DEMONSTRATIO MATHEMATICA
Vol. XXXVIII No 2 2005

Joso Vukman

# ON $\alpha$-DERIVATIONS OF PRIME AND SEMIPRIME RINGS

**Abstract.** In this paper we investigate identities with $\alpha$-derivations on prime and semiprime rings. We prove, for example, the following result. If $D : R \to R$ is an $\alpha$-derivation of a 2 and 3$-$torsion free semiprime ring $R$ such that $[D(x), x^2] = 0$ holds, for all $x \in R$, then $D$ maps $R$ into its center. The results of this paper are motivated by the work of Thaheem and Samman [20].

## Introduction

Throughout, $R$ is an associative ring with center $Z(R)$. Given an integer $n \geq 2$, a ring $R$ is said to be $n-$torsion free if for $x \in R$, $nx = 0$ implies $x = 0$. As usual we write $[x, y]$ for $xy - yx$ and make use of the commutator identities $[xy, z] = [x, z]y + x[y, z]$, $[x, yz] = [x, y]z + y[x, z]$. We denote by $I$ the identity mapping of a ring $R$. Recall that a ring $R$ is *prime* if for $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is *semiprime* in case $aRa = (0)$ implies $a = 0$. For explanation of the extended centroid $C(R)$ of a semiprime ring $R$ we refer to [1]. An additive mapping $D : R \to R$ is called a *derivation* if $D(xy) = D(x)y + xD(y)$ holds, for all pairs $x, y \in R$. Let $\alpha$ be an automorphism of a ring $R$. An additive mapping $D : R \to R$ is called an $\alpha - $*derivation* if $D(xy) = D(x)a(y) + xD(y)$ holds, for all pairs $x, y \in R$. Note that the mapping, $D = \alpha - I$ is an $\alpha$-derivation. Of course, the concept of $\alpha$-derivation generalizes the concept of derivation, since any $I-$ derivation is a derivation. $\alpha$-derivations are further generalized as $(\alpha, \beta)-$derivations. Let $\alpha, \beta$ be automorphisms of $R$, then an additive mapping $D$ of $R$ into itself is called an $(\alpha, \beta) - $*derivation* if $D(xy) = D(x)\alpha(y) + \beta(x)D(y)$ holds for all $x, y \in R$. $\alpha$-derivations and $(\alpha, \beta)-$derivations have been applied in various situations; in particular, in the solution of some functional equa-

2000 *Mathematics Subject Classification*: 16A12, 1670, 16A72.
*Key words and phrases*: Prime ring, semiprime ring, derivation, automorphism, $\alpha-$derivation, $(\alpha, \beta)-$derivation, commuting mapping, centralizing mapping, skew-commuting mapping, skew-centralizing mapping.
This research has been supported by the Research Council of Slovenia.

tions (see, e. g. Brešar [4]). For more information on $\alpha-$derivations and $(\alpha,\beta)-$derivations, we refer to [3, 4, 8, 9, 10, 11, 12, 13, 14, 20]. In this paper we are concerned with $\alpha$-derivations. A mapping $f$ of $R$ into itself is called *centralizing* on $R$ if $[f(x),x] \in Z(R)$ holds, for all $x \in R$; in the special case when $[f(x),x] = 0$ holds, for all $x \in R$, the mapping $f$ is said to be *commuting* on $R$. The history of commuting and centralizing mappings goes back to 1955 when Divinsky [16] proved that a simple Artinian ring is commutative if it has a commuting nontrivial automorphism. Two years later Posner [19] has proved that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative (Posner's second theorem). Luh [17] generalized the Divinsky result, we have just mentioned above, to arbitrary prime rings. Mayne [18] has proved that in case there exists a nontrivial centralizing automorphism on a prime ring, then the ring is commutative. A result of Brešar [5], which states that every additive commuting mapping $f$ of prime ring $R$ is of the form $f(x) = \lambda x + \zeta(x)$ where $\lambda$ is an element of $C(R)$ and $\zeta : R \to C(R)$ is an additive mapping, should be mentioned. A mapping $f : R \to R$ is called *skew − centralizing* on $R$ if $f(x)x+xf(x) \in Z(R)$ holds for all $x \in R$; in particular, if $f(x)x+xf(x) = 0$ holds for all $x \in R$, then it is called *skew − commuting* on $R$. Brešar [6] has proved that if $R$ is a 2-torsion free semiprime ring and $f : R \to R$ is an additive skew-commuting mapping on $R$, then $f = 0$.

In [20], Thaheem an Samman have proved the following result.

