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B. G. Pachpatte

INEQUALITIES APPLICABLE TO CERTAIN PARTIAL FINITE SUM-DIFFERENCE EQUATIONS

Abstract. The aim of this paper is to establish explicit bounds on certain discrete inequalities in two independent variables involving iterated sums, which can be used to study the qualitative behavior of solutions of some classes of partial finite sum-difference equations. Applications are also given to illustrate the usefulness of one of our results.

1. Introduction

In [2] Bykov and Salpagarov (see, also [1,3,4]) have given the explicit bounds on the following inequalities

$$u(t) \leq c + \int_{\alpha}^{t} k(t,s)u(s)ds + \int_{\alpha}^{t}\left(\int_{\alpha}^{s} h(t,s,\sigma)u(\sigma)d\sigma\right)ds, \tag{1.1}$$

$$\begin{aligned} u(t) \leq c &+ \int_{\alpha}^{t} b(s)u(s)ds + \int_{\alpha}^{t}\left(\int_{\alpha}^{s} k(s,\tau)u(\tau)d\tau\right)ds \\ &+ \int_{\alpha}^{t}\left(\int_{\alpha}^{s}\left(\int_{\alpha}^{\tau} h(s,\tau,\sigma)u(\sigma)d\sigma\right)d\tau\right)ds, \end{aligned} \tag{1.2}$$

under some suitable conditions on the functions involved therein. Recently in [6] it is shown that the explicit bounds on the discrete versions of the above inequalities and their generalizations are equally important in certain applications. In view of the successful utilizations of the explicit bounds on such inequalities (see[5]) it is natural to expect that the explicit bounds on the two independent variable discrete generalizations of (1.1), (1.2) would be useful in certain new applications. The main purpose of this paper is to establish explicit bounds on two independent variable discrete general-

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izations of the above inequalities, which can be used as handy tools in the study of certain new classes of partial finite sum-difference equations. Some applications of one of our results are also given.

2. Statement of Results

In what follows R denotes the set of real numbers and $R_+ = [0, \infty)$, $N_0 = \{0, 1, 2, \ldots\}$ are the given subsets of R. We denote by

$$D = \{(x, y, s, t) \in N_0^4 : 0 \leq s \leq x < \infty, 0 \leq t \leq y < \infty\},$$
$$E = \{(x, y, s, t, \sigma, \tau) \in N_0^6 : 0 \leq \sigma \leq s \leq x < \infty,\ 0 \leq \tau \leq t \leq y < \infty\}.$$

For any function $u(x, y)$, $x, y \in N_0$, we define the operators $\Delta_1 u(x, y) = u(x+1, y) - u(x, y)$, $\Delta_2 u(x, y) = u(x, y+1) - u(x, y)$ and $\Delta_2\Delta_1 u(x, y) = \Delta_2(\Delta_1 u(x, y))$. We use the usual conventions that the empty sums and products are taken to be 0 and 1 respectively and assume that all the sums and products involved throughout the discussion exist on the respective domains of their definitions.

Our main results are established in the following theorems.

THEOREM 1. *Let $u(x, y) : N_0^2 \to R_+$, $k(x, y, s, t)$, $\Delta_1 k(x, y, s, t)$, $\Delta_2 k(x, y, s, t)$, $\Delta_2\Delta_1 k(x, y, s, t) : D \to R_+$, $h(x, y, s, t, \sigma, \tau)$, $\Delta_1 h(x, y, s, t, \sigma, \tau)$, $\Delta_2 h(x, y, s, t, \sigma, \tau)$, $\Delta_2\Delta_1 h(x, y, s, t, \sigma, \tau) : E \to R_+$ and $c \geq 0$ is a constant.*

