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# HORIZONTAL EXTENSION OF CONNECTIONS INTO (2)-CONNECTIONS

**Abstract.** We discuss the prolongation of connections to to some non product preserving bundles. We introduce the concept of $(r)$-connection on a fibered manifold $Y$ and for a given connection $\Gamma$ on $Y$ we construct its horizontal extension $\Gamma^{(2)}$. We also prove that $\Gamma^{(2)}$ is the unique (2)-connection on $Y$ canonically dependent on $\Gamma$.

## 0. Introduction

Let $T^{(r)}M$ be the $r$-th order tangent vector bundle defined by $T^{(r)}M = (J^r(M,\mathbb{R})_0)^*$. We recall that a general connection on a fibered manifold $p : Y \to M$ is a smooth section $\Gamma : Y \to J^1Y$ of the first jet prolongation of $Y$, which can be also interpreted as the lifting map (denoted by the same symbol)

$$\Gamma : Y \times_M TM \to TY.$$

DEFINITION. An $(r)$-*connection* on a fibered manifold $p : Y \to M$ is a fiber linear map

$$\widetilde{\Gamma} : Y \times_M T^{(r)}M \to T^{(r)}Y$$

over the identity of $Y$ such that $T^{(r)}p \circ \widetilde{\Gamma}(y,v) = v$ for every $(y,v) \in Y \times_M T^{(r)}M$.

Clearly, for $r = 1$ we obtain the notion of a connection, because there is an identification $T^{(1)}M \widetilde{=} TM$. In this note, for a connection $\Gamma : Y \times_M TM \to TY$ we introduce its horizontal extension $\Gamma^{(2)} : Y \times_M T^{(2)}M \to T^{(2)}Y$. The main result is the following theorem.

THEOREM 1. *The horizontal extension* $\Gamma^{(2)} : Y \times_M T^{(2)}M \to T^{(2)}Y$ *of* $\Gamma$ *is the unique* (2)*-connection on* $Y$ *canonically dependent on* $\Gamma$.

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Section 1 is devoted to the problem of the existence of a natural operator transforming connections on $p : Y \to M$ into connections on $Fp : FY \to FM$, where $F$ is a non product preserving bundle functor. We introduce examples of such natural operators for concrete non product preserving functors. In Section 2 we construct the horizontal extension of a connection $\Gamma$ on $p : Y \to M$. Section 3 is devoted to the proof of Theorem 1.

All manifolds and maps are assumed to be of class $\mathcal{C}^\infty$. Unless otherwise specified, we use the terminology and notation from the book [2].

## 1. Prolongation of connections to some non product preserving bundles

The motivation of the present paper is the following. Let $F$ be a bundle functor on the category $\mathcal{M}f$ of smooth manifolds and all smooth maps and let $\Gamma : Y \to J^1Y$ be a general connection on the fibered manifold $p : Y \to M$. It is well known that if $F$ preserves products, then $\Gamma$ induces a connection on $Fp : FY \to FM$, see [2]. By the general theory, every product preserving functor $F$ on $\mathcal{M}f$ is a Weil functor $F = T^A$ determined by a Weil algebra $A$. In what follows the connection on $T^Ap : T^AY \to T^AM$ induced by a connection $\Gamma$ on $p : Y \to M$ will be denoted by $\mathcal{T}^A\Gamma$. Clearly, $\mathcal{T}^A\Gamma : T^AY \to J^1T^AY$ and the lifting map is of the form

$$\mathcal{T}^A\Gamma : T^AY \times_{T^AM} TT^AM \to TT^AY.$$

We remark that the connection $\mathcal{T}^A\Gamma$ has been constructed by I. Kolář, [1], in the case of higher order velocities functors and then by J. Slovák, [3], in the general case of an arbitrary Weil functor.

Write $\mathcal{FM}$ for the category of fibered manifolds and fibered manifold morphisms. Denoting by $B : \mathcal{FM} \to \mathcal{M}f$ the base functor, the geometrical construction of a connection on $Fp : FY \to FM$ by means of a connection on $p : Y \to M$ is a natural operator of the form $J^1 \rightsquigarrow J^1(F \to FB)$. By [2], if the functor $F$ does not preserve products, then there is an open problem on the existence of a natural operator $J^1 \rightsquigarrow J^1(F \to FB)$. Now we present examples of such natural operators for concrete non product preserving functors.

