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COEFFICIENT ESTIMATES FOR CERTAIN CLASS OF ANALYTIC FUNCTIONS OF COMPLEX ORDER AND TYPE BETA

Abstract. Let $f(x) = z + \sum_{n=k+1}^{\infty} a_n z^n$ belong to the class $S(1-b),\beta)$, $b \neq 0$, complex and $0 < \beta \leq 1$. In this paper, we determine sharp coefficient estimates for functions of the form $f(z)^t = z^t + \sum_{n=t+k}^{\infty} a_n^{(t)} z^n$, where t is a positive integer. The results obtained generalize the work of many authors.

1. Introduction

Let A denote the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. A function $f(z)$ in A is said to be starlike of complex order $1-b$ and type β if it satisfies

$$\left| \frac{\frac{zf'(x)}{f(z)} - 1}{2\beta(\frac{zf'(z)}{f(z)} - 1 + b) - (\frac{zf'(z)}{f(z)} - 1)} \right| < 1 \quad (z \in U) \tag{1.2}$$

for some b ($b \neq 0$, complex) and (β ($0 < \beta \leq 1$). We denote by $S(1-b,\beta)$ the subclass of A consisting of all such functions. The class $S(1-b,\beta)$ was studied by Aouf, Owa and Obradovic' [3].

MacGregor [11] obtained upper bounds for the moduli of the coefficients of a starlike function whose power series representation in U is of the form

$$f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n. \tag{1.3}$$

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Boyd [5], Srivastava [22], Mogra and Juneja [15], Aouf [1, 2] and Owa and Aouf [18] extanded MacGregor's result to many classes of analytic functions.

In this paper, we determine sharp coefficient bounds for functions of the form

$$f(z)^t = z^t + \sum_{n=k+1}^{\infty} a_n^{(t)} z^n \tag{1.4}$$

where $f(z) \in S(1-b,\beta)$ and t is a positive integer. The results obtained refine and generalize the corresponding results of MacGregor [11], Boyd [5], Libera [10], Srivastava [22], Gopalakrishna and Shetiya [7], Aouf [1, 2], Owa and Aouf [18], Mogra [13] and Mogra and Juneja [15].

2. Coefficient estimates

We shall use the following lemmas in our investigation.

LEMMA 1. *If k, t and q are positive integers, then we have*

$$\begin{aligned} 4\beta^2t^2|b|^2 + \sum_{m=1}^{q-1}\{|(2\beta-1)mk + 2\beta tb|^2 - m^2k^2\} \\ \times \left\{\frac{1}{m!}\prod_{j=0}^{m-1}\left|(2\beta-1)j + \frac{2\beta tb}{k}\right|\right\}^2 \\ = \left\{\frac{k}{(q-1)!}\prod_{j=0}^{q-1}\left|(2\beta-1)j + \frac{2\beta tb}{k}\right|\right\}^2. \end{aligned} \tag{2.1}$$

This lemma can be easily proved by induction on q.

LEMMA 2. *If k, t and q are positive integers, then we have*

$$(n-t)^2 \geq \frac{(qk)^2\{|(2\beta-1)(n-t)+2\beta tb|^2-(n-t)^2\}}{|(2\beta-1)qk+2\beta tb|^2-(qk)^2} \tag{2.2}$$

for $n \geq t + qk$.

The proof is strightforward and we omit it.

THEOREM 1. *Suppose that $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n \in S(1-b,\beta)$ and for integer $t \geq 1$, let*

$$f(z)^t = z^t + \sum_{n=t+k}^{\infty} a_n^{(t)} z^n.$$

(a) *If $\beta t[(n-t)\operatorname{Re}\{b\} + t|b|^2] > (1-\beta)(n-t)[(n-t) + t\operatorname{Re}\{b\}]$. Then*

$$|a_n^{(t)}| \leq \frac{k}{(n-t)(m-1)!}\prod_{j=0}^{m-1}\left|(2\beta-1)j + \frac{2\beta tb}{k}\right| \tag{2.3}$$

for $t+mk \le n \le t+(m+1)k-1$, $m = 1,2,3\ldots,\ M+1$ *and*

$$\text{(2.4)} \qquad |a_n^{(t)}| \le \frac{k}{(n-t)(M+1)!}\prod_{j=0}^{M+1}\left|(2\beta-1)j + \frac{2\beta tb}{k}\right|$$

for $n > t+(M+2)k-1$, *where* $M = [G]$ *(Gauss symbol) and*

$$G = \frac{\beta t[(n-t)\operatorname{Re}\{b\} + t|b|^2]}{(1-\beta)(n-t)[(n-t)+t\operatorname{Re}\{b\}]}.$$