THEOREM A ([20], Proposition 2.3). *Let $D : R \to R$ be an $\alpha$-derivation, where $R$ is a semiprime ring. If $D$ is commuting on $R$, then $D$ maps $R$ into $Z(R)$.*

The result above was the inspiration for our first theorem.

THEOREM 1. *Let $D : R \to R$ be an $\alpha$-derivation, where $R$ is a 2 and $3-$torsion free semiprime ring. Suppose that the mapping $x \mapsto [D(x),x]$ skew-commuting on $R$. In this case $D$ maps $R$ into $Z(R)$.*

Neglecting the fact that in the theorem above we have an additional assumptions that a ring is 2 and 3−torsion free, Theorem 1 generalizes Theorem A. In the proof of Theorem 1 we need Theorem A and the following lemma.

LEMMA 1 ([22], Lemma 1). *Let $R$ be a semiprime ring. Suppose that the relation $axb+bxc = 0$ holds, for all $x \in R$ and some $a,b,c \in R$. In this case $(a+c)xb = 0$ and $bx(a+c) = 0$ is satisfied, for all $x \in R$.*

Proof of Theorem 1. We have therefore the relation

$$(1) \qquad [D(x),x]x + x[D(x),x] = 0, \quad \text{for all } x \in R,$$

which can be written as

(2) $$[D(x), x^2] = 0, \quad \text{for all } x \in R.$$

The linearization of the above relation gives

$$[D(y), x^2] + [D(x), xy + yx] + [D(x), y^2] + [D(y), xy + yx] = 0, \quad \text{for all } x, y \in R.$$

Putting in the above relation $-x$ for $x$ and comparing the relation so obtained with the above relation one obtains

(3) $$[D(y), x^2] + [D(x), xy + yx] = 0, \quad \text{for all } x, y \in R.$$

Putting in the above relation $xy$ for $y$ and applying (2) we obtain

$$\begin{aligned} 0 &= [D(x)a(y) + xD(y), x^2] + [D(x), x(xy + yx)] \\ &= D(x)[\alpha(y), x^2] + x[D(y), x^2] + [D(x), x](xy + yx) + x[D(x), xy + yx] \\ &= D(x)[\alpha(y), x^2] + [D(x), x](xy + yx), \quad \text{for all } x, y \in R. \end{aligned}$$

We have therefore proved that

(4) $$D(x)[\alpha(y), x^2] + [D(x), x](xy + yx) = 0$$

holds, for all $x, y \in R$. The substitution $xy$ for $y$ in the above relation leads to

(5) $$D(x)[\alpha(x)a(y), x^2] + [D(x), x]x(xy + yx) = 0, \quad \text{for all } x, y \in R.$$

Left multiplication of the relation (4) by $x$ gives

(6) $$xD(x)[a(y), x^2] + x[D(x), x](xy + yx) = 0, \quad \text{for all } x, y \in R.$$

Combining (5) and (6) and applying the relation (1) we obtain

$$D(x)[\alpha(x)a(y), x^2] + xD(x)[a(y), x^2] = 0, \quad \text{for all } x, y \in R.$$

We have therefore $D(x)[\alpha(x)y, x^2] + xD(x)[y, x^2] = 0$, for all $x, y \in R$, which can be written in the form

(7) $$A(x)y + B(x)[y, x^2] = 0, \quad \text{for all } x, y \in R,$$

where $A(x)$ and $B(x)$ denotes $D(x)[a(x), x^2]$ and $xD(x) + D(x)\alpha(x)$, respectively. Putting in the above relation $yz$ for $y$ and applying the relation (7) one obtains

$$B(x)y[z, x^2] = 0, \quad \text{for all } x, y, z \in R.$$

The substitution $[z, x^2]yB(x)$ for $y$ in the above relation gives

$$(B(x)[z, x^2])y(B(x)[z, x^2]) = 0, \quad \text{for all } x, y, z \in R,$$

whence it follows $B(x)[z, x^2] = 0$, for all $x, z \in R$, which reduces the relation (7) to $A(x)y = 0$, for all $x, y \in R$, which makes it possible to conclude that

(8) $$A(x) = 0, \quad \text{for all } x \in R.$$

Putting in the relation (4), $y = x$ and applying the above relation we obtain

(9) $$[D(x), x]x^2 = 0, \qquad \text{for all } x \in R.$$

Right multiplication of the relation (1) by $x$ gives according to the above relation