Example 1. Let $Q$ be a fixed manifold and define a bundle functor $F^Q$ on $\mathcal{M}f$ by

$$F^QM = M \times Q, \quad F^Qf = f \times \mathrm{id}_Q : F^QM \to F^QN.$$

If $\mathrm{card}(Q) > 1$, then $F^Q$ is a non product preserving bundle functor of order zero. Conversely, an arbitrary bundle functor $F$ on $\mathcal{M}f$ of order zero is naturally equivalent to some $F^Q$. Given a connection $\Gamma : Y \to J^1Y$ on

$p : Y \to M$, we can define a map

$$A^Q(\Gamma) = \Gamma \times \mathrm{id}_Q : Y \times Q \to J^1Y \times Q \subset J^1(Y \times Q),$$

where the inclusion $J^1Y \times Q \subset J^1(Y \times Q)$ is given by $(j^1_x\sigma, q) \mapsto j^1_{(x,q)}(\sigma \times \mathrm{id}_Q)$. Clearly, $A^Q(\Gamma) : Y \times Q \to J^1(Y \times Q)$ is a connection on $F^Qp : F^QY \to F^QM$.

EXAMPLE 2. Consider a Weil functor $T^B$ and the functor $F^Q$ from Example 1. Then the composition $F = F^Q \circ T^B$ is a non product preserving bundle functor. Let $\mathcal{T}^B\Gamma : T^BY \to J^1T^BY$ be the connection on $T^Bp : T^BY \to T^BM$ determined by the connection $\Gamma$ on $p : Y \to M$, [2]. Write

$$A^{Q,B}(\Gamma) = A^Q(\mathcal{T}^B\Gamma).$$

Obviously, $A^{Q,B}(\Gamma)$ is the connection on $Fp : FY \to FM$ and the geometrical construction $\Gamma \mapsto A^{Q,B}(\Gamma)$ is an example of a natural operator transforming connections to non product preserving bundles.

We recall that a bundle functor $F$ on $\mathcal{M}f$ is said to have the point property, if $F(\mathrm{pt}) = \mathrm{pt}$, where pt denote a one-point manifold. For example, every product preserving bundle functor $F = T^A$ has the point property, while the functors $F^Q$ and $F^Q \circ T^B$ from Example 1 and Example 2 have not. Using such a point of view, the problem on the existence of a natural operator $J^1 \rightsquigarrow J^1(F \to FB)$ reduces to the following question:

*Does there exist a natural operator transforming connections on $p : Y \to M$ into connections on $Fp : FY \to FM$ for any concrete non product preserving functor $F : \mathcal{M}f \to \mathcal{FM}$ with the point property?*

The simple example of such a functor is the second order tangent functor $T^{(2)}$. So there is a problem on the existence of a connection

$$A(\Gamma) : T^{(2)}Y \times_{T^{(2)}M} TT^{(2)}M \to TT^{(2)}Y$$

on $T^{(2)}p : T^{(2)}Y \to T^{(2)}M$ canonically dependent on a connection $\Gamma$ on $Y \to M$. If such $A(\Gamma)$ exists, then we can construct a (2)-connection $\tilde{A}(\Gamma) : Y \times_M T^{(2)}M \to T^{(2)}Y$ on $Y$ by

$$\tilde{A}(\Gamma)(y, v) = \mathrm{pr}_2 \circ A(\Gamma)(0_y, \frac{d}{dt}_{t=0}(0_x + tv)), \quad (y, v) \in Y_x \times T^{(2)}_x M, \quad x \in M,$$

where $0_y \in T^{(2)}_y Y$ and $0_x \in T^{(2)}_x M$ are the zero elements, $\mathrm{pr}_2 : T_{0_y}T^{(2)}Y \tilde{=} T_yY \times T^{(2)}_y Y \to T^{(2)}_y Y$ is the projection onto the second factor and $\tilde{=}$ is the obvious identification

$$T_yY \times T^{(2)}_y Y \ni (u, w) \tilde{=} T^{(2)}\tilde{u}(0_y) + \frac{d}{dt}_{t=0}(tw) \in T_{0_y}T^{(2)}Y.$$

Here $\mathcal{T}^{(2)}\tilde{u}$ denote the flow prolongation of a vector field $\tilde{u} \in \mathcal{X}(Y)$ with $\tilde{u}(y) = u$. Clearly, $\mathcal{T}^{(2)}\tilde{u}(0_y)$ is independent of the choice of $\tilde{u}$ with $\tilde{u}(y) = u$. The above construction indicates that Theorem 1 may be the first step in direction to solve the problem formulated above.

## 2. Construction of the horizontal extension

First we prove the following general assertion.

