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A CLASS OF RATIONAL MAPPINGS IN SEVERAL COMPLEX VARIABLES

Abstract. In this paper, we will give coefficient conditions for mappings of the form $f(z) = z/(1+\sum_{k=1}^{\infty} b_k z_1^k)$ to be starlike or convex on the Euclidean unit ball B in $\mathbb{C}^n$. Our results give concrete examples of strongly starlike mappings of order α, starlike mappings of order α and convex mappings on B.

1. Introduction

In this paper, we are concerned with finding coefficient conditions for mappings of the form

$$f(z) = \frac{z}{1 + \sum_{k=1}^{\infty} b_k z_1^k} \tag{1.1}$$

to be starlike or convex on the Euclidean unit ball B in $\mathbb{C}^n$.

Reade-Silverman-Todorov [8], Obradović [5] and Obradović-Ponnusamy-Singh-Vasundhra [7] gave coefficient conditions for functions of the form (1.1) to be starlike of order α $(0 \le \alpha \le 1)$ or convex of order β $(0 \le \beta < 1)$ on the unit disc U in $\mathbb{C}$.

In this paper, we will generalize the above results to mappings of the form (1.1) on the Euclidean unit ball B in $\mathbb{C}^n$. Our results give concrete examples of strongly starlike mappings of order α, starlike mappings of order α and convex mappings on B.

2. Preliminaries

Let $\mathbb{C}^n$ denote the space of n complex variables $z = (z_1, \dots, z_n)$ with the Euclidean inner product $\langle z, w \rangle = \sum_{j=1}^{n} z_j \overline{w}_j$ and the Euclidean norm $\|z\| = \langle z, z \rangle^{1/2}$.

Let $z' = (z_2, \dots, z_n)$ so that $z = (z_1, z')$. Let $B = \{z \in \mathbb{C}^n : \|z\| < 1\}$ be the Euclidean unit ball in $\mathbb{C}^n$. In the case of one complex variable, B

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is denoted by U. If $G \subset \mathbb{C}^n$ is an open set, let $H(G)$ denote the set of holomorphic mappings from G into $\mathbb{C}^n$. If $f \in H(B)$, we say that f is normalized if $f(0) = 0$ and $Df(0) = I$.

A mapping $f \in H(B)$ with $f(0) = 0$ is called starlike if f is biholomorphic on B and $f(B)$ is a starlike domain with respect to zero.

A mapping $f \in H(B)$ is called convex if f is biholomorphic on B and $f(B)$ is a convex domain.

DEFINITION 1. A normalized locally biholomorphic mapping f on B is said to be strongly starlike of order $\alpha \in (0,1]$, if

$$\left|\arg \langle [Df(z)]^{-1} f(z), z \rangle\right| < \alpha \frac{\pi}{2}, \quad z \in B \setminus \{0\}.$$

DEFINITION 2. A normalized locally biholomorphic mapping f on B is said to be starlike of order $\alpha \in (0,1]$, if

$$\left| \frac{1}{\|z\|^2} \langle [Df(z)]^{-1} f(z), z \rangle - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha}, \quad z \in B \setminus \{0\}.$$

In this paper, we will consider mappings of the form

$$f(z) = \frac{z}{\phi(z_1)}, \tag{2.1}$$

where

$$\phi(z_1) = 1 + \sum_{k=1}^{\infty} b_k z_1^k. \tag{2.2}$$

Then

$$Df(z) = \begin{bmatrix} \frac{\phi - z_1\phi'}{\phi^2} & 0 \\ -\frac{\phi'}{\phi^2} z' & \frac{1}{\phi} I_{n-1} \end{bmatrix}$$

and

$$[Df(z)]^{-1} = \begin{bmatrix} \frac{\phi^2}{\phi - z_1\phi'} & 0 \\ \frac{\phi\phi'}{\phi - z_1\phi'} z' & \phi I_{n-1} \end{bmatrix}.$$

