following result by A. Gorny (see [10, pp. 138–139]): let $h \in C^p([0, 1], \mathbf{R})$ and let $M_j := \|h^{(j)}\|_0$, $0 \le j \le p$. Then for all $1 \le j \le p - 1$ it holds that

$$M_j \le 4e^{2j} \binom{p}{j}^j M_0^{1-(j/p)} \bar{M}_p^{j/p},$$

where $\bar{M}_p := \max\{ M_p, p! M_0 \}$. This inequality and the arithmetic-geometric mean inequality ensure that

$$\begin{aligned} M_j &\le 4e^{2j} \binom{p}{j}^j M_0^{1-(j/p)} [M_p^{j/p} + (p! M_0)^{j/p}] \\ &\le 4e^{2j} \binom{p}{j}^j (p!)^{j/p} \left(M_0 + M_0^{1-(j/p)} M_p^{j/p} \right) \\ &\le 4e^{2j} \binom{p}{j}^j (p!)^{j/p} \left[M_0 + \left(1 - \frac{j}{p}\right) M_0 + \frac{j}{p} M_p \right], \end{aligned}$$

whence

$$M_j \le C_{p,j}(M_0 + M_p) \qquad \text{for all } 1 \le j \le p - 1, \tag{2.12}$$

where $C_{p,j} := 8e^{2j} \binom{p}{j}^j (p!)^{j/p}$.

Passing now to the subject of Lemma 2.6, let (u, v) be an arbitrary point in the closure of the graph of L. Then there exists a sequence (u_k) in $D(L) = C^{2n}([0, 1], \mathbf{R})$ such that the sequence $((u_k, L(u_k)))$ converges to (u, v) in $Y \times Y$. This means that the sequences (u_k) and $\left(u_k^{(2n)}\right)$ converge uniformly to u and v, respectively. According to (2.12), we have

$$\|u_k^{(j)} - u_\ell^{(j)}\|_0 \le C_{2n,j}(\|u_k - u_\ell\|_0 + \|u_k^{(2n)} - u_\ell^{(2n)}\|_0)$$

for all positive integers k, ℓ and all $1 \le j \le 2n-1$. Consequently, there exist the functions $v_1, \dots, v_{2n-1}$ such that the sequence $\left(u_k^{(j)}\right)$ converges uniformly to v_j for all $1 \le j \le 2n-1$. A standard theorem ensures now that all the functions $u, v_1, \dots, v_{2n-1}$ are differentiable and that

$$\begin{aligned} u' &= v_1, \\ v_j' &= v_{j+1}, \qquad \text{for } 1 \le j \le 2n-2 \\ v_{2n-1}' &= v. \end{aligned}$$

Therefore, $v = u^{(2n)} = L(u)$, hence (u, v) belongs to the graph of L. ∎

The result of Theorem 2.5 is interesting especially at resonance. We illustrate this remark by the following

COROLLARY 2.7. *Let* $g : [0,1] \times \mathbf{R} \to \mathbf{R}$ *be a continuous function, whose partial derivative with respect to the second argument, denoted by* g_u', *exists and is continuous on* $[0,1] \times \mathbf{R}$. *If there exists a continuous nonincreasing function* $\omega : \mathbf{R}_+ \to]0, \infty[$, *such that*

$$g_u'(x,u) \ge \omega(|u|) \qquad \textit{for all} \ \ (x,u) \in [0,1] \times \mathbf{R}$$

and $\int_0^\infty \omega(x)dx = \infty$, *then for every continuous function* $e : [0,1] \to \mathbf{R}$ *the boundary value problem* (1.3) *has a unique solution.*

Proof. Apply Theorem 2.5 for $n = 2$ and $f(x,u) = \pi^4 u - g(x,u) + e(x)$. Then

$$f_u'(x,u) = \pi^4 - g_u'(x,u) \le \pi^4 - \omega(|u|) \qquad \text{for all} \ \ (x,u) \in [0,1] \times \mathbf{R},$$

so all the assumptions of Theorem 2.5 are fulfilled if $\eta : \mathbf{R}_+ \to]-\infty, \pi^4[$ is the function defined by $\eta(u) := \pi^4 - \omega(u)$. ∎

Comparing Corollary 2.7 with Theorem 1.5 we point out that none of them can be derived from the other. Indeed, if $g(x,u) = \arctan u$, then Theorem 1.5 can be applied but not Corollary 2.7. On the other hand, for

$$g(x,u) := \begin{cases} u+1 & \text{if } (x,u) \in [0,1] \times \ ]-\infty, 1[\\ 2 + \ln u & \text{if } (x,u) \in [0,1] \times [1, \infty[, \end{cases}$$

the hypotheses of Corollary 2.7 are satisfied if ω is defined by

$$\omega(u) := \begin{cases} 1 & \text{if } u \in [0,1] \\ 1/u & \text{if } u \in]1, \infty[. \end{cases}$$

Consequently, Corollary 2.7 can be applied but neither Theorem 1.5 nor Theorem 1.4. ∎

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