(10) $$x[D(x), x]x = 0, \qquad \text{for all } x \in R.$$

From the relation (9) one obtains after some calculations (see [15])

$$[D(x), x](xy + yx) + ([D(x), y] + [D(y), x])x^2 = 0, \qquad \text{for all } x, y \in R.$$

Multiplying the relation above from the right side by $[D(x), x]x$ and applying the relation (10), we arrive at $[D(x), x]xy[D(x), x]x = 0$, for all $x, y \in R$, which gives

(11) $$[D(x), x]x = 0, \qquad \text{for all } x \in R.$$

Combining the relation (1) with the above relation, we obtain

(12) $$x[D(x), x] = 0, \qquad \text{for all } x \in R.$$

From the relation (11) one obtains easily

$$[D(x), x]y + [D(y), x]x + [D(x), y]x = 0, \qquad \text{for all } x, y \in R.$$

Right multiplication of the above relation by $[D(x), x]$ gives, according to (12), $[D(x), x]y[D(x), x] = 0$, for all $x, y \in R$, whence it follows that $[D(x), x] = 0$, for all $x \in R$. Now Theorem A completes the proof of the theorem. ∎

Our next theorem is inspired by the following result proved by Brešar and Hvala [7]. Suppose there exists an additive mapping $f : R \to R$, where $R$ is a prime ring of characteristic different from two, satisfying the relation $f(x)^2 = x^2$ for all $x \in R$. In this case either $f = I$ or $f = -I$.

THEOREM 2. *Let $D : R \to R$ be an a–derivation, where $R$ is a 2 and 3–torsion free semiprime ring. Suppose that $D(x)^2 = x^2$ holds, for all $x \in R$. In this case $D = 0$.*

Proof. We have therefore the relation

(13) $$D(x)^2 = x^2, \qquad \text{for all } x \in R.$$

The linearization of the above relation gives

(14) $$D(x)D(y) + D(y)D(x) = xy + yx, \qquad \text{for all } x, y \in R.$$

Applying the relation (13) we obtain $[D(x), x^2] = [D(x), D(x)^2] = 0$, for all $x \in R$, which makes it possible to conclude that $D$ maps $R$ into $Z(R)$, according to Theorem 1. The fact that $D(x) \in Z(R)$, for any $x \in R$, means that we can write the relation (14) in the form

(15) $$2D(x)D(y) = xy + yx, \qquad \text{for all } x, y \in R.$$

The substitution $xy$ for $y$ in the above relation gives

$$2D(x)^2a(y) + 2D(x)xD(y) = x(xy + yx), \quad \text{for all } x, y \in R.$$

Multiplying the relation (15) from the left side by $x$ and subtracting the relation so obtained from the above relation, we obtain $2D(x)^2a(y) = 0$, for all $x, y \in R$ (recall that $[D(x), x] = 0$). We have therefore $D(x)^2y = 0$, for all $x, y \in R$, which means that

$$D(x)^2 = 0, \quad \text{for all } x \in R.$$

From the above relation it follows that $D(x) = 0$, for all $x \in R$, since in semiprime rings there are no nonzero central nilpotent elements. The proof of the theorem is complete. ∎

Throughout the proof of our last result we need the following lemma.

LEMMA 2 ([20], Proposition 2.1). *Let $D : R \to R$ be an $\alpha$-derivation, where $R$ is a prime ring, and let $a$ be an element of $R$. Suppose that either $aD(x) = 0$ or $D(x)a = 0$ holds, for all $x \in R$. In this case either $a = 0$ or $D = 0$.*

It is well-known that in case derivations $D, G : R \to R$, of a prime ring $R$ of characteristic different from two, satisfying the relation $D(x)G(x) = 0$, for all $x \in R$, then either $D = 0$ or $G = 0$ (see, Corollary 1 in [2]). The following theorem generalizes the result we have just mentioned.

THEOREM 3. *Let $R$ be a prime ring of characteristic different from two and let $D, G : R \to R$ be $\alpha$-derivations. Suppose that $D(x)G(x) = 0$ holds, for all $x \in R$. In this case either $D = 0$ or $G = 0$.*

Proof. We have the relation

$$D(x)G(x) = 0, \quad \text{for all } x \in R. \tag{16}$$

Let us assume that neither $D$ nor $G$ maps $R$ into $Z(R)$. The linearization of the above relation gives

$$D(x)G(y) + D(y)G(x) = 0, \quad \text{for all } x, y \in R. \tag{17}$$

The substitution $yz$ for $y$ in the above relation gives

$$D(x)G(y)\alpha(z) + D(x)yG(z) + D(y)\alpha(z)G(x) + yD(z)G(x) = 0, \quad \text{for all } x, y, z \in R.$$