(a_1) If

$$(2.1)\quad u(x, y) \leq c + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} k(x, y, s, t)u(s, t) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\Big(\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} h(x, y, s, t, \sigma, \tau)u(\sigma, \tau)\Big)$$

for $x, y \in N_0$, then

$$(2.2)\qquad u(x, y) \leq c\prod_{m=0}^{x-1}[1 + \sum_{n=0}^{y-1}[A(m, n) + B(m, n)]],$$

for $x, y \in N_0$, where

$$(2.3)\quad A(x, y) = k(x+1, y+1, x, y) + \sum_{s=0}^{x-1}\Delta_1 k(x, y+1, s, y)$$
$$+ \sum_{t=0}^{y-1}\Delta_2 k(x+1, y, x, t) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\Delta_2\Delta_1 k(x, y, s, t),$$

$$(2.4)\quad B(x,y)=\sum_{\sigma=0}^{x-1}\sum_{\tau=0}^{y-1}h(x+1,y+1,x,y,\sigma,\tau)$$
$$+\sum_{s=0}^{x-1}\Big(\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{y-1}\Delta_1 h(x,y+1,s,y,\sigma,\tau)\Big)$$
$$+\sum_{t=0}^{y-1}\Big(\sum_{\sigma=0}^{x-1}\sum_{\tau=0}^{t-1}\Delta_2 h(x+1,y,x,t,\sigma,\tau)\Big)$$
$$+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\Big(\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1}\Delta_2\Delta_1 h(x,y,s,t,\sigma,\tau)\Big).$$

(a_2) *Let $g(u)$ be a nondecreasing continuous function defined on R_+ with $g(u)>0$ for $u>0$. If*

$$(2.5)\qquad u(x,y)\le c+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}k(x,y,s,t)g(u(s,t))$$
$$+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\Big(\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1}h(x,y,s,t,\sigma,\tau)g(u(\sigma,\tau))\Big),$$

for $x,y\in N_0$, then for $0\le x\le x_1$, $0\le y\le y_1$, $x,x_1,y,y_1\in N_0$,

$$(2.6)\qquad u(x,y)\le G^{-1}[G(c)+\sum_{m=0}^{x-1}\sum_{n=0}^{y-1}[A(m,n)+B(m,n)]],$$

where $A(x,y),B(x,y)$ are given by (2.3), (2.4),

$$(2.7)\qquad G(r)=\int_{r_0}^{r}\frac{dw}{g(w)},\quad r>0,$$

$r_0>0$ is arbitrary, G^{-1} is the inverse of G and $x_1,y_1\in N_0$ be chosen so that

$$G(c)+\sum_{m=0}^{x-1}\sum_{n=0}^{y-1}[A(m,n)+B(m,n)]\in Dom(G^{-1}),$$

for all $x,y\in N_0$ such that $0\le x\le x_1$, $0\le y\le y_1$.

THEOREM 2. *Let $u(x,y)$, $k(x,y,s,t)$, $h(x,y,s,t,\sigma,\tau)$, c be as in Theorem 1 and $b(x,y):N_0^2\to R_+$.*

(b_1) *If*

$$(2.8)\quad u(x,y)\le c+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}b(s,t)u(s,t)+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\Big(\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1}k(s,t,\sigma,\tau)u(\sigma,\tau)\Big)$$
$$+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\Big(\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1}\Big(\sum_{m=0}^{\sigma-1}\sum_{n=0}^{\tau-1}h(s,t,\sigma,\tau,m,n)u(m,n)\Big)\Big),$$

for $x, y \in N_0$, *then*

$$u(x,y) \leq c \prod_{s=0}^{x-1} [1 + \sum_{t=0}^{y-1} Q(s,t)], \tag{2.9}$$

for $x, y \in N_0$, *where*

$$\begin{aligned} Q(x,y) = b(x,y) &+ \sum_{\sigma=0}^{x-1} \sum_{\tau=0}^{y-1} k(x,y,\sigma,\tau) \\ &+ \sum_{\sigma=0}^{x-1} \sum_{\tau=0}^{y-1} \Big(\sum_{m=0}^{\sigma-1} \sum_{n=0}^{\tau-1} h(x,y,\sigma,\tau,m,n) \Big). \end{aligned} \tag{2.10}$$

