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ON A PROBLEM OF H. S. AL-AMIRI AND M. O. READE

Abstract. In the present paper, the authors investigate the univalence of the functions f, analytic in E, $f(0) = 0, f'(0) = 1$ and which satisfy

$$\operatorname{Re}\left[(1-\alpha)f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > \beta,\ z \in E,$$

where $\alpha > 0$ and $0 < \beta < 1$. The univalence of such functions has already been established in the case when $\alpha \leq 0$ and $\beta = 0$ by H. S. Al-Amiri and M. O. Reade in 1975.

1. Introduction

Let H denote the class of functions analytic in the unit disc $E = \{z : |z| < 1\}$. For $a \in \mathbb{C}$ ($\mathbb{C}$ is the set of complex numbers), we let

$$\mathcal{A}(a) = \{f \in H : f(z) = a + a_1 z + a_2 z^2 + \ldots\};$$

and

$$\mathcal{A} = \{f \in H : f(z) = z + a_2 z^2 + a_3 z^3 + \ldots\}.$$

We denote by S and $St(\gamma), 0 \leq \gamma < 1$, the subclasses of $\mathcal{A}$ consisting of univalent functions and starlike functions of order γ, respectively. We shall write $St(0) = St$, the usual class of starlike (with respect to the origin) functions in E.

Let $f \in \mathcal{A}$, with $\frac{f(z)f'(z)}{z} \neq 0$ in E and let α be a real number. Then f is said to be α-convex in E if and only if the inequality

$$\operatorname{Re}\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > 0,$$

holds in E. If $M(\alpha)$ denotes the class of α-convex functions, then Miller, Mocanu and Reade [7,8] proved that the functions in $M(\alpha)$ are starlike

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univalent in E for all real α, and if $\alpha \geq 1$, then $M(\alpha)$ consists of convex univalent functions.

Al-Amiri and Reade [3], in 1975, studied the class $H(\alpha)$ of functions $f \in \mathcal{A}$ with $f'(z) \neq 0$ in E which satisfy the condition

$$\text{(1)} \qquad \operatorname{Re}\left[(1-\alpha)f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > 0, \ z \in E.$$

They proved that when $\alpha \leq 0$, the function f in $H(\alpha)$ satisfies $\operatorname{Re} f'(z) > 0$, $z \in E$,which in turn implies that f is close-to-convex and hence, univalent in E (Noshiro [10] and Warchawaski [11]). Note that $H(1)$ consists of convex functions which are obviously univalent in E. Ahuja and Silverman [2] gave an example to show that $H(1) \not\subset H(\alpha)$, $\alpha \neq 1$. To the best of our knowledge, the question of univalence of the functions of the class $H(\alpha)$, for $\alpha > 0$, remains unsettled till date. In the present paper, we have persued this problem, although we have succeeded only partially.

We define a class $H(\alpha, \beta)$ of functions $f \in \mathcal{A}$ with $f'(z) \neq 0$ in E for which

$$\text{(2)} \qquad \operatorname{Re}\left[(1-\alpha)f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > \beta, \ z \in E.$$

We claim that when $\beta = \alpha$ and $0 < \alpha < 1$, $H(\alpha, \beta)$ consists of univalent functions. Further, when $\beta = 1/2$ and α, $\alpha \geq 0$, is any real number, we show that the functions f in $H(\alpha, \beta)$ satisfy $\operatorname{Re} f'(z) > 1/2$ in E.

2. Preliminaries

We shall need the following definition and lemmas to prove our results.

DEFINITION 2.1. Let f and g be analytic in E. We say that f is subordinate to g in E, written as $f \prec g$ or $f(z) \prec g(z)$ in E, if g is univalent in E, $f(0) = g(0)$ and $f(E) \subset g(E)$.

LEMMA 2.1 [6, page 11]. *Let $n \geq 0$ be an integer and let $\gamma \in \mathbb{C}$, with $\operatorname{Re}\gamma > -n$. If $f(z) = \sum_{m \geq n} a_m z^m$ is analytic in E and F is defined by*

$$F(z) = \frac{1}{z^\gamma}\int_0^z f(t)t^{\gamma-1}dt = \sum_{m \geq n} \frac{a_m z^m}{m+\gamma},$$

then, F is analytic in E.