PROPOSITION 1. (a) *Given an element $w = j^2_{x_o}\gamma \in T^{2*}_{x_o}M \tilde{=} (T^{(2)}_{x_o}M)^*$, $x_o \in M$, $\gamma : M \to \mathbb{R}$, $\gamma(x_o) = 0$, define two maps $\Phi_w : \mathcal{X}(M) \to \mathbb{R}$ and $\Psi_w : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathbb{R}$ by*

$$\Phi_w(V) = V\gamma(x_o) \text{ and } \Psi_w(V,W) = \frac{1}{2}(VW\gamma(x_o) + WV\gamma(x_o))$$

*for $V, W \in \mathcal{X}(M)$. Then $\Phi_w$ is linear over $\mathbb{R}$, $\Psi_w$ is symmetric and 2-linear over $\mathbb{R}$,*

$$\Phi_w(fV) = f(x_o)\Phi_w(V) \tag{1}$$

*for $V \in \mathcal{X}(M)$ and $f \in C^\infty(M)$ and*

$$\begin{aligned}\Psi_w(fV, gW) = &\frac{1}{2}(f(x_o)Vg(x_o)\Phi_w(W) + g(x_o)Wf(x_o)\Phi_w(V)) \\ &+ f(x_o)g(x_o)\Psi_w(V,W)\end{aligned} \tag{2}$$

*for $V, W \in \mathcal{X}(M)$ and $f, g \in C^\infty(M)$.*

(b) *Conversely, suppose that we have a linear (over $\mathbb{R}$) map $\Phi : \mathcal{X}(M) \to \mathbb{R}$ and a symmetric 2-linear (over $\mathbb{R}$) map $\Psi : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathbb{R}$ such that for some $x_o \in M$*

$$\Phi(fV) = f(x_o)\Phi(V) \tag{1'}$$

*for $V \in \mathcal{X}(M)$, $f \in C^\infty(M)$ and*

$$\begin{aligned}\Psi(fV, gW) = &\frac{1}{2}(f(x_o)Vg(x_o)\Phi(W) + g(x_o)Wf(x_o)\Phi(V)) \\ &+ f(x_o)g(x_o)\Psi(V,W)\end{aligned} \tag{2'}$$

*for $V, W \in \mathcal{X}(M)$ and $f, g \in C^\infty(M)$. Then there exists one and only one element $w \in T^{2*}_{x_o}M$ such that $\Phi = \Phi_w$ and $\Psi = \Psi_w$.*

Proof. Clearly, it suffices to prove the part (b). Since $\Phi(V)$ and $\Psi(V,W)$ depend only on $germ_{x_o}(V)$ and $germ_{x_o}(W)$, we can assume that $M = \mathbb{R}^m$ and $x_o = 0$. Define $w = j^2_0\gamma$ by

$$\gamma : \mathbb{R}^m \to \mathbb{R}, \quad \gamma(0) = 0, \quad \frac{\partial}{\partial x^i}\gamma(0) = \Phi\left(\frac{\partial}{\partial x^i}\right)$$

and

$$\frac{\partial^2}{\partial x^i \partial x^j}\gamma(0) = \Psi\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$$

for $i, j = 1, \ldots, m$. Obviously, the definition of $w$ is correct as $\Psi$ is symmetric. It remains to prove that $\Phi = \Phi_w$ and $\Psi = \Psi_w$. We see that $\Phi(\frac{\partial}{\partial x^i}) = \Phi_w(\frac{\partial}{\partial x^i})$ and $\Psi(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \Psi_w(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ for $i, j = 1, \ldots, m$. Further, using conditions (1), (2), (1′) and (2′) and the linearity we can see that $\Phi(V) = \Phi_w(V)$ and $\Psi(V, W) = \Psi_w(V, W)$ for any $V = \sum f_i \frac{\partial}{\partial x^i}$ and $W = \sum g_i \frac{\partial}{\partial x^i}$. The proof of Proposition 1 is complete. ∎

Using Proposition 1 we can present the following construction of a (2)-connection by means of a general connection $\Gamma : Y \times_M TM \to TY$ on a fibered manifold $p : Y \to M$.

Consider $y_o \in Y_{x_o} = p^{-1}(x_o)$, $x_o \in M$ and take an arbitrary element $w \in T^{2*}_{y_o}Y$. Let $\Phi_w : \mathcal{X}(Y) \to \mathbb{R}$ and $\Psi_w : \mathcal{X}(Y) \times \mathcal{X}(Y) \to \mathbb{R}$ be the maps corresponding to $w$ in the sense of Proposition 1. Define $\Phi^\Gamma_w : \mathcal{X}(M) \to \mathbb{R}$ and $\Psi^\Gamma_w : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathbb{R}$ by

$$\Phi^\Gamma_w(V) = \Phi_w(V^\Gamma) \text{ and } \Psi^\Gamma_w(V, W) = \Psi_w(V^\Gamma, W^\Gamma), \quad V, W \in \mathcal{X}(M),$$

where $V^\Gamma \in \mathcal{X}(Y)$ is the $\Gamma$-horizontal lift of the vector field $V$.

LEMMA 1. *Maps $\Phi := \Phi^\Gamma_w$ and $\Psi := \Psi^\Gamma_w$ satisfy the assumptions of Proposition* 1(b).