(b_2) *Let* $g(u)$ *be as in Theorem 1, part* (a_2). *If*

$$\begin{aligned} &u(x,y) \\ &\leq c + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} b(s,t) g(u(s,t)) + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} \Big(\sum_{\sigma=0}^{s-1} \sum_{\tau=0}^{t-1} k(s,t,\sigma,\tau) g(u(\sigma,\tau)) \Big) \\ &+ \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} \Big(\sum_{\sigma=0}^{s-1} \sum_{\tau=0}^{t-1} \Big(\sum_{m=0}^{\sigma-1} \sum_{n=0}^{\tau-1} h(s,t,\sigma,\tau,m,n) g(u(m,n)) \Big) \Big), \end{aligned} \tag{2.11}$$

for $x, y \in N_0$, *then for* $0 \leq x \leq x_2$, $0 \leq y \leq y_2$; $x, x_2, y, y_2 \in N_0$,

$$u(x,y) \leq G^{-1}[G(c) + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} Q(s,t)], \tag{2.12}$$

where $Q(x,y)$ *is given by* (2.10), G, G^{-1} *are as in Theorem 1, part* (a_2) *and* $x_2, y_2 \in N_0$, *be chosen so that*

$$G(c) + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} Q(s,t) \in Dom(G^{-1}),$$

for all $x, y \in N_0$ *such that* $0 \leq x \leq x_2$, $0 \leq y \leq y_2$.

3. Proofs of Theorems 1 and 2

(a_1) We first assume that $c > 0$ and define a function $z(x,y)$ by the right hand side of (2.1). Then $z(x,y) > 0$, $z(0,y) = z(x,0) = c$, and

$$\begin{aligned} &\Delta_1 z(x,y) = z(x+1,y) - z(x,y) \\ &= \sum_{s=0}^{x} \sum_{t=0}^{y-1} k(x+1,y,s,t) u(s,t) + \sum_{s=0}^{x} \sum_{t=0}^{y-1} \Big(\sum_{\sigma=0}^{s-1} \sum_{\tau=0}^{t-1} h(x+1,y,s,t,\sigma,\tau) u(\sigma,\tau) \Big) \end{aligned} \tag{3.1}$$

$$-\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}k(x,y,s,t)u(s,t)-\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\Big(\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1}h(x,y,s,t,\sigma,\tau)u(\sigma,\tau)\Big)$$

$$=\sum_{t=0}^{y-1}k(x+1,y,x,t)u(x,t)+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}k(x+1,y,s,t)u(s,t)$$

$$-\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}k(x,y,s,t)u(s,t)+\sum_{t=0}^{y-1}\Big(\sum_{\sigma=0}^{x-1}\sum_{\tau=0}^{t-1}h(x+1,y,x,t,\sigma,\tau)u(\sigma,\tau)\Big)$$

$$+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\Big(\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1}h(x+1,y,s,t,\sigma,\tau)u(\sigma,\tau)\Big)$$

$$-\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\Big(\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1}h(x,y,s,t,\sigma,\tau)u(\sigma,\tau)\Big)$$

$$=\sum_{t=0}^{y-1}k(x+1,y,x,t)u(x,t)+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\Delta_1k(x,y,s,t)u(s,t)$$

$$+\sum_{t=0}^{y-1}\Big(\sum_{\sigma=0}^{x-1}\sum_{\tau=0}^{t-1}h(x+1,y,x,t,\sigma,\tau)u(\sigma,\tau)\Big)$$

$$+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\Big(\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1}\Delta_1h(x,y,s,t,\sigma,\tau)u(\sigma,\tau)\Big).$$