According to (17) one can replace in the above relation $D(x)G(y)$ by $-D(y)G(x)$ and $D(z)G(x)$ by $-D(x)G(z)$, which gives

$$D(y)[\alpha(z), G(x)] + [D(x), y]G(z) = 0, \quad \text{for all } x, y, z \in R. \tag{18}$$

In particular for $y=D(x)$ the above relation reduces to $D^2(x)[\alpha(z),G(x)]=0$, for all $x,z\in R$, which means that we have

$$D^2(x)[y,G(x)]=0, \quad \text{for all } x,y\in R.$$

The substitution $yz$ for $y$ in the above relation leads to

$$D^2(x)y[z,G(x)]=0, \quad \text{for all } x,y,z\in R. \tag{19}$$

The linearization of the above relation gives

$$D^2(x)y[z,G(w)]+D^2(w)y[z,G(x)]=0, \quad \text{for all } x,y,z\in R. \tag{20}$$

There exist $x$ and $z$ such that $[z,G(x)]\neq 0$, since we have assumed that $G$ does not map $R$ into $Z(R)$. Therefore, it follows from the relation (19) that $D^2(x)=0$, which reduces the relation (20) to $D^2(w)y[z,G(x)]=0$, for all $y,w\in R$, whence one can conclude that

$$D^2(x)=0, \quad \text{for all } x\in R. \tag{21}$$

Putting in the relation (17) $D(y)$ for $y$ and applying the above relation we obtain $D(x)D(G(y))=0$, for all $x,y\in R$, whence we obtain, since $D\neq 0$ (recall that we have assumed that $D$ does not map $R$ into $Z(R)$), by applying Lemma 2

$$G(D(x))=0, \quad \text{for all } x\in R. \tag{22}$$

Putting in the relation (21) $xy$ for $y$ and applying the relation (21) we obtain $D(x)(D(\alpha(y))+\alpha(D(y)))=0, x,y\in R$, whence, using the same arguments as in the proof of the above relation, it follows that

$$D(\alpha(x))+\alpha(D(x))=0, \quad \text{for all } x\in R. \tag{23}$$

For $z=D(x)$, the relation (18) reduces, because of (22) to

$$D(y)[\alpha(D(z)),G(x)]=0, \quad \text{for all } x,y,z\in R.$$

According to the relation (21) one can replace in the above relation $\alpha(D(z))$ by $-D(\alpha(z))$, which gives $D(y)[D(\alpha(z)),G(x)]=0$, for all $x,y,z\in R$. We have therefore $D(y)[D(z),G(x)]=0$, for all $x,y,z\in R$. Applying again Lemma 2 one can conclude that

$$[D(x),G(y)]=0, \quad \text{for all } x,y\in R. \tag{24}$$

In particular, for $y=x$, the above relation reduces to

$$G(x)D(x)=0, \quad \text{for all } x\in R, \tag{25}$$

because of the relation (16). Right multiplication of the relation (17) by $D(x)$ gives because of (25)

$$D(x)G(y)D(x)=0, \quad \text{for all } x,y\in R.$$

According to the relation (24) one can replace in the above relation $D(x)G(y)$ by $G(y)D(x)$, which leads to

$$G(y)D(x)^2 = 0, \quad \text{for all } x, y \in R.$$

Since $G \neq 0$, the above relation implies

$$D(x)^2 = 0, \quad \text{for all } x \in R.$$

The linearization of the above relation gives

$$D(x)D(y) + D(y)D(x) = 0, \quad \text{for all } x, y \in R. \tag{26}$$

Putting in the relation (18) $D(y)$ for $y$, and applying the relation (21) we arrive at $[D(x), D(y)]G(z) = 0$, for all $x, y, z \in R$, whence it follows

$$[D(x), D(y)] = 0, \quad \text{for all } x, y \in R,$$

since $G \neq 0$. Combining the above relation with the relation (26) we obtain $D(x)D(y) = 0$, for all $x, y \in R$, which gives $D = 0$, contrary to the assumption that $D$ does not map $R$ into $Z(R)$. We have therefore proved that either $D$ or $G$ maps $R$ into $Z(R)$. Suppose that $D$ maps $R$ into $Z(R)$. In this case left multiplication of the relation (16) by $y$ gives

$$D(x)yG(x) = 0, \quad \text{for all } x, y \in R.$$

Suppose that $D(x) \neq 0$, for some $x \in R$. Now it follows from the relation above that $G(x) = 0$, which reduces the relation (14) to $D(x)G(y) = 0$, for all $y \in R$, whence it follows $G = 0$. Since the proof in case $G$ maps $R$ into $Z(R)$ goes through in the same way, we can conclude that the proof of the theorem is complete. ∎