Proof. Denote by $f^V := f \circ p : Y \to \mathbb{R}$ the vertical lift of a function $f : M \to \mathbb{R}$. Then the assertion follows from the properties of $\Phi_w$ and $\Psi_w$, from the linearity of the $\Gamma$-horizontal lift and from the formulas $(fW)^\Gamma = f^V W^\Gamma$ and $W^\Gamma f^V = (Wf)^V$, $W \in \mathcal{X}(M)$, $f \in C^\infty(M)$. ∎

So, by Proposition 1 there is one and only one element $w^\Gamma \in T^{2*}_{x_o}M$ corresponding to $(\Phi^\Gamma_w, \Psi^\Gamma_w)$.

LEMMA 2. *The mapping $(\Gamma^{(2)})^*_{y_o} : T^{2*}_{y_o}Y \to T^{2*}_{x_o}M$, $(\Gamma^{(2)})^*_{y_o}(w) = w^\Gamma$, is linear.*

Proof. It follows from the fact that $(\Phi_w, \Psi_w)$ depends linearly on $w$. ∎

Define a map $\Gamma^{(2)} : Y \times_M T^{(2)}M \to T^{(2)}Y$ as the dual map of $(\Gamma^{(2)}_y)^*$ for any $y \in Y$.

PROPOSITION 2. *The mapping $\Gamma^{(2)} : Y \times_M T^{(2)}M \to T^{(2)}Y$ is a (2)-connection on $p : Y \to M$ canonically dependent on the connection $\Gamma$.*

Proof. It remains to observe that $T^{(2)}p \circ \Gamma^{(2)}(y, v) = v$ for any $(y, v) \in Y \times_M T^{(2)}M$ and that we have not used charts in the construction of $\Gamma^{(2)}$. ∎

DEFINITION. The (2)-connection $\Gamma^{(2)}$ is called *the horizontal extension of* $\Gamma$.

REMARK 1. Obviously, the restriction and corestriction of $\Gamma^{(2)}$ to $Y \times_M TM \subset Y \times_M T^{(2)}M$ and $TY \subset T^{(2)}Y$ is equal to $\Gamma$. So $\Gamma^{(2)}$ is in fact an extension of $\Gamma$.

## 3. Proof of Theorem 1

The existence has been already proved in Section 2, so that it suffices to prove the uniqueness. Let $A(\Gamma), B(\Gamma) : Y \times_M T^{(2)}M \to T^{(2)}Y$ be (2)-connections canonically dependent on a connection $\Gamma : Y \times_M TM \to TY$ on a fibered manifold $p : Y \to M$ with $\dim(M) = m$ and $\dim(Y) = m + n$. In other words, we have two $\mathcal{FM}_{m,n}$-natural operators $A : \Gamma \to A(\Gamma)$ and $B : \Gamma \to B(\Gamma)$ in the sense of the book [2].

PROPOSITION 3. *We have* $A = B$.

The proof of Proposition 3 will occupy the rest of this section. From now on let $\mathbb{R}^{m,n} = \mathbb{R}^m \times \mathbb{R}^n$ be the trivial fiber bundle over $\mathbb{R}^m$ with fiber $\mathbb{R}^n$, and $x^1, \dots, x^m, y^1, \dots, y^n$ be the standard coordinates on $\mathbb{R}^{m,n}$.

Define a map $\Phi_A : Con(\mathbb{R}^{m,n}) \times T_0^{(2)}\mathbb{R}^m \to \mathbb{R}$ by

$$\Phi_A(\Gamma, v) := \langle A(\Gamma)((0,0), v), j^2_{(0,0)}y^1 \rangle$$

for any connection $\Gamma$ on $\mathbb{R}^{m,n}$ and any $v \in T_0^{(2)}\mathbb{R}^m$.

LEMMA 3. *If* $\Phi_A = \Phi_B$, *then* $A = B$.

Proof. It is a consequence of $\mathcal{FM}_{m,n}$-naturality of $A$ and $B$ and of the fact that the $\mathcal{FM}_{m,n}$-orbit of $j^2_{(0,0)}y^1$ is dense in $T^{2*}Y$ for any $\mathcal{FM}_{m,n}$-object $Y$. ∎

LEMMA 4. *Suppose that*

$$\Phi_A\Big(\sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i=1}^m \sum_{j=1}^n \sum_{|\alpha|+|\beta| \le K} \Gamma^j_{i\alpha\beta} x^\alpha y^\beta dx^i \otimes \frac{\partial}{\partial y^j}, (j_0^2 x^\rho)^*\Big) =$$

$$= \Phi_B\Big(\sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i=1}^m \sum_{j=1}^n \sum_{|\alpha|+|\beta| \le K} \Gamma^j_{i\alpha\beta} x^\alpha y^\beta dx^i \otimes \frac{\partial}{\partial y^j}, (j_0^2 x^\rho)^*\Big)$$