LEMMA 2.2 [9]. *Let c be a complex number with $\operatorname{Re} c > 0$. Let $P \in \mathcal{A}(c)$ be such that $\operatorname{Re} P(z) > 0$, $z \in E$. If $p \in \mathcal{A}(1/c)$ satisfies the differential equation*

$$zp'(z) + P(z)p(z) = 1 \ z \in E,$$

then $\operatorname{Re} p(z) > 0$, $z \in E$.

LEMMA 2.3. *Let $\phi(u,v)$ be a complex function: $\phi : \mathbb{D} \to \mathbb{C}$ ($\mathbb{D}$ is a subset of $\mathbb{C} \times \mathbb{C}$) and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that ϕ satisfies the following conditions:*

(i) *$\phi(u,v)$ is continuous in $\mathbb{D}$;*
(ii) *$(1,0) \in \mathbb{D}$ and Re $\phi(1,0) > 0$; and*
(iii) *Re $\{\phi(iu_2, v_1)\} \le 0$ for all $(iu_2, v_1) \in \mathbb{D}$ such that $v_1 \le -(1+u_2^2)/2$.*

Let $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ be regular in the unit disc E, such that $(p(z), zp'(z)) \in \mathbb{D}$ for all $z \in E$. If

$$\text{Re}[\phi(p(z), zp'(z))] > 0, \ z \in E,$$

then,

$$\text{Re}p(z) > 0, \ z \in E.$$

Lemma 2.3 is due to Miller [5].

LEMMA 2.4. *Let p be analytic in E and let it satisfy the differential subordination*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \frac{1 + (1-2\delta)z}{1-z} = h(z)$$

with $\beta > 0$ and $-\text{Re}(\gamma/\beta) \le \delta < 1$. Then the differential equation

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z), \quad q(0) = 1,$$

has a univalent solution $q(z)$. In addition, $p(z) \prec q(z) \prec h(z)$ and q is the best dominant.

Lemma 2.4 is due to Eenigenburg, Miller, Mocanu and Reade [4].

LEMMA 2.5 [12]. *Let μ be a positive measure on $[0,1]$ and let g be a complex-valued function defined on $E \times [0,1]$ such that $g(.,t)$ is analytic in E for each $t \in [0,1]$ and $g(z,.)$ is μ-integrable on $[0,1]$ for all $z \in E$. In addition, suppose that $\text{Re}g(z,t) > 0, g(-r,t)$ is real and*

$$\text{Re}\frac{1}{g(z,t)} \ge \frac{1}{g(-r,t)}$$

for all $|z| \le r < 1$ and $t \in [0,1]$. If

$$g(z) = \int_0^1 g(z,t)d\mu(t),$$

then

$$\text{Re}\frac{1}{g(z)} \ge \frac{1}{g(-r)}.$$

3. Main results and their proofs

We begin with the following lemma.

LEMMA 3.1. *Let α be a real number with $0 < \alpha < 1$ and let $\phi \in \mathcal{A}$ be a starlike function of order α. If $F(z) = I_\alpha[\phi](z)$ is defined by*

$$F(z) = \left[\left(\frac{1-\alpha}{\alpha}\right) z \int_0^z \phi^{1/\alpha}(t) t^{-2} dt\right]^\alpha, \tag{3}$$

then $F(z)/z \neq 0$ in E and $F \in St(\alpha)$. Here, all the powers are principal ones.

Proof. Let us define a function q as under:

$$\begin{aligned} q(z) &= \frac{z}{\phi^{1/\alpha}(z)} \int_0^z (\phi(t))^{1/\alpha} t^{-2} dt \\ &= \frac{\alpha}{1-\alpha} + q_1 z + \dots . \end{aligned} \tag{4}$$

In view of Lemma 2.1 (with $n = 0$), $q \in \mathcal{A}(\alpha/(1-\alpha))$. Differentiating (4), we get

$$\left(\frac{1}{\alpha}\frac{z\phi'(z)}{\phi(z)} - 1\right) q(z) + zq'(z) = 1. \tag{5}$$

Since ϕ is a starlike function of order α, therefore,

$$\operatorname{Re}\left(\frac{1}{\alpha}\frac{z\phi'(z)}{\phi(z)} - 1\right) > 0, \ z \in E.$$

Using Lemma 2.2 with $c = (1-\alpha)/\alpha$ and $P(z) = \frac{1}{\alpha}\frac{z\phi'(z)}{\phi(z)} - 1$, we conclude that $q(z) \neq 0$ and $\operatorname{Re} q(z) > 0$, for all $z \in E$.