From (3.1) and using the facts that $u(x,y)\le z(x,y)$, $z(x,y)$ is nondecreasing in $x,y\in N_0$, we have

$$\Delta_2\Delta_1z(x,y)=\Delta_1z(x,y+1)-\Delta_1z(x,y)\tag{3.2}$$

$$=\sum_{t=0}^{y}k(x+1,y+1,x,t)u(x,t)+\sum_{s=0}^{x-1}\sum_{t=0}^{y}\Delta_1k(x,y+1,s,t)u(s,t)$$

$$+\sum_{t=0}^{y}\Big(\sum_{\sigma=0}^{x-1}\sum_{\tau=0}^{t-1}h(x+1,y+1,x,t,\sigma,\tau)u(\sigma,\tau)\Big)$$

$$+\sum_{s=0}^{x-1}\sum_{t=0}^{y}\Big(\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1}\Delta_1h(x,y+1,s,t,\sigma,\tau)u(\sigma,\tau)\Big)$$

$$-\sum_{t=0}^{y-1}k(x+1,y,x,t)u(x,t)-\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\Delta_1k(x,y,s,t)u(s,t)$$

$$
\begin{aligned}
&-\sum_{t=0}^{y-1}\Big(\sum_{\sigma=0}^{x-1}\sum_{\tau=0}^{t-1} h(x+1,y,x,t,\sigma,\tau)u(\sigma,\tau)\Big)\\
&-\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\Big(\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} \Delta_1 h(x,y,s,t,\sigma,\tau)u(\sigma,\tau)\Big)\\
=\ & k(x+1,y+1,x,y)u(x,y)+\sum_{t=0}^{y-1} k(x+1,y+1,x,t)u(x,t)\\
&+\sum_{s=0}^{x-1}\Delta_1 k(x,y+1,s,y)u(s,y)+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\Delta_1 k(x,y+1,s,t)u(s,t)\\
&-\sum_{t=0}^{y-1} k(x+1,y,x,t)u(x,t)-\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\Delta_1 k(x,y,s,t)u(s,t)\\
&+\sum_{\sigma=0}^{x-1}\sum_{\tau=0}^{y-1} h(x+1,y+1,x,y,\sigma,\tau)u(\sigma,\tau)\\
&+\sum_{t=0}^{y-1}\Big(\sum_{\sigma=0}^{x-1}\sum_{\tau=0}^{t-1} h(x+1,y+1,x,t,\sigma,\tau)u(\sigma,\tau)\Big)\\
&+\sum_{s=0}^{x-1}\Big(\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{y-1} \Delta_1 h(x,y+1,s,y,\sigma,\tau)u(\sigma,\tau)\Big)\\
&+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\Big(\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} \Delta_1 h(x,y+1,s,t,\sigma,\tau)u(\sigma,\tau)\Big)\\
&-\sum_{t=0}^{y-1}\Big(\sum_{\sigma=0}^{x-1}\sum_{\tau=0}^{t-1} h(x+1,y,x,t,\sigma,\tau)u(\sigma,\tau)\Big)\\
&-\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\Big(\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} \Delta_1 h(x,y,s,t,\sigma,\tau)u(\sigma,\tau)\Big)\\
=\ & k(x+1,y+1,x,y)u(x,y)+\sum_{s=0}^{x-1}\Delta_1 k(x,y+1,s,y)u(s,y)\\
&+\sum_{t=0}^{y-1}\Delta_2 k(x+1,y,x,t)u(x,t)+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\Delta_2\Delta_1 k(x,y,s,t)u(s,t)
\end{aligned}
$$

$$+\sum_{\sigma=0}^{x-1}\sum_{\tau=0}^{y-1} h(x+1,y+1,x,y,\sigma,\tau)u(\sigma,\tau)$$

$$+\sum_{s=0}^{x-1}\Big(\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{y-1} \Delta_1 h(x,y+1,s,y,\sigma,\tau)u(\sigma,\tau)\Big)$$

$$+\sum_{t=0}^{y-1}\Big(\sum_{\sigma=0}^{x-1}\sum_{\tau=0}^{t-1} \Delta_2 h(x+1,y,x,t,\sigma,\tau)u(\sigma,\tau)\Big)$$