*for any* $(j_0^2 x^\rho)^* \in T_0^{(2)}\mathbb{R}^m$, *any* $K \in \mathbb{N}$ *and any* $\Gamma^j_{i\alpha\beta}$ *for* $i, j, \alpha, \beta$ *as indicated, where* $((j_0^2 x^\rho)^*)_{1 \le |\rho| \le 2}$ *is the basis of* $T_0^{(2)}\mathbb{R}^m$ *dual to* $(j_0^2 x^\rho)_{1 \le |\rho| \le 2}$. *Then* $A = B$.

Proof. It follows from Lemma 3 and from the corollary of non-linear Peetre theorem (Corollary 19.8 in [2]). ∎

LEMMA 5. *Suppose that*

$$\Phi_A\Big(\sum_{i=1}^m dx^i\otimes\frac{\partial}{\partial x^i}+\sum_{i=1}^m\sum_{j=1}^n\sum_{|\alpha|+|\beta|\le 1}\Gamma^j_{i\alpha\beta}x^\alpha y^\beta dx^i\otimes\frac{\partial}{\partial y^j},(j^2_0x^\rho)^*\Big)=$$
$$=\Phi_B\Big(\sum_{i=1}^m dx^i\otimes\frac{\partial}{\partial x^i}+\sum_{i=1}^m\sum_{j=1}^n\sum_{|\alpha|+|\beta|\le 1}\Gamma^j_{i\alpha\beta}x^\alpha y^\beta dx^i\otimes\frac{\partial}{\partial y^j},(j^2_0x^\rho)^*\Big)$$

*for any* $(j^2_0x^\rho)^*\in T^{(2)}_0\mathbb{R}^m$ *and any* $\Gamma^j_{i\alpha\beta}$ *for* $i,j,\alpha,\beta$ *as indicated. Then* $A=B$.

Proof. Using the invariance of $A$ with respect to the homotheties $\frac{1}{t}\,\mathrm{id}_{\mathbb{R}^{m,n}}$ for $t\neq 0$ we obtain the homogeneity condition

$$\Phi_A\Big(\sum_{i=1}^m dx^i\otimes\frac{\partial}{\partial x^i}+\sum_{i=1}^m\sum_{j=1}^n\sum_{|\alpha|+|\beta|\le K}t^{|\alpha|+|\beta|}\Gamma^j_{i\alpha\beta}x^\alpha y^\beta dx^i\otimes\frac{\partial}{\partial y^j},(j^2_0x^\rho)^*\Big)$$
$$=t^{|\rho|-1}\Phi_A\Big(\sum_{i=1}^m dx^i\otimes\frac{\partial}{\partial x^i}+\sum_{i=1}^m\sum_{j=1}^n\sum_{|\alpha|+|\beta|\le K}\Gamma^j_{i\alpha\beta}x^\alpha y^\beta dx^i\otimes\frac{\partial}{\partial y^j},(j^2_0x^\rho)^*\Big)$$

for $t>0$. Since $1\le|\rho|\le 2$, the homogeneous function theorem (see [2]) reads

$$\Phi_A\Big(\sum_{i=1}^m dx^i\otimes\frac{\partial}{\partial x^i}+\sum_{i=1}^m\sum_{j=1}^n\sum_{|\alpha|+|\beta|\le K}\Gamma^j_{i\alpha\beta}x^\alpha y^\beta dx^i\otimes\frac{\partial}{\partial y^j},(j^2_0x^\rho)^*\Big)=$$
$$=\Phi_A\Big(\sum_{i=1}^m dx^i\otimes\frac{\partial}{\partial x^i}+\sum_{i=1}^m\sum_{j=1}^n\sum_{|\alpha|+|\beta|\le 1}\Gamma^j_{i\alpha\beta}x^\alpha y^\beta dx^i\otimes\frac{\partial}{\partial y^j},(j^2_0x^\rho)^*)\Big).$$

From this equality for $A$ and for $B$ instead of $A$ in the assumption of our lemma there follows the assumption of Lemma 4, and applying Lemma 4 we complete the proof. ∎

LEMMA 6. *Suppose that*

$$(3)\ \Phi_A\Big(\sum_{i=1}^m dx^i\otimes\frac{\partial}{\partial x^i}+x^{k_o}dx^{i_o}\otimes\frac{\partial}{\partial y^{j_o}}+\sum_{i=1}^m\sum_{j,l=1}^n\Gamma^j_{il}y^l dx^i\otimes\frac{\partial}{\partial y^j},(j^2_0x^\rho)^*\Big)$$
$$=\Phi_B\Big(\sum_{i=1}^m dx^i\otimes\frac{\partial}{\partial x^i}+x^{k_o}dx^{i_o}\otimes\frac{\partial}{\partial y^{j_o}}+\sum_{i=1}^m\sum_{j,l=1}^n\Gamma^j_{il}y^l dx^i\otimes\frac{\partial}{\partial y^j},(j^2_0x^\rho)^*\Big)$$