From (3) and (4), we see that F can be represented as

$$\begin{aligned} F(z) &= \left(\frac{1-\alpha}{\alpha} q(z)\right)^\alpha \phi(z) \\ &= z + a_2 z^2 + \dots, \text{ say.} \end{aligned} \tag{6}$$

Since q and ϕ are analytic, we conclude that $F \in \mathcal{A}$ and $F(z)/z \neq 0$ in E. Differentiation of (6) logarithmically leads to

$$\frac{1}{\alpha}\frac{zF'(z)}{F(z)} = \frac{zq'(z)}{q(z)} + \frac{1}{\alpha}\frac{z\phi'(z)}{\phi(z)}$$

or

$$\frac{1}{\alpha}\frac{zF'(z)}{F(z)} - 1 = \frac{1}{q(z)}, \text{ using (5).}$$

As $\operatorname{Re} q(z) > 0$, $z \in E$, we have $F \in St(\alpha)$.

THEOREM 3.1. *Let $0 < \alpha < 1$. Suppose that an analytic function p, $p(0)=1$, satisfies*

$$\operatorname{Re}\left[(1-\alpha)p(z)+\alpha\left(1+\frac{zp'(z)}{p(z)}\right)\right] > \alpha, \quad z \in E, \tag{7}$$

then, there exists a function $\phi \in \mathcal{A}$, starlike of order α, such that p has the representation

$$p(z) = \left[\frac{1-\alpha}{\alpha}\int_0^1 \left(\frac{\phi(tz)}{\phi(z)}\right)^{\frac{1}{\alpha}} t^{-2}dt\right]^{-1} \tag{8}$$

and Re $p(z) > 0$, $z \in E$.

Proof. In view of (7), we can write

$$(1-\alpha)p(z)+\alpha\left(1+\frac{zp'(z)}{p(z)}\right) = \frac{z\phi'(z)}{\phi(z)}$$

for some $\phi \in St(\alpha)$. Let F be as defined in (3) and let

$$p(z) = \frac{1}{1-\alpha}\left(\frac{zF'(z)}{F(z)} - \alpha\right). \tag{9}$$

Then p is analytic in E with $p(0) = 1$ and Re $p(z) > 0$ in E(as $F \in St(\alpha)$, by Lemma 3.1). In view of (3) and (9), one can easily verify that

$$p(z) = \left(\frac{\phi(z)}{F(z)}\right)^{1/\alpha} = \left[\frac{1-\alpha}{\alpha}\int_0^1 \left(\frac{\phi(tz)}{\phi(z)}\right)^{\frac{1}{\alpha}} t^{-2}dt\right]^{-1}.$$

This completes the proof of our theorem. ∎

Taking $p(z) = f'(z)$ in Theorem 3.1, we immediately get the following result:

THEOREM 3.2. *Let α be a real number such that $0 < \alpha < 1$. Let $f \in \mathcal{A}$ satisfy*

$$\operatorname{Re}\left[(1-\alpha)f'(z)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right] > \alpha, \quad z \in E. \tag{10}$$

