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A NOTE ON CENTRALIZERS IN SEMIPRIME RINGS

Abstract. The purpose of this paper is to prove the following result: Let R be a $(m+n+2)!$ and $3m^2n+3mn^2+4m^2+4n^2+10mn$-torsion free semiprime ring with an identity element and let $T : R \to R$ be an additive mapping such that

$$3T(x^{m+n+1}) = T(x)x^{m+n} + x^m T(x)x^n + x^{m+n}T(x)$$

is fulfilled for all $x \in R$ and some fixed nonnegative integers m and n, $m+n \neq 0$. In this case T is a centralizer.

This research has been motivated by the work of Vukman and Kosi-Ulbl [7]. Throughout, R will represent an associative ring with center $Z(R)$. A ring R is n-torsion free, where n is an integer, in case $nx = 0, x \in R$ implies $x = 0$. As usual the commutator $xy - yx$ will be denoted by $[x, y]$. We shall use basis commutator identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$. Recall that R is prime if $aRb = (0)$ implies $a = 0$ or $b = 0$, and is semiprime if $aRa = (0)$ implies $a = 0$. An additive mapping $D : R \to R$, where R is an arbitrary ring is called a derivation in case $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$ and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ holds for all $x \in R$. Obviously, any derivation is a Jordan derivation. The converse is in general not true. Herstein [4] proved that any Jordan derivation on a 2-torsion free prime ring is a derivation. A brief proof of Herstein theorem can be found in [2]. Cusack [3] has extended Herstein theorem to 2-torsion free semiprime rings (see also [1] for an alternative proof). An additive mapping $T : R \to R$ is called a left (right) centralizer in case $T(xy) = T(x)y$ $(T(xy) = xT(y))$ holds for all $x, y \in R$. We follow Zalar [8] and call T a centralizer in case T is both left and right centralizer. In case R has an identity element $T : R \to R$ is left (right) centralizer iff T is of the form $T(x) = ax$ $(T(x) = xa)$ for

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some fixed element $a \in R$. An additive mapping $T : R \to R$ is called a left (right) Jordan centralizer in case $T(x^2) = T(x)x$ ($T(x^2) = xT(x)$) holds for all $x \in R$. Following ideas from [1] Zalar [8] has proved that any left (right) Jordan centralizer on a 2-torsion free semiprime ring is a left (right) centralizer. Recently Vukman [5] has proved that in case $T : R \to R$ is an additive mapping, where R is a 2-torsion free semiprime ring, which satisfies the identity $2T(x^2) = T(x)x + xT(x)$ for all $x \in R$, then T is a centralizer. See also [6] for some other results concerning centralizers in semiprime rings. We start with the following conjecture:

CONJECTURE. Let R be a semiprime ring with suitable torsion restrictions. Any additive mapping $T : R \longrightarrow R$ such that

$$3T(x^{m+n+1}) = T(x)x^{m+n} + x^m T(x)x^n + x^{m+n}T(x)$$

holds for all $x \in R$ and some fixed nonnegative integers m and n, $m+n \neq 0$ is a centralizer.

The result below gives an affirmative answer to the question above in case R has an identity element.

THEOREM. *Let R be a $(m+n+2)!$ and $3m^2n + 3mn^2 + 4m^2 + 4n^2 + 10mn$-torsion free semiprime ring with an identity element and let $T : R \to R$ be an additive mapping such that*

$$3T(x^{m+n+1}) = T(x)x^{m+n} + x^m T(x)x^n + x^{m+n}T(x)$$

is fulfilled for all $x \in R$ and some fixed nonnegative integers m and n, $m + n \neq 0$. In this case T is a centralizer.

Let us see the background of the conjecture and the theorem above. In [7] we find the following result:

THEOREM A. *Let R be a 2 and 3-torsion free semiprime ring and let $T : R \to R$ be an additive mapping. Suppose that*

$$3T(xyx) = T(x)yx + xT(y)x + xyT(x)$$

holds for all $x, y \in R$. In this case T is a centralizer.

For $y = x$ the above equation reduces to

$$3T(x^3) = T(x)x^2 + xT(x)x + x^2T(x), \quad x \in R.$$

This observation leads to the conjecture and the theorem mentioned above.

Proof of Theorem. We have the relation

(1) $$3T(x^{m+n+1}) = T(x)x^{m+n} + x^m T(x)x^n + x^{m+n}T(x), \quad x \in R.$$

Putting $x + c$ for x in the above relation, where c is any element of the center $Z(R)$, we obtain

$$(2)\quad 3\sum_{i=0}^{m+n+1}\binom{m+n+1}{i}T(x^{m+n+1-i}c^i)$$
$$=\sum_{j=0}^{m+n}\binom{m+n}{j}[T(x)+T(c)]x^{m+n-j}c^j+$$
$$+\sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}x^{m-k}c^k[T(x)+T(c)]x^{n-l}c^l+$$
$$+\sum_{p=0}^{m+n}\binom{m+n}{p}x^{m+n-p}c^p[T(x)+T(c)],\qquad x\in R.$$

Using (1) and rearranging the equation (2) in sense of collecting together terms involving equal number of factors c and $T(c)$ we obtain:

$$(3)\qquad \sum_{i=1}^{m+n} f_i(x,c)=0,\quad x\in R,$$

where $f_i(x,c)$ stand for the expression of terms involving i factors of c.