*for any* $i_o,k_o=1,\dots,m$, $j_o=1,\dots,n$, *any* $\Gamma^j_{il}\in\mathbb{R}$ *for* $i,j,l$ *as indicated* *and any* $(j^2_0 x^\rho)^* \in T^{(2)}_0\mathbb{R}^m$. *Suppose also that*

$$(4)\quad \Phi_A\Big(\sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + dx^{i_o} \otimes \frac{\partial}{\partial y^{j_o}} + \sum_{i=1}^m \sum_{j,l=1}^n \Gamma^j_{il} y^l dx^i \otimes \frac{\partial}{\partial y^j}, (j^2_0 x^\rho)^*\Big)$$
$$= \Phi_B\Big(\sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + dx^{i_o} \otimes \frac{\partial}{\partial y^{j_o}} + \sum_{i=1}^m \sum_{j,l=1}^n \Gamma^j_{il} y^l dx^i \otimes \frac{\partial}{\partial y^j}, (j^2_0 x^\rho)^*\Big)$$

*for any* $i_o = 1, \dots, m$, $j_o = 1, \dots, n$, *any* $\Gamma^j_{il} \in \mathbb{R}$ *for* $i, j, l$ *as indicated and any* $(j^2_0 x^\rho)^* \in T^{(2)}_0\mathbb{R}^m$. *Then* $A = B$.

P r o o f. By the invariance of $A$ with respect to fiber homotheties $(x^1, \dots, x^m, ty^1, \dots, ty^n)$ for $t \neq 0$ we obtain the homogeneity condition

$$\Phi_A\Big(\sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i=1}^m \sum_{j=1}^n \sum_{|\alpha|+|\beta|\le 1} t^{1-|\beta|}\Gamma^j_{i\alpha\beta} x^\alpha y^\beta dx^i \otimes \frac{\partial}{\partial y^j}, (j^2_0 x^\rho)^*\Big)$$
$$= t\Phi_A\Big(\sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i=1}^m \sum_{j=1}^n \sum_{|\alpha|+|\beta|\le 1} \Gamma^j_{i\alpha\beta} x^\alpha y^\beta dx^i \otimes \frac{\partial}{\partial y^j}, (j^2_0 x^\rho)^*\Big)$$

for $t > 0$. By the homogeneous function theorem,

$$\Phi_A\Big(\sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i=1}^m \sum_{j=1}^n \sum_{|\alpha|+|\beta|\le 1} \Gamma^j_{i\alpha\beta} x^\alpha y^\beta dx^i \otimes \frac{\partial}{\partial y^j}, (j^2_0 x^\rho)^*\Big)$$

is a linear combination of $\Gamma^j_{i\alpha\beta}$ for $|\beta| = 0$ with coefficients being smooth maps in $\Gamma^j_{i\alpha\beta}$ for $|\beta| = 1$. Write $e_k = (0, \dots, 0, 1, 0, \dots, 0)$ and $(0) = (0, \dots, 0)$. Clearly, if $|\beta| = 0$ (i.e. $\beta = (0)$), then we have $|\alpha| = 1$ or $|\alpha| = 0$. This yields

$$\Phi_A = \sum C^{ie_k}_j \Gamma^j_{ie_k(0)} + \sum D^i_j \Gamma^j_{i(0)(0)}$$

where the coefficients $C^{ie_k}_j$ and $D^i_j$ are smooth maps of all $\Gamma^j_{i(0)\beta}$ with $|\beta| = 1$. Obviously, $\Gamma^{j_o}_{i_o e_{k_0}(0)} = 1$ and all other $\Gamma^j_{i\alpha(0)} = 0$ correspond to (3) and $\Gamma^{j_o}_{i_o(0)(0)} = 1$ and all other $\Gamma^j_{i\alpha(0)} = 0$ correspond to (4). Therefore Lemma 6 is a simple consequence of Lemma 5. ∎

LEMMA 7. *Suppose that*

$$\Phi_A\Big(\sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + x^{k_o} dx^{i_o} \otimes \frac{\partial}{\partial y^{j_o}}, (j^2_0 x^\rho)^*\Big)$$
$$= \Phi_B\Big(\sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + x^{k_o} dx^{i_o} \otimes \frac{\partial}{\partial y^{j_o}}, (j^2_0 x^\rho)^*\Big)$$

*for any* $i_o, k_o = 1, \dots, m$, $j_o = 1, \dots, n$ *and any* $(j_0^2 x^\rho)^* \in T_0^{(2)}\mathbb{R}^m$. *Then* $A = B$.