Then Re$f'(z) > 0$ *in E and hence, f is univalent in E.*

To prove our next result, we shall need the definition and some properties of Gaussian hypergeometric function. The function

$${}_2F_1(a,b,c;z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \dots,$$

is called Gaussian hypergeometric function where a, b and c are complex constants with $c \neq 0, -1, -2, \dots$. Following properties of ${}_2F_1(a,b,c;z)$ are well-known (see [1, p. 556–558])

$${}_2F_1(a,b,c;z) = {}_2F_1(b,a,c;z); \tag{11}$$

(12) $$ {}_2F_1(a,b,c;z) = (1-z)^{-a}\ {}_2F_1(a,c-b,c;z/(z-1)); $$

and if $\operatorname{Re} c > \operatorname{Re} b > 0$, then there is a probability measure on $[0,1]$ given by

$$ d\mu(t) = \frac{\Gamma(c)t^{b-1}(1-t)^{c-b-1}}{\Gamma(b)\Gamma(c-b)}dt, $$

such that

(13) $$ {}_2F_1(a,b,c;z) = \int_0^1 (1-tz)^{-a} d\mu(t). $$

In the next theorem, we obtain the best dominant for $f'(z)$, when $f \in H(\alpha,\alpha), 0<\alpha<1$.

THEOREM 3.3. *Let $f \in H(\alpha,\alpha)$, $0<\alpha<1$. Then for all $z \in E$, we have*

$$ f'(z) \prec \frac{1}{{}_2F_1(\frac{2}{\alpha}-2,1,\frac{1}{\alpha};\frac{z}{z-1})} \prec \frac{1+z}{1-z}. $$

Proof. Let $f'(z) = p(z)$. Since $f \in H(\alpha,\alpha)$, we can write

$$ (1-\alpha)p(z) + \alpha\left(1+\frac{zp'(z)}{p(z)}\right) \prec \frac{1+(1-2\alpha)z}{1-z}. $$

A little simplification leads us to

$$ p(z) + \frac{\alpha}{1-\alpha}\frac{zp'(z)}{p(z)} \prec \frac{1+z}{1-z}. $$

This is Briot-Bouquet differential subordination of the form defined in Lemma 2.4, with $\beta = \frac{1-\alpha}{\alpha}$, $\delta = 0$ and $\gamma = 0$. Since $0<\alpha<1$, an application of Lemma 2.4 gives that $p(z) \prec q(z) \prec (1+z)/(1-z)$, where $q(z)$ is the univalent solution of the differential equation

$$ q(z) + \frac{\alpha}{1-\alpha}\frac{zq'(z)}{q(z)} = \frac{1+z}{1-z}. $$

On solving this differential equation, we get

$$ \begin{aligned} q(z) &= \frac{z^{\frac{1}{\alpha}-1}(1-z)^{2-\frac{2}{\alpha}}}{\frac{1-\alpha}{\alpha}\int_0^z t^{\frac{1}{\alpha}-2}(1-t)^{2-\frac{2}{\alpha}}dt} = \frac{z^{\frac{1}{\alpha}-1}(1-z)^{2-\frac{2}{\alpha}}}{\frac{1-\alpha}{\alpha}\int_0^1 (sz)^{\frac{1}{\alpha}-2}(1-sz)^{2-\frac{2}{\alpha}}z\,ds} \\ &= \frac{(1-z)^{2-\frac{2}{\alpha}}}{\frac{\Gamma(\frac{1}{\alpha})}{\Gamma(\frac{1}{\alpha}-1)\Gamma(1)}\int_0^1 s^{\frac{1}{\alpha}-2}(1-sz)^{-(\frac{2}{\alpha}-2)}ds} \\ &= \frac{(1-z)^{2-\frac{2}{\alpha}}}{{}_2F_1(\frac{2}{\alpha}-2,\frac{1}{\alpha}-1,\frac{1}{\alpha};z)} \qquad \text{(using (13))} \\ &= \frac{1}{{}_2F_1(\frac{2}{\alpha}-2,1,\frac{1}{\alpha};\frac{z}{z-1})}. \qquad \text{(using (12))} \end{aligned} $$

Thus, we have

$$f'(z) \prec \frac{1}{{}_2F_1(\frac{2}{\alpha}-2,1,\frac{1}{\alpha};\frac{z}{z-1})} \prec \frac{1+z}{1-z}.$$

THEOREM 3.4. *If $f \in H(\alpha,\alpha)$, $1/2<\alpha<1$, then*

$$\text{(14)} \qquad \mathrm{Re} f'(z) > \frac{1}{{}_2F_1(\frac{2}{\alpha}-2,1,\frac{1}{\alpha};\frac{1}{2})}, \qquad \textit{for all } z \in E.$$