(b) *If* $\beta t[(n-t)\operatorname{Re}\{b\} + t|b|^2] \le (1-\beta)(n-t)[(n-t)+t\operatorname{Re}\{b\}]$, *then*

$$\text{(2.5)} \qquad |a_n^{(t)}| \le \frac{2\beta t|b|}{(n-t)} \quad \text{for } n \ge t+k.$$

The estimates (2.3) *are sharp for* $n = t+mk$, $m = 1,2,\ldots$; $t = 1,2\ldots$, *while the estimates in* (2.5) *are sharp for all* n.

Proof. Let

$$g(z) = \frac{1}{t}\frac{z(f(z)^t)'}{f(z)^t} = \frac{zf'(z)}{f(z)}.$$

Define

$$\text{(2.6)} \qquad \begin{aligned} h(z) &= \frac{g(z)-1}{2\beta b + (2\beta-1)(g(z)-1)} \\ &= \frac{z(f(z)^t)' - tf(z)^t}{2\beta tbf(z)^t + (2\beta-1)[z(f(z)^t)' - tf(z)^t]} \\ &= \sum_{n=k}^{\infty} h_n z^n. \end{aligned}$$

By (1.2), $h(z)$ is regular in U and satisfies $|h(z)| < 1$. Equating the coefficients of the same powers on both sides of the equation

$$z(f(z)^t)' - tf(z)^t = h(z)\{2\beta btf(z)^t + (2\beta-1)[z(f(z)^t)' - tf(z)^t]\}$$

or

$$\text{(2.7)} \qquad \begin{aligned} &\sum_{n=t+k}^{\infty}(n-t)a_n^{(t)}z^n \\ &\quad = \Big(\sum_{n=k}^{\infty}h_n z^n\Big)\Big\{2\beta bz^t + \sum_{n=t+k}^{\infty}[(2\beta-1)(n-t)+2\beta bt]a_n^{(t)}z^n\Big\}, \end{aligned}$$

we obtain the relations

$$\text{(2.8)} \qquad va_{t+v}^{(t)} = 2\beta bth_v, \qquad v = k, k+1, \ldots, 2k-1.$$

Since $|h(z)| < 1$, we have $\sum_{n=k}^{\infty} |h_n|^2 \leq 1$ and therefore

$$\sum_{n=k}^{2k-1} |h_n|^2 \leq 1. \tag{2.9}$$

From (2.8) and (2.9) we obtain

$$\sum_{v=k}^{2k-1} v^2 |a_{t+v}^{(t)}|^2 \leq 4\beta^2 t^2 |b|^2. \tag{2.10}$$

We rewrite (2.7) in the form

$$\sum_{n=t+k}^{p} (n-t) a_n^{(t)} z^n + \sum_{n=p+1}^{\infty} d_n z^n$$
$$= h(z)\left\{2\beta b z^t + \sum_{n=t+k}^{p-k} [(2\beta-1)(n-t) + 2\beta t b] a_n^{(t)} z^n\right\}.$$

Multiplying each sides of this by its conjugate, integrating around $|z|=r<1$ and letting $r \to 1$ yields the inequality

$$\sum_{n=t+k}^{p} (n-t)^2 |a_n^{(t)}|^2 \tag{2.11}$$
$$\leq 4\beta^2 t^2 |b|^2 + \sum_{n=t+k}^{p-k} |(2\beta-1)(n-t) + 2\beta t b|^2 |a_n^{(t)}|^2$$

from which we obtain

$$\sum_{n=p-k+1}^{p} (n-t)^2 |a_n^{(t)}|^2 \tag{2.12}$$
$$\leq 4\beta^2 t^2 |b|^2 + \sum_{n=t+k}^{p-k} \{|(2\beta-1)(n-t) + 2\beta t b|^2 - (n-t)^2\} |a_n^{(t)}|^2.$$

Now two cases arise:

(a) If $\beta t[(n-t)\operatorname{Re}\{b\} + t|b|^2] > (1-\beta)(n-t)[(n-t) + t\operatorname{Re}\{b\}]$,

then by an inductive argument we will establish the inequalities

$$\sum_{n=t+mk}^{t+(m+1)k-1} (n-t)^2 |a_n^{(t)}|^2 \leq \left\{\frac{k}{(m-1)!} \prod_{j=0}^{m-1} \left|(2\beta-1)j + \frac{2\beta t b}{k}\right|\right\}^2 \tag{2.13a}$$

and

$$\text{(2.13b)} \qquad \sum_{n=t+mk}^{t+(m+1)k-1} \{|(2\beta-1)(n-t)2\beta tb|^2-(n-t)^2\}|a_n^{(t)}|^2$$

$$\leq \{|(2\beta-1)mk+2\beta tb|^2-m^2k^2\}\left\{\frac{1}{m!}\prod_{j=0}^{m-1}\left|(2\beta-1)j+\frac{2\beta tb}{k}\right|\right\}^2$$

for $m=1,2,\ldots,M+1$, where $M=[G]$, G is given by

$$G=\frac{\beta t[(n-t)\operatorname{Re}\{b\}+t|b|^2]}{(1-\beta)(n-t)[(n-t)+t\operatorname{Re}\{b\}]}$$

and $[G]$ is the greatest integer not greater than G.

For $m=1$, (2.13a) gives

$$\sum_{n=t+k}^{t+2k-1}(n-t)^2|a_n^{(t)}|^2\leq 4\beta^2t^2|b|^2,$$

which is true by (2.10). Thus (2.13a) is valid for $m=1$. To prove (2.13b) for $m=1$, we use Lemma 2 and (2.10) as follows:

$$\sum_{n=t+k}^{t+2k-1}\{|(2\beta-1)(n-t)+2\beta tb|^2-(n-t)^2\}|a_n^{(t)}|^2$$

$$\leq \frac{\{|(2\beta-1)k+2\beta tb|^2-k^2\}}{k^2}\sum_{n=t+k}^{t+2k-1}(n-t)^2|a_n^{(t)}|^2$$

$$\leq \frac{4\beta^2t^2|b|^2}{k^2}\{|(2\beta-1)k+2\beta tb|^2-k^2\}.$$

Now suppose that (2.13a) and (2.13b) hold for $m=1,2,\ldots,q-1$. Using (2.12) with $p=t+(q+1)k-1$ and the inductive hypothesis concerning (2.13a) we obtain the inequalities

$$\sum_{n=t+qk}^{t+(q+1)k-1}(n-t)^2|a_n^{(t)}|^2$$

$$\leq 4\beta^2t^2|b|^2+\sum_{n=t+k}^{t+qk-1}\{|(2\beta-1)(n-t)+2\beta tb|^2-(n-t)^2\}|a_n^{(t)}|^2$$

$$=4\beta^2t^2|b|^2+\sum_{m=1}^{q-1}\sum_{n=t+mk}^{t+(m+1)k-1}\{|(2\beta-1)(n-t)+2\beta tb|^2-(n-t)^2\}|a_n^{(t)}|^2$$

$$\leq 4\beta^2t^2|b|^2 + \sum_{m=1}^{q-1}\{|(2\beta-1)mk+2\beta tb|^2 - m^2k^2\}$$

$$\times \left\{\frac{1}{m}\prod_{j=0}^{m-1}\left|(2\beta-1)j+\frac{2\beta tb}{k}\right|\right\}^2 \quad \text{by using Lemma 1.}$$

This last sequence of inequalities implies (2.13a) where $m = k$. Continuing our argument, we use Lemma 2 and (2.13a) with $m = q$ to deduce (2.13b) for $m = q$ as follows:

$$\sum_{n=t+qk}^{t+(q+1)k-1}\{|(2\beta-1)(n-t)+2\beta tb|^2 - (n-t)^2\}|a_n^{(t)}|^2$$

$$\leq \frac{|(2\beta-1)qk+2\beta tb|^2 - q^2k^2}{q^2k^2}\left\{\frac{k}{(q-1)!}\prod_{j=0}^{q-1}\left|(2\beta-1)j+\frac{2\beta tb}{k}\right|\right\}^2$$

$$= \{|(2\beta-1)qk+2\beta tb|^2 - q^2k^2\}\left\{\frac{1}{q!}\prod_{j=0}^{q-1}\left|(2\beta-1)j+\frac{2\beta tb}{k}\right|\right\}^2.$$

This completes the proof of (2.13a) and (2.13b). Now (2.3) follows from (2.13a).