Proof. It remains to verify the assumptions of Lemma 6.

Step 1. *The first assumption of Lemma 6.* By the invariance of $A$ with respect to the homotheties $(\frac{1}{t}x^1, \dots, \frac{1}{t}x^m, \tau y^1, \dots, \tau y^n)$ for $t \neq 0$ and $\tau \neq 0$ we obtain the homogeneity condition

$$\begin{aligned}(5)\qquad \Phi_A\Big(\sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + \tau t^2 a x^{k_o} dx^{i_o} \otimes \frac{\partial}{\partial y^{j_o}} \\ + \sum_{i=1}^m \sum_{j,l=1}^n t\Gamma_{il}^j y^l dx^i \otimes \frac{\partial}{\partial y^j}, (j_0^2 x^\rho)^*\Big) \\ = \tau t^{|\rho|} \Phi_A\Big(\sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + a x^{k_o} dx^{i_o} \otimes \frac{\partial}{\partial y^{j_o}} \\ + \sum_{i=1}^m \sum_{j,l=1}^n \Gamma_{il}^j y^l dx^i \otimes \frac{\partial}{\partial y^j}, (j_0^2 x^\rho)^*\Big)\end{aligned}$$

for $t > 0$, $\tau > 0$. Then by the homogeneous function theorem

$$\begin{aligned}\Phi_A\Big(\sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + x^{k_o} dx^{i_o} \otimes \frac{\partial}{\partial y^{j_o}} + \sum_{i=1}^m \sum_{j,l=1}^n \Gamma_{il}^j y^l dx^i \otimes \frac{\partial}{\partial y^j}, (j_0^2 x^\rho)^*\Big) \\ = \Phi_A\Big(\sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + x^{k_o} dx^{i_o} \otimes \frac{\partial}{\partial y^{j_o}}, (j_0^2 x^\rho)^*\Big)\end{aligned}$$

if $|\rho| = 2$, and

$$\Phi_A\Big(\sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + x^{k_o} dx^{i_o} \otimes \frac{\partial}{\partial y^{j_o}} + \sum_{i=1}^m \sum_{j,l=1}^n \Gamma_{il}^j y^l dx^i \otimes \frac{\partial}{\partial y^j}, (j_0^2 x^\rho)^*\Big) = 0$$

if $|\rho| = 1$. Using this fact for $A$ and for $B$ instead of $A$ we see that the assumption of our lemma implies the first assumption of Lemma 6.

Further, write $t = 1$ in the homogeneity condition (5). Then $\Phi_A$ is linear in $a$ with coefficients being smooth maps of the remaining terms. For $a = 0$ we obtain

$$(6)\qquad \Phi_A\Big(\sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i=1}^m \sum_{j,l=1}^n \Gamma_{il}^j y^l dx^i \otimes \frac{\partial}{\partial y^j}, (j_0^2 x^\rho)^*\Big) = 0$$

for any $(j_0^2 x^\rho)^* \in T_0^{(2)}\mathbb{R}^m$.

Step 2. *The second assumption of Lemma 6.* By the invariance of $A$ with respect to $(x^1, \dots, x^m, y^1, \dots, y^{j_o} - x^{i_o}, \dots, y^n)$ (only $j_o$-position is exceptional) we obtain

$$\begin{aligned}\Phi_A\Big(\sum_{i=1}^{m} dx^i\otimes\frac{\partial}{\partial x^i}+dx^{i_o}\otimes\frac{\partial}{\partial y^{j_o}}+\sum_{i=1}^{m}\sum_{j,l=1}^{n}\Gamma^j_{il}y^l dx^i\otimes\frac{\partial}{\partial y^j},(j^2_0x^\rho)^*\Big)\\ =\Phi_A\Big(\sum_{i=1}^{m} dx^i\otimes\frac{\partial}{\partial x^i}+\sum_{i=1}^{m}\sum_{j,l=1}^{n}\Gamma^j_{il}y^l dx^i\otimes\frac{\partial}{\partial y^j},(j^2_0x^\rho)^*\Big)\\ +\delta^{j_o}_1\langle (j^2_0x^\rho)^*,j^2_0x^{i_o}\rangle\\ =\delta^{j_o}_1\langle (j^2_0x^\rho)^*,j^2_0x^{i_o}\rangle.\end{aligned}$$

Here $\delta^{j_o}_1$ denote the Kronecker delta and we have also used (6) and the fact that $A(\Gamma)$ is a (2)-connection. The similar is true also for $B$ instead of $A$. This proves our claim. ■