It would be interesting to know whether the results presented in this paper can be generalized to $(\alpha, \beta)-$derivations. Let us point out that Chaudhry and Thaheem [12] proved the following result. Let $\alpha, \beta$ be centralizing automorphisms and let $D$ be an $(\alpha, \beta)-$derivation of a $2-$torsion free semiprime ring $R$, respectively. If $[[D(x), x], x] = 0$ holds for all $x \in R$, then $D$ maps $R$ into its center. The result, we have just mentioned, generalizes a result proved by Vukman in [21].

**Acknowledgement.** The author would like to thank to the referee for helpful suggestions.

## References


[1] K. I. Beidar, W. S. Martindale III, A. V. Mikhalev, *Rings with Generalized Identities*, Marcel Dekker, Inc. New York 1996.

[2] M. Brešar, J. Vukman, *Orthogonal derivations and an extension of a theorem of Posner*, Radovi Mat. Vol. 5 (1989), 237–246.

[3] M. Brešar, J. Vukman, *Jordan $(\vartheta, \phi)$-derivations*, Glasnik Mat. 26 (1991), 13–17.
[4] M. Brešar, *On the composition of $(\alpha, \beta)$-derivations of rings, and an application to von Neumann algebras*, Acta Sci. Math. (1992), 369–376.
[5] M. Brešar, *Centralizing mappings and derivations in prime rings*, J. Algebra 156 (1993), 385–394.
[6] M. Brešar,*On skew-commuting mappings of rings, Bull*, Austral. Math. Soc. 47 (1993), 291–296.
[7] M. Brešar, B. Hvala, *On additive maps of prime rings*, Bull. Austral. Math. Soc. Vol. 51 (1995), 377–381.
[8] J. C. Chang, *$\alpha$-derivations with invertible values*, Bull. Inst. Math. Acad. Sinica, Vol. 13 (1985), 323–333.
[9] J. C. Chang, *On fixed power central $(\alpha, \beta)$-derivations*, Bull. Inst. Math. Acad. Sinica, Vol. 15 (1987), 163–178.
[10] J. C. Chang, *A note on $(\alpha, \beta)$-derivations*, Chinese J. Math. Vol. 19 (1991), 277–285.
[11] J. C. Chang, *On $(\alpha, \beta)$-derivations of prime rings*, Chinese J. Math. Vol. 22 (1994), 21–30.
[12] M. A. Chaudhry, A. B. Thaheem, *$(\alpha, \beta)$-derivations on semiprime rings*, Intern. Math. Journal, Vol. 3 (2003), 1033–1042.
[13] M. A. Chaudhry, A. B. Thaheem, *On $(\alpha, \beta)$-derivations of semiprime rings*, Demonstratio Math. Vol. 36, 2 (2003), 283–287.
[14] T. C. Chen, *Special identities with $(\alpha, \beta)$-derivations*, Riv. Mat. Univ. Parma 5 (1996), 109–119.
[15] L. O. Chung, J. Luh, *Semiprime rings with nilpotent derivatives*, Canad. Math. Bull. 24 (1981), 415–421.
[16] I. N. Divinsky, *On commuting automorphisms of rings*, Trans. Roy. Canada Sect. III, 49 (1955), 19–22.
[17] J. Luh, *A note on commuting automorphisms of rings*, Amer. Math. Monthly 77 (1970), 61–62.
[18] J. H. Mayne, *Centralizing automorphisms of prime rings*, Canad. Math. Bull. Vol. 19 (1976), 113–115.
[19] E. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. 8 (1957), 1093–1100.
[20] A. B. Thaheem, M. S. Samman, *A note on $\alpha$-derivations on semiprime rings*, Demonstratio Math. Vol. 34, 4 (2001), 783–788.
[21] J. Vukman, *Derivations on semiprime rings*, Bull. Austral. Math. Soc. 53 (1995), 353–359.
[22] J. Vukman, *Centralizers on semiprime rings*, Comment. Math. Univ. Carolinae 42 (2001), 783–788.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARIBOR
PEF, Koroška 160
2000 MARIBOR, SLOVENIA
E-mail: joso.vukman@uni-mb.si

*Received March 9, 2004; revised version May 31, 2004.*