To prove (2.4), suppose $n > t+(M+2)k-1$. Putting $p = t+(q+1)k-1$ in (2.12), we have

$$\sum_{n=t+qk}^{t+(q+1)k-1}(n-t)^2|a_n^{(t)}|^2$$

$$\leq 4\beta^2t^2|b|^2 + \sum_{n=t+k}^{t+qk-1}\{|(2\beta-1)(n-t)+2\beta tb|^2 - (n-t)^2\}|a_n^{(t)}|^2.$$

Hence for $n > t+(M+2)k-1$, we have

$$(n-t)^2|a_n^{(t)}|^2 \leq 4\beta^2t^2|b|^2 + \sum_{n=t+k}^{t+qk-1}\{|(2\beta-1)(n-t)+2\beta tb|^2 - (n-t)^2\}|a_n^{(t)}|^2$$

$$= 4\beta^2t^2|b|^2 + \sum_{n=t+k}^{t+(M+2)k-1}\{|(2\beta-1)(n-t)+2\beta tb|^2 - (n-t)^2\}|a_n^{(t)}|^2$$

$$+ \sum_{n=t+(M+2)k}^{t+qk-1}\{|(2\beta-1)(n-t)+2\beta tb|^2 - (n-t)^2\}|a_n^{(t)}|^2$$

$$\leq 4\beta^2t^2|b|^2 + \sum_{n=t+k}^{t+(M+2)k-1} \{|(2\beta-1)(n-t)+2\beta tb|^2-(n-t)^2\}|a_n^{(t)}|^2$$

$$= 4\beta^2t^2|b|^2 + \sum_{m=1}^{M+1} \sum_{n=t+mk}^{t+(m+1)k-1} \{|(2\beta-1)(n-t)+2\beta tb|^2-(n-t)^2\}|a_n^{(t)}|^2$$

$$\leq 4\beta^2t^2|b|^2 + \sum_{m=1}^{M+1} \{|(2\beta-1)mk+2\beta tb|^2-m^2k^2\}$$

$$\times \left\{\frac{1}{m!}\prod_{j=0}^{m-1}\left|(2\beta-1)j+\frac{2\beta tb}{k}\right|\right\}^2$$

$$= \left\{\frac{k}{(M+1)!}\prod_{j=0}^{M+1}\left|(2\beta-1)j+\frac{2\beta tb}{k}\right|\right\}^2 \quad \text{by using Lemma 1.}$$

This gives

$$|a_n^{(t)}| \leq \frac{k}{(n-t)(M+1)!}\prod_{j=0}^{M+1}\left|(2\beta-1)j+\frac{2\beta tb}{k}\right|$$

for $n > t+(M+2)k-1$ which proves (2.4).

(b) If $\beta t[(n-t)\operatorname{Re}\{b\}+t|b|^2] \leq (1-\beta)(n-t)[(n-t)+t\operatorname{Re}\{b\}]$, then (2.11) gives

$$\sum_{n=t+k}^{p} (n-t)^2|a_n^{(t)}|^2 \leq 4\beta^2t^2|b|^2$$

or

$$(n-t)^2|a_n^{(t)}|^2 \leq 4\beta^2t^2|b|^2 \quad \text{if } n \geq t+k,$$

that is

$$|a_n^{(t)}| \leq \frac{2\beta t|b|}{(n-t)} \quad \text{if } n \geq t+k,$$

which gives (2.5).

The function $f(z)$, given by

$$f(z) = \frac{z}{(1-(2\beta-1)z^k)^{\frac{2\beta tb}{(2\beta-1)k}}}, \tag{2.14}$$

where $\beta t[(n-t)\operatorname{Re}\{b\}+t|b|^2] > (1-\beta)(n-t)[(n-t)+t\operatorname{Re}\{b\}]$, shows the estimates in (2.3) are sharp for $n = t+mk$, $m = 1, 2, \ldots$, $t = 1, 2, \ldots$, while the estimates in (2.5) are sharp for the function

$$f_n(z) = z \exp\left(\frac{2\beta t b}{(n-1)} z^{n-1}\right), \tag{2.15}$$

where $\beta t[(n-t)\operatorname{Re}\{b\} + t|b|^2] \le (1-\beta)(n-t)[(n-t) + t\operatorname{Re}\{b\}]$ and $n \ge t+k$.