$$+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\Big(\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} \Delta_2\Delta_1 h(x,y,s,t,\sigma,\tau)u(\sigma,\tau)\Big)$$

$$\leq [A(x,y)+B(x,y)]z(x,y).$$

Now by following the proof of Theorem 4.2.1 given in [5] we get

$$z(x,y) \leq c\prod_{m=0}^{x-1}\Big[1+\sum_{n=0}^{y-1}[A(m,n)+B(m,n)]\Big], \tag{3.3}$$

for $x, y \in N_0$. Using (3.3) in $u(x,y) \leq z(x,y)$, we get the required inequality in (2.2).

If $c \geq 0$, we carry out the above procedure with $c+\epsilon$ instead of c, where $\epsilon > 0$ is an arbitrary small constant, and subsequently pass to the limit as $\epsilon \to 0$ to obtain (2.2).

(a_2) Assume that $c > 0$ and define a function $z(x,y)$ by the right hand side of (2.5). Then $z(x,y) > 0$, $z(x,0) = z(0,y) = c$, $u(x,y) \leq z(x,y)$ and $z(x,y)$ is nondecreasing in $x, y \in N_0$. By following the arguments as in the proof of (a_1) upto (3.2) with suitable modifications we get

$$\Delta_2\Delta_1 z(x,y) \leq [A(x,y)+B(x,y)]g(z(x,y)). \tag{3.4}$$

The remaining proof can be completed as in the proof of Theorem 5.2.1 given in [5,p.388].

(b_1) Let $c > 0$ and define a function $z(x,y)$ by the right hand side of (2.8). Then $z(x,y) > 0$, $z(x,0) = z(0,y) = c$, $u(x,y) \leq z(x,y)$, $z(x,y)$ is nondecreasing in $x, y \in N_0$ and

$$\Delta_2\Delta_1 z(x,y) = b(x,y)u(x,y) + \sum_{\sigma=0}^{x-1}\sum_{\tau=0}^{y-1} k(x,y,\sigma,\tau)u(\sigma,\tau) \tag{3.5}$$

$$+\sum_{\sigma=0}^{x-1}\sum_{\tau=0}^{y-1}\Big(\sum_{m=0}^{\sigma-1}\sum_{n=0}^{\tau-1} h(x,y,\sigma,\tau,m,n)u(m,n)\Big)$$

$$\leq Q(x,y)z(x,y).$$

The rest of the proof can be completed by following the proof of Theorem 4.2.1 given in [5].

(b_2) The proof can be completed by following the proof of (b_1) and closely looking at the proof of Theorem 5.2.1 given in [5]. Here, we leave the details to the reader.

4. Applications

In this section, we present applications of the inequality established in Theorem 2, part (b_1) to study certain properties of solutions of the partial finite sum-difference equation of the form

$$\Delta_2\Delta_1 u(x,y) = F\Big(x,y,u(x,y),\sum_{\sigma=0}^{x-1}\sum_{\tau=0}^{y-1} P(x,y,\sigma,\tau,u(\sigma,\tau)),\\ \sum_{\sigma=0}^{x-1}\sum_{\tau=0}^{y-1}\Big(\sum_{m=0}^{\sigma-1}\sum_{n=0}^{\tau-1} H(x,y,\sigma,\tau,m,n,u(m,n))\Big)\Big), \tag{4.1}$$

with the given initial conditions at $x = 0, y = 0$ as

$$u(x,0) = d(x),\ u(0,y) = e(y),\ u(0,0) = 0, \tag{4.2}$$

where $d,e : N_0 \to R$, $P : D \times R \to R$, $H : E \times R \to R$, $F : N_0^2 \times R^3 \to R$. It is easy to observe that the problem (4.1) – (4.2) is equivalent to the following sum-difference equation

$$u(x,y) = d(x) + e(y) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} F\Big(s,t,u(s,t),\\ \sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} P(s,t,\sigma,\tau,u(\sigma,\tau)),\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1}\Big(\sum_{m=0}^{\sigma-1}\sum_{n=0}^{\tau-1} H(s,t,\sigma,\tau,m,n,u(m,n))\Big)\Big). \tag{4.3}$$

The following theorem deals with the boundedness of the solutions of (4.1)-(4.2).