Therefore, we obtain

$$[Df(z)]^{-1} f(z) = \frac{\phi(z_1)}{\phi(z_1) - z_1\phi'(z_1)} z$$

and

$$\langle [Df(z)]^{-1} f(z), z \rangle = \frac{\phi(z_1)}{\phi(z_1) - z_1\phi'(z_1)} \|z\|^2. \tag{2.3}$$

Also, we have

$$D^2 f(z)(x,x) = \begin{bmatrix} \frac{-z_1\phi\phi'' - 2(\phi - z_1\phi')\phi'}{\phi^3} x_1^2 \\ \frac{-\phi\phi'' + 2(\phi')^2}{\phi^3} x_1^2 z' - \frac{2\phi'}{\phi^2} x_1 x' \end{bmatrix}.$$

Then,

$$[Df(z)]^{-1}D^2f(z)(x,x) = -\frac{\phi'' x_1^2}{\phi - z_1\phi'}z - \frac{2\phi' x_1}{\phi}x.$$

Therefore, we have

$$\langle [Df(z)]^{-1}D^2f(z)(x,x), z\rangle = -\frac{\phi'' x_1^2}{\phi - z_1\phi'}\|z\|^2 - \frac{2\phi' x_1}{\phi}\langle x, z\rangle. \tag{2.4}$$

3. Sufficient conditions for starlikeness

Let f be a holomorphic function of the form (2.1) and (2.2) on the unit disc U in $\mathbb{C}$. Obradović [5, Theorem 3] gave coefficient conditions in terms of b_k's for f to be starlike on U. We will generalize the result to mappings of the form (2.1) and (2.2) on the Euclidean unit ball B in $\mathbb{C}^n$ and give coefficient conditions in terms of b_k's for f to be strongly starlike of order α on B.

THEOREM 3.1. *Let $0 < \alpha \le 1$. Let f be a holomorphic mapping of the form* (2.1) *and* (2.2) *on B. If*

$$\sum_{k=2}^{\infty}(k-1)|b_k| \le M$$

and

$$\sum_{k=1}^{\infty}|b_k| \le \sqrt{1-M^2}\sin\left(\frac{\pi\alpha}{2}\right) - M\cos\left(\frac{\pi\alpha}{2}\right)$$

for some $0 < M < \sin(\pi\alpha/2)$, then f is strongly starlike of order α on B.

Proof. Let

$$L = \sqrt{1-M^2}\sin\left(\frac{\pi\alpha}{2}\right) - M\cos\left(\frac{\pi\alpha}{2}\right).$$

Since

$$|\phi - 1| < \sum_{k=1}^{\infty}|b_k| \le L$$

and

$$|\phi - z_1\phi' - 1| < \sum_{k=2}^{\infty}(k-1)|b_k| \le M,$$

we have

$$\left|\arg\langle [Df(z)]^{-1}f(z), z\rangle\right| \le |\arg\phi| + |\arg(\phi - z_1\phi')| < \arcsin L + \arcsin M$$

from (2.3). We use an argument similar to that in the proof of [6, Corollary 1.10]. If $x^2 + y^2 \le 1$, $x \ge 0$ and $y \ge 0$, then

$$\cos(\arcsin x + \arcsin y) = \sqrt{1-x^2}\sqrt{1-y^2} - xy \ge 0.$$

Therefore, $0 \le \arcsin x + \arcsin y \le \pi/2$ and we have

$$\arcsin x + \arcsin y = \arcsin(x\sqrt{1-y^2} + y\sqrt{1-x^2}).$$

Since

$$1 - (L^2 + M^2) = \left(\sqrt{1-M^2}\cos\left(\frac{\pi\alpha}{2}\right) + M\sin\left(\frac{\pi\alpha}{2}\right)\right)^2 \ge 0$$

and

$$M\sqrt{1-L^2} + L\sqrt{1-M^2} = \sin(\pi\alpha/2),$$

we obtain that

$$\left|\arg\left\langle [Df(z)]^{-1}f(z), z\right\rangle\right| < \arcsin\left(\sin\left(\frac{\pi\alpha}{2}\right)\right) = \frac{\pi\alpha}{2}.$$