3. Remarks on Theorem 1

(1) (i) Putting $b = (1-\rho)\cos\alpha e^{-i\alpha}$ ($|\alpha| < \frac{\pi}{2}$ and $0 \le \rho < 1$) and $\beta = 1$, in Theorem 1, we get the result due to Bhatia and Rajasekaran [4].

(ii) Putting $b = (1-\alpha)$ $(0 \le \alpha < 1)$ in Theorem 1, we get the result due to Rajasekaran and Bhatia [19].

(2) (i) Putting $t = 1$, in Theorem 1, we get the result due to Owa and Aouf [18].

(ii) Putting $t = 1$, $b = 1$ and $\beta = 1$ in Theorem 1, we get the result due to MacGregor [11].

(iii) Putting $t = 1$, $b = (1-\alpha)$ $(0 \le \alpha < 1)$ and $\beta = 1$ in Theorem 1, we get the result due to Boyd [5].

(iv) Putting $t = 1$, $b = (1-\alpha)\cos\lambda e^{-i\lambda}$ ($|\lambda| < \frac{\pi}{2}$, $0 \le \lambda < 1$) and $\beta = 1$ in Theorem 1, we get the result due to Gopalakrishna and Shetiya [7].

(v) Putting $t = 1$, $b = (1-\alpha)$ $(0 \le \alpha < 1)$ in Theorem 1, we get the result due to Mogra and Juneja [15].

(3) Putting $t = 1$,

$$(1-b,\beta) = (1-b,1),$$

$$(1-b,\beta) = \left(1-b, \frac{2M-1}{2M}\right),\ M > \frac{1}{2},$$

$$(1-b,\beta) = (1-(1-\alpha)\cos\lambda e^{-i\lambda}, \beta),\ |\lambda| < \frac{\pi}{2},\ 0 \le \alpha < 1, \text{ and } 0 < \beta \le 1,$$

$$(1-b,\beta) = \left(1-(1-\alpha)\cos\lambda e^{-i\lambda}, \frac{2M-1}{2M}\right),\ |\lambda| < \frac{\pi}{2}, \text{ and } M > \frac{1}{2},$$

$$(1-b,\beta) = (1-(1-\alpha)\cos\lambda e^{-i\lambda}, 1),\ |\lambda| < \frac{\pi}{2},\ 0 \le \alpha < 1,$$

$$(1-b,\beta) = \left(1-\cos\lambda e^{-i\lambda}, \frac{2-\cos\lambda}{2}\right),\ |\lambda| < \frac{\pi}{2},$$

$$(1-b,\beta) = \left(1-(1-\alpha), \frac{1}{2}\right),\ 0 \le \alpha < 1,$$

$$(1-b,\beta) = \left(0, \frac{1}{2}\right),$$

$$(1-b,\beta) = \left(0, \frac{2M-1}{2M}\right),\ M > \frac{1}{2},$$

$$(1-b,\beta)=(1-b,\beta),$$
$$(1-b,\beta)=(1-(1-\alpha),\beta),\ 0\le\alpha<1,$$

in Theorem 1, we get the corresponding coefficient estimates for the functions of the form $f(z)=z+\sum_{n=k+1}^{\infty}a_nz^n$ belonging to the classes introduced by Nasr and Aouf [16, 17], Mogra and Ahuja [14], Kulshrestha [9], Libera [10], Goel [6], Wright [23], McCarty [12], Singh [20], Singh and Singh [21], Aouf, Owa and Obradovic' [3] and Juneja and Mogra [8], respectively.

(4) The results due to Nasr and Aouf [16, 17], Mogra and Ahuja [14], Kulshrestha [9], Libera [10], Goel [6], Wright [23], McCarty [12], Singh [20], Singh and Singh [21], Aouf, Owa and Obradovic' [3], and Juneja and Mogra [8] can be obtained by taking different values of the parameters b,β with $k=t=1$ in Theorem 1.

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