THEOREM 3. *Suppose that the functions d, e, F, P, H satisfy*

$$|d(x)| + |e(y)| \le c, \tag{4.4}$$

$$|F(x,y,u,v,w)| \le b(x,y)\,|u| + |v| + |w|, \tag{4.5}$$

$$|P(x,y,\sigma,\tau,u)| \le k(x,y,\sigma,\tau)\,|u|, \tag{4.6}$$

$$|H(x,y,\sigma,\tau,m,n,u)| \le h(x,y,\sigma,\tau,m,n)\,|u|, \tag{4.7}$$

where c, b, k, h are as in Theorem 2 and

$$\prod_{m=0}^{x-1}\left[1+\sum_{n=0}^{y-1} Q(m,n)\right] < \infty, \tag{4.8}$$

in which $Q(x,y)$ is given by (2.10), *then the solution $u(x,y)$ of* (4.1) – (4.2) *is bounded and*

$$|u(x,y)| \leq c\prod_{m=0}^{x-1}\left[1+\sum_{n=0}^{y-1} Q(m,n)\right], \tag{4.9}$$

for $x, y \in N_0$.

Proof. Let $u(x,y)$ be a solution of (4.1) – (4.2). Then $u(x,y)$ also satisfies (4.3). Using (4.4) – (4.7) in (4.3) we have

$$\begin{aligned}|u(x,y)| \leq{}& c+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1} b(s,t)|u(s,t)| \\ &+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\Big(\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} k(s,t,\sigma,\tau)|u(\sigma,\tau)|\Big) \\ &+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\Big(\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1}\Big(\sum_{m=0}^{\sigma-1}\sum_{n=0}^{\tau-1} h(s,t,\sigma,\tau,m,n)|u(m,n)|\Big)\Big).\end{aligned} \tag{4.10}$$

Now an application of Theorem 2, part (b_1) to (4.10) yields the bound in (4.9), which in view of condition (4.8) implies the boundedness of solutions of (4.1)–(4.2).

Next, we establish the uniqueness of solutions of (4.1)–(4.2).

THEOREM 4. *Suppose that the functions F, P, H in* (4.1) *satisfy the conditions*

$$|F(x,y,u,v,w) - F(x,y,\bar{u},\bar{v},\bar{w})| \leq b(x,y)|u-\bar{u}|+|v-\bar{v}|+|w-\bar{w}|, \tag{4.11}$$

$$|P(x,y,\sigma,\tau,u) - P(x,y,\sigma,\tau,\bar{u})| \leq k(x,y,\sigma,\tau)|u-\bar{u}|, \tag{4.12}$$

$$\begin{aligned}&|H(x,y,\sigma,\tau,m,n,u) - H(x,y,\sigma,\tau,m,n,\bar{u})| \\ &\qquad\qquad \leq h(x,y,\sigma,\tau,m,n)|u-\bar{u}|,\end{aligned} \tag{4.13}$$

where b, k, h are as in Theorem 2. Then (4.1)–(4.2) *has at most one solution on N_0^2.*

Proof. Let $u(x,y)$ and $v(x,y)$ be two solutions of (4.1)–(4.2) for $x, y \in N_0$. Using the facts that $u(x,y)$ and $v(x,y)$ are the solutions of (4.3) and the conditions (4.11)–(4.13) we have

$$(4.14)\qquad |u(x,y)-v(x,y)| \le \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} b(s,t)|u(s,t)-v(s,t)|$$
$$+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\Big(\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} k(s,t,\sigma,\tau)|u(\sigma,\tau)-v(\sigma,\tau)|\Big)$$
$$+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\Big(\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1}\Big(\sum_{m=0}^{\sigma-1}\sum_{n=0}^{\tau-1} h(s,t,\sigma,\tau,m,n)|u(m,n)-v(m,n)|\Big)\Big).$$

Now a suitable application of Theorem 2, part (b_1) (when $c=0$) to (4.14) yields $u(x,y)=v(x,y)$ i.e. there is at most one solution to (4.1)–(4.2) on N_0^2.