Thus, f is strongly starlike of order α. This completes the proof. ∎

Obradović-Ponnusamy-Singh-Vasundhra [7, Theorem 4.3] also obtained coefficient conditions in terms of b_k's for functions f of the form (2.1) and (2.2) to be starlike on U. We will generalize the result to mappings of the form (2.1) and (2.2) on the Euclidean unit ball B in $\mathbb{C}^n$ and give coefficient conditions in terms of b_k's for f to be strongly starlike of order α on B.

THEOREM 3.2. *Assume that* $\sum_{k=2}^{\infty}(k-1)|b_k| \le \lambda$. *Let* f *be a holomorphic mapping of the form* (2.1) *and* (2.2) *on* B. *Then* f *is strongly starlike of order* α *on* B *for*

$$(3.1)\qquad 0 < \lambda \le \frac{-|b_1| + \sqrt{|b_1|^2 + 2(\sin^2(\pi\alpha/2) - |b_1|^2)/(1+\cos(\pi\alpha/2))}}{2}.$$

Proof. If (3.1) holds, then we have

$$2\lambda^2 + 2|b_1|\lambda - \frac{\sin^2(\pi\alpha/2) - |b_1|^2}{1+\cos(\pi\alpha/2)} \le 0.$$

This condition is equivalent to the condition

$$(3.2)\qquad \lambda\cos\left(\frac{\pi\alpha}{2}\right) \le \sqrt{1-\lambda^2}\sin\left(\frac{\pi\alpha}{2}\right) - (|b_1| + \lambda).$$

Therefore, we have

$$\sum_{k=1}^{\infty}|b_k| \le |b_1| + \lambda \le \sqrt{1-\lambda^2}\sin\left(\frac{\pi\alpha}{2}\right) - \lambda\cos\left(\frac{\pi\alpha}{2}\right).$$

Moreover, the inequality

$$\lambda \le \sqrt{1-\lambda^2}\sin\left(\frac{\pi\alpha}{2}\right) - \lambda\cos\left(\frac{\pi\alpha}{2}\right)$$

implies that

$$\lambda \le \frac{\sin(\pi\alpha/2)}{\sqrt{2+2\cos(\pi\alpha/2)}}.$$

Thus, we obtain Theorem 3.2 from Theorem 3.1. This completes the proof. ∎

Reade-Silverman-Todorov [8, Theorem 2] obtained coefficient conditions in terms of b_k's for functions f of the form (2.1) and (2.2) to be starlike of order α on U. We will generalize the result to mappings of the form (2.1) and (2.2) on the Euclidean unit ball B in $\mathbb{C}^n$.

THEOREM 3.3. *Let f be a holomorphic mapping of the form* (2.1) *and* (2.2) *on B. If*

$$(3.3)\qquad \sum_{k=2}^{\infty}(k-1+\alpha)|b_k| \le \begin{cases} (1-\alpha)-(1-\alpha)|b_1|, & 0<\alpha\le 1/2,\\ (1-\alpha)-\alpha|b_1|, & 1/2<\alpha\le 1,\end{cases}$$

then f is starlike of order α on B.