Proof. Since $f \in H(\alpha,\alpha)$, by Theorem 3.3, we get

$$f'(z) \prec \frac{1}{{}_2F_1(\frac{2}{\alpha}-2,1,\frac{1}{\alpha};\frac{z}{z-1})}.$$

In order to prove inequality (14), we use Lemma 2.5 alongwith properties (11) and (13) of ${}_2F_1(a,b,c;z)$. If we write $a=2/\alpha-2, b=1, c=1/\alpha$, we obtain

$$q(z)=\frac{1}{g(z)},$$

where

$$g(z)={}_2F_1\left(a,1,c;\frac{z}{z-1}\right)={}_2F_1\left(1,a,c;\frac{z}{z-1}\right).$$

Since $1/2<\alpha<1$, we get $c>a>0$, and therefore, from (13), we obtain

$$g(z)=\int_0^1 \frac{1-z}{1-(1-t)z}d\mu(t),$$

where $\mu(t)$ is the probability measure on $[0,1]$. Let us define a function $g(z,t)$ on $E\times[0,1]$ by

$$g(z,t)=\frac{1-z}{1-(1-t)z}.$$

Then, obviously, $g(.,t)$ is analytic in E for each $t\in[0,1]$ and $g(z,.)$ is μ-integrable on $[0,1]$ for all $z\in E$. Further, $\mathrm{Re}\, g(z,t)>0, g(-r,t)$ is real and

$$\mathrm{Re}\frac{1}{g(z,t)} \geq \frac{1+(1-t)r}{1+r}=\frac{1}{g(-r,t)},$$

for $|z|\leq r<1$ and $t\in[0,1]$. An application of Lemma 2.5 gives

$$\mathrm{Re}\frac{1}{g(z)} \geq \frac{1}{g(-r)}.$$

Letting $r\to 1^-$, we get

$$\mathrm{Re}\frac{1}{g(z)} \geq \frac{1}{g(-1)} = {}_2F_1\left(\frac{2}{\alpha}-2,1,\frac{1}{\alpha};\frac{1}{2}\right).$$

This completes the proof of our theorem.

THEOREM 3.5. *Let $\alpha \geq 0$ be a real number. Suppose that an analytic function $f \in \mathcal{A}$ satisfies*

$$\text{(15)} \qquad \mathrm{Re}\left[(1-\alpha)f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > 1/2, \; z \in E.$$

Then Re $f'(z) > 1/2$ in E.

Proof. Let us define a function p in E by

$$f'(z) = \frac{1 + p(z)}{2},$$

where $p(z) = 1 + p_1 z + \ldots$. We shall show that Re $p(z) > 0$, $z \in E$. Obviously,

$$(1-\alpha)f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = (1-\alpha)\frac{1+p(z)}{2} + \alpha\left[1 + \frac{zp'(z)}{1+p(z)}\right].$$

Define $\Psi(u,v) : \mathbb{D} = (\mathbb{C}\backslash\{-1\}) \times \mathbb{C} \to \mathbb{C}$ as under:

$$\Psi(u,v) = 2\left[(1-\alpha)\frac{1+u}{2} + \alpha\left(1 + \frac{v}{1+u}\right)\right] - 1.$$

Then $\Psi(u,v)$ is continuous in $\mathbb{D}$, $(1,0) \in \mathbb{D}$ and Re$\Psi(1,0) > 0$. Further, (15) implies that Re$\Psi(p(z), zp'(z)) > 0$, $z \in E$.

Also, for $(ir_2, s_1) \in \mathbb{D}$, with $s_1 \leq -(1+r_2^2)/2$, we have

$$\begin{aligned}\mathrm{Re}\Psi(ir_2, s_1) &= \mathrm{Re}\left[(1-\alpha)(1+r_2 i) + 2\alpha\left(1 + \frac{s_1}{1+r_2 i}\right) - 1\right] \\ &= -\alpha + 2\alpha\left[1 + \frac{s_1}{1+r_2^2}\right] \leq -\alpha + 2\alpha\left[1 - \frac{1+r_2^2}{2(1+r_2^2)}\right] = 0.\end{aligned}$$

Result now follows by Lemma 2.3.

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