Our next result shows the dependency of solutions of equation (4.1) on given initial values.

THEOREM 5. *Let $u_1(x,y)$ and $u_2(x,y)$ be the solutions of* (4.1) *with the given initial conditions*

$$(4.15)\qquad u_1(x,0)=d_1(x),\ u_1(0,y)=e_1(y),\ u_1(0,0)=0,$$

and

$$(4.16)\qquad u_2(x,0)=d_2(x),\ u_2(0,y)=e_2(y),\ u_2(0,0)=0,$$

respectively, where $d_1,d_2,e_1,e_2 : N_0 \to R$ and

$$(4.17)\qquad |d_1(x)+e_1(y)-d_2(x)-e_2(y)| \le c,$$

where $c \ge 0$ is a constant. Suppose that the functions F,P,H in (4.1) *satisfy the conditions* (4.11), (4.12), (4.13). *Then*

$$(4.18)\qquad |u_1(x,y)-u_2(x,y)| \le c\prod_{m=0}^{x-1}\Big[1+\sum_{n=0}^{y-1}Q(m,n)\Big],$$

for $x,y \in N_0$, where $Q(x,y)$ is given by (2.10).

Proof. From the hypotheses, it is easy to observe that

$$(4.19)\qquad |u_1(x,y)-u_2(x,y)| \le |d_1(x)+e_1(y)-d_2(x)-e_2(y)|$$
$$+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\Big|F\Big(s,t,u_1(s,t),\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1}P(s,t,\sigma,\tau,u_1(\sigma,\tau)),$$
$$\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1}\Big(\sum_{m=0}^{\sigma-1}\sum_{n=0}^{\tau-1}H(s,t,\sigma,\tau,m,n,u_1(m,n))\Big)\Big)$$

$$-F\Big(s,t,u_2(s,t),\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1}P(s,t,\sigma,\tau,u_2(\sigma,\tau)),$$

$$\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1}\Big(\sum_{m=0}^{\sigma-1}\sum_{n=0}^{\tau-1}H(s,t,\sigma,\tau,m,n,u_2(m,n))\Big)\Big)\Big|$$

$$\leq c+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}b(s,t)|u_1(s,t)-u_2(s,t)|$$

$$+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\Big(\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1}k(s,t,\sigma,\tau)|u_1(\sigma,\tau)-u_2(\sigma,\tau)|\Big)$$

$$+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\Big(\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1}\Big(\sum_{m=0}^{\sigma-1}\sum_{n=0}^{\tau-1}h(s,t,\sigma,\tau,m,n)|u_1(m,n)-u_2(m,n)|\Big)\Big).$$

Now an application of Theorem 2, part (b_1) to (4.19) yields the estimate (4.18), which shows the dependency of solutions of (4.1) on given initial values.

In conclusion, we note that the inequality given in Theorem 1, part (a_1) can be used to study similar properties as above for the following sum-difference equation

$$u(x,y)=f(x,y)+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}F(x,y,s,t,u(s,t))$$
$$+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\Big(\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1}H(x,y,s,t,\sigma,\tau,u(\sigma,\tau))\Big), \tag{4.20}$$

under some suitable conditions on the functions involved in (4.20). Various other applications of the inequalities in this paper will be given elsewhere.

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57, SHRI NIKETEN COLONEY, NEAR ABHINAY TALKIES,
AURANGABAD, 431001, MAHARASHTRA, INDIA
Email: bgpachpatte@hotmail.com

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