Proof. Put

$$g(z_1)=\frac{\phi(z_1)}{\phi(z_1)-z_1\phi'(z_1)}.$$

Then, from (2.3), f is starlike of order α if and only if

$$\left|g(z_1)-\frac{1}{2\alpha}\right|<\frac{1}{2\alpha}.$$

This condition is equivalent to

$$\left|\frac{1-(1/g-\alpha)/(1-\alpha)}{1+(1/g-\alpha)/(1-\alpha)}\right|<1.$$

Let

$$q(z_1)=\frac{1-(1/g(z_1)-\alpha)/(1-\alpha)}{1+(1/g(z_1)-\alpha)/(1-\alpha)}.$$

Then

$$q(z_1)=\frac{\sum_{k=1}^{\infty}kb_kz_1^k}{2(1-\alpha)+(1-2\alpha)b_1z_1-\sum_{k=2}^{\infty}(k-2+2\alpha)b_kz_1^k}.$$

Therefore, if the condition (3.3) holds, then we have $|q(z_1)|<1$. This completes the proof. ∎

As corollaries to Theorems 3.1, 3.2 and 3.3, we obtain the following theorem.

THEOREM 3.4. *Let f be a holomorphic mapping of the form* (2.1) *and* (2.2) *on B. If one of the following conditions is satisfied, then f is starlike on B.*

(1) $\sum_{k=2}^{\infty}(k-1)|b_k|\le(-|b_1|+\sqrt{2-|b_1|^2})/2$;

(2) $\sum_{k=2}^{\infty}(k-1)|b_k|\le 1-|b_1|$;

(3) $\sum_{k=2}^{\infty}(k-1)|b_k|\le M$ *and* $\sum_{k=1}^{\infty}|b_k|\le\sqrt{1-M^2}$ *for some* $0<M<1$.

REMARK 1. Since

$$\frac{-|b_1|+\sqrt{2-|b_1|^2}}{2}\le 1-|b_1|$$

for $0 \le |b_1| < 1$, the condition (2) of Theorem 3.4 is weaker than the condition (1) of Theorem 3.4. Also, from the proof of Theorem 3.2, the condition (3) of Theorem 3.4 is weaker than the condition (1) of Theorem 3.4.

We will give examples which show that the assumptions of Theorem 3.4 are not sharp.

EXAMPLE 1. (i) Let

$$f_1(z) = \frac{z}{1 + b_1 z_1 + (\lambda/2) z_1^3},$$

where $\lambda > 0$ is a constant such that

$$b_1 = \sqrt{1-\lambda^2} \sin\left(\frac{\pi\alpha}{2}\right) - \lambda \cos\left(\frac{\pi\alpha}{2}\right) - \frac{\lambda}{2} > 0.$$

Then, we have

$$\lambda \cos\left(\frac{\pi\alpha}{2}\right) - \sqrt{1-\lambda^2} \sin\left(\frac{\pi\alpha}{2}\right) + (|b_1| + \lambda) = \frac{\lambda}{2} > 0.$$

Thus, from (3.2), we obtain that f_1 does not satisfy the condition of Theorem 3.2. However, since

$$\sum_{k=2}^{\infty} (k-1)|b_k| = \lambda$$

and

$$\sum_{k=1}^{\infty} |b_k| = \sqrt{1-\lambda^2} \sin\left(\frac{\pi\alpha}{2}\right) - \lambda \cos\left(\frac{\pi\alpha}{2}\right),$$

f_1 satisfies the assumptions of Theorem 3.1. So, the assumption of Theorem 3.1 is strictly weaker than that of Theorem 3.2. Especially, the condition (3) in Theorem 3.4 is strictly weaker than that of (1). This implies that the assumption of Theorem 3.2 and the condition (1) in Theorem 3.4 are not sharp.

(ii) Let

$$f_2(z) = \frac{z}{1 + b_1 z_1 + b_2 z_1^2},$$

where $|b_1| + |b_2| = 1$ and $b_2 \neq 0$. Then f_2 does not satisfy the condition (3) of Theorem 3.4 and satisfies the condition (2) of Theorem 3.4. This implies that the condition (3) of Theorem 3.4 is not sharp. Also, let

$$f_3(z) = \frac{2z}{2 + z_1 + a z_1^3},$$

where $1/2 < a \le 3/5$. Then f_3 does not satisfy the condition (2) of Theorem 3.4 and satisfies the condition (3) of Theorem 3.4 by [5, Remark 1]. This implies that the condition (2) of Theorem 3.4 is not sharp.