Let e stand for the identity element. Replacing c by $2e$, $3e$, $(m+n)e$ in turn in the equation (3), and expressing the resulting system of $m+n$ homogeneous equations, we see that the coefficient matrix of the system is a van der Monde matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^{m+n} \\ \vdots & \vdots & \vdots & \vdots \\ m+n & (m+n)^2 & \cdots & (m+n)^{m+n} \end{bmatrix}.$$

Since any prime factor of this determinant is also a factor of $(m+n-1)!$, it follows that the system has only the trivial solution.

In particular,

$$f_{m+n-1}(x,e)=3\binom{m+n+1}{m+n-1}T(x^2)-\binom{m+n}{m+n-1}T(x)x-$$
$$-\binom{m+n}{m+n-2}ax^2-\binom{m}{m-2}x^2a-\binom{n}{n-2}ax^2-$$
$$-\binom{m}{m-1}xT(x)-\binom{n}{n-1}T(x)x-$$
$$-\binom{m}{m-1}\binom{n}{n-1}xax-\binom{m+n}{m+n-1}xT(x)-$$
$$-\binom{m+n}{m+n-2}x^2a=0,\quad x\in R$$

and

$$f_{m+n}(x,e) = 3\binom{m+n+1}{m+n}T(x) - \binom{m+n}{m+n}T(x) - \binom{m+n}{m+n-1}ax-$$
$$-\binom{m}{m-1}xa - \binom{n}{n-1}ax - \binom{m}{m}\binom{n}{n}T(x)-$$
$$-\binom{m+n}{m+n}T(x) - \binom{m+n}{m+n-1}xa = 0, \quad x \in R,$$

where a stands for $T(e)$. The above equations reduce to

$$\begin{aligned}(4) \quad & 3(m+n+1)(m+n)T(x^2)\\ &= (2m+4n)T(x)x + (4m+2n)xT(x)\\ &\quad + (m^2+2n^2+2mn-m-2n)ax^2\\ &\quad + (2m^2+n^2+2mn-2m-n)x^2a + 2mnxax, \quad x \in R\end{aligned}$$

and

$$(5) \qquad 3(m+n)T(x) = (m+2n)ax + (2m+n)xa, \quad x \in R,$$

respectively. According to (5) one obtains the relation

$$3(m+n)T(x^2) = (m+2n)ax^2 + (2m+n)x^2a, \quad x \in R.$$

Using the above connection one can replace the expression $3(m+n)T(x^2)$ with $(m+2n)ax^2 + (2m+n)x^2a$ in the relation (4). Thus we have after some calculation

$$\begin{aligned}(6) \quad & (mn+2m+4n)ax^2 + (mn+4m+2n)x^2a-\\ & -(2m+4n)T(x)x - (4m+2n)xT(x) - 2mnxax = 0, \quad x \in R.\end{aligned}$$

Left and right multiplication of the relation (5) by x gives

$$(7) \qquad 3(m+n)xT(x) = (m+2n)xax + (2m+n)x^2a, \quad x \in R$$

and

$$(8) \qquad 3(m+n)T(x)x = (m+2n)ax^2 + (2m+n)xax, \quad x \in R,$$

respectively.

Using the relations (7) and (8) in the relation (6) multiplied by $3(m+n)$ gives after some calculation

$$(9) \qquad A(m,n)ax^2 + A(m,n)x^2a - 2A(m,n)xax = 0, \quad x \in R,$$

where $A(m,n)$ stands for $3m^2n + 3mn^2 + 4m^2 + 4n^2 + 10mn$. Since R is a $(m+n+2)!$-torsion free ring, we obtain from the above relation

$$x^2a + ax^2 - 2xax = 0, \quad x \in R.$$

The above relation can be written in the form

$$[[a, x], x] = 0, \quad x \in R. \tag{10}$$

Linearization of the above relation gives

$$[[a, x], y] + [[a, y], x] = 0, \quad x, y \in R. \tag{11}$$

Putting xy for y in relation (11) we obtain because of (10) and (11):

$$\begin{aligned} 0 &= [[a, x], xy] + [[a, xy], x] = \\ &= [[a, x], x]y + x[[a, x], y] + [[a, x]y + x[a, y], x] = \\ &= x[[a, x], y] + [[a, x], x]y + [a, x][y, x] + x[[a, y], x] = \\ &= [a, x][y, x], \quad x, y \in R. \end{aligned}$$

Thus we have

$$[a, x][y, x] = 0, \quad x, y \in R.$$

The substitution ya for y in the above relation gives

$$[a, x]y[a, x] = 0, \quad x, y \in R. \tag{12}$$

Let us point out that so far we have not used the assumption that R is semiprime. Since R is semiprime, it follows from the relation (12) that $[a, x] = 0$, $x \in R$. In other words, $a \in Z(R)$, which reduces the relation (5) to $T(x) = ax$, and $T(x) = xa$ for all $x \in R$. The proof of the theorem is complete.

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