REMARK 2. The results in this section can be generalized to holomorphic mappings of the form (2.1) and (2.2) on the unit ball $\mathbf{B}$ with respect to an arbitrary norm on $\mathbb{C}^n$ such that $\mathbf{B} \subset U \times \mathbb{C}^{n-1}$. Also, the results in this section can be generalized to holomorphic mappings of the form (2.1) and (2.2) on the unit ball in a complex Hilbert space H with inner product $\langle \cdot, \cdot \rangle$. In this case, we define $z_1 = \langle z, u \rangle$, where u is a unit vector in H.

4. A sufficient condition for convexity

Reade-Silverman-Todorov [8, Theorem 4] obtained coefficient conditions in terms of b_k's for functions f of the form (2.1) and (2.2) to be convex on U. The following theorem generalizes their result to several complex variables.

THEOREM 4.1. *Let f be a holomorphic mapping of the form* (2.1) *and* (2.2) *on B. If there exist p, $q > 0$ with $1/p + 1/q \leq 1$ such that*

$$\max\left\{ \sum_{k=1}^{\infty}(2kp+1)|b_k|, \sum_{k=2}^{\infty}(k-1)(kq+1)|b_k| \right\} \leq 1,$$

then f is convex on B.

Proof. By Kikuchi [4, Theorem 2.1], Gong-Wang-Yu [1, Theorem 2] (cf. [2], [3]), f is convex if and only if

$$1 - \Re\langle [Df(z)]^{-1} D^2 f(z)(x,x), z \rangle \geq 0 \tag{4.1}$$

for all $x \in \mathbb{C}^n$ with $\|x\| = 1$ and $z \in B \setminus \{0\}$ with $\Re\langle x, z \rangle = 0$. From (2.4), we have

$$1 - \langle [Df(z)]^{-1} D^2 f(z)(x,x), z \rangle = 1 + \frac{\phi'' x_1^2}{\phi - z_1 \phi'} \|z\|^2 + \frac{2\phi' x_1}{\phi} \langle x, z \rangle.$$

Since

$$\left| \frac{\phi'' x_1^2}{\phi - z_1 \phi'} \right| \|z\|^2 \leq \frac{\sum_{k=2}^{\infty} k(k-1)|b_k|}{1 - \sum_{k=2}^{\infty}(k-1)|b_k|} \leq \frac{1}{q}$$

and

$$\left| \frac{2\phi' x_1}{\phi} \langle x, z \rangle \right| \leq \frac{2\sum_{k=1}^{\infty} k|b_k|}{1 - \sum_{k=1}^{\infty} |b_k|} \leq 1/p,$$

we obtain the inequality (4.1). This completes the proof. ∎

REMARK 3. Theorem 4.1 can be generalized to holomorphic mappings of the form (2.1) and (2.2) on the unit ball in a complex Hilbert space H with inner product $\langle \cdot, \cdot \rangle$. In this case, we define $z_1 = \langle z, u \rangle$, where u is a unit vector in H. However, we cannot generalize the theorem to the unit polydisc U^n in $\mathbb{C}^n$, since any normalized convex mapping f on U^n has the form $f(z) = T(f_1(z_1), \ldots, f_n(z_n))$, where T is a nonsingular linear transformation and f_j is a convex mapping on U for each j [9, Theorem 3].

Obradović-Ponnusamy-Singh-Vasundhra [7, Corollary 4.7] showed that if

$$\sum_{k=2}^{\infty}(k-1)k|b_k| \leq 2\lambda,$$

where

$$\lambda = \frac{7+|b_1| - \sqrt{33+30|b_1|+|b_1|^2}}{8},$$

then f is convex on U. It would be interesting to generalize this result to several complex variables.

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