LEMMA 8. *Suppose that*

$$\begin{aligned}\Phi_A\Big(\sum_{i=1}^{m} dx^i\otimes\frac{\partial}{\partial x^i}+x^{k_o}dx^{i_o}\otimes\frac{\partial}{\partial y^{j_o}},(j^2_0(x^{i_o}x^{k_o}))^*\Big)\\ =\Phi_B\Big(\sum_{i=1}^{m} dx^i\otimes\frac{\partial}{\partial x^i}+x^{k_o}dx^{i_o}\otimes\frac{\partial}{\partial y^{j_o}},(j^2_0(x^{i_o}x^{k_o}))^*\Big)\end{aligned}$$

*for any $i_o,k_o=1,\dots,m$ and $j_o=1,\dots,n$. Then $A=B$.*

Proof. By the invariance of $A$ with respect to $(\frac{1}{t^1}x^1,\dots,\frac{1}{t^m}x^m,\tau y^1,\dots$ $\dots,\tau y^n)$ for $t=(t^1,\dots,t^m)\in(\mathbb{R}_+\setminus\{0\})^m$ and $\tau\neq 0$ we obtain the homogeneity condition

$$\begin{aligned}\Phi_A\Big(\sum_{i=1}^{m} dx^i\otimes\frac{\partial}{\partial x^i}+\tau t^{i_o}t^{k_o}ax^{k_o}dx^{i_o}\otimes\frac{\partial}{\partial y^{j_o}},(j^2_0x^\rho)^*\Big)\\ =\tau t^\rho\Phi_A\Big(\sum_{i=1}^{m} dx^i\otimes\frac{\partial}{\partial x^i}+ax^{k_o}dx^{i_o}\otimes\frac{\partial}{\partial y^{j_o}},(j^2_0x^\rho)^*\Big)\end{aligned}$$

for any $t\in(\mathbb{R}_+\setminus\{0\})^m$ and $\tau>0$. Then by the homogeneous function theorem

$$\Phi_A\Big(\sum_{i=1}^{m} dx^i\otimes\frac{\partial}{\partial x^i}+x^{k_o}dx^{i_o}\otimes\frac{\partial}{\partial y^{j_o}},(j^2_0x^\rho)^*\Big)=0$$

if only $x^\rho\neq x^{k_o}x^{i_o}$. The similar fact holds for $B$ instead of $A$. Hence Lemma 8 is a simple consequence of Lemma 7. ■

LEMMA 9. *Suppose that*

$$\Phi_A(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + x^{k_o} dx^{i_o} \otimes \frac{\partial}{\partial y^{j_o}} + x^{i_o} dx^{k_o} \otimes \frac{\partial}{\partial y^{j_o}}, (j_0^2(x^{i_o} x^{k_o}))^*) =$$

$$= \Phi_B(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + x^{k_o} dx^{i_o} \otimes \frac{\partial}{\partial y^{j_o}} + x^{i_o} dx^{k_o} \otimes \frac{\partial}{\partial y^{j_o}}, (j_0^2(x^{i_o} x^{k_o}))^*)$$

*for any* $i_o, k_o = 1, \ldots, m$ *and* $j_o = 1, \ldots, n$. *Then* $A = B$.

Proof. By the invariance of $A$ with respect to $(x^1, \ldots, x^m, ty^1, \ldots, ty^n)$ for $t \neq 0$ we obtain the homogeneity condition

$$\Phi_A\left(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + tax^{k_o} dx^{i_o} \otimes \frac{\partial}{\partial y^{j_o}} + tbx^{i_o} dx^{k_o} \otimes \frac{\partial}{\partial y^{j_o}}, (j_0^2(x^{i_o} x^{k_o}))^*\right)$$

$$= t\Phi_A\left(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + ax^{k_o} dx^{i_o} \otimes \frac{\partial}{\partial y^{j_o}} + bx^{i_o} dx^{k_o} \otimes \frac{\partial}{\partial y^{j_o}}, (j_0^2(x^{i_o} x^{k_o}))^*\right).$$

Further, by the homogeneous function theorem we obtain

$$\Phi_A\left(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + x^{k_o} dx^{i_o} \otimes \frac{\partial}{\partial y^{j_o}} + x^{i_o} dx^{k_o} \otimes \frac{\partial}{\partial y^{j_o}}, (j_0^2(x^{i_o} y^{j_o}))^*\right)$$

$$= \Phi_A\left(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + x^{k_o} dx^{i_o} \otimes \frac{\partial}{\partial y^{j_o}}, (j_0^2(x^{i_o} y^{j_o}))^*\right)$$

$$+ \Phi_A\left(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + x^{i_o} dx^{k_o} \otimes \frac{\partial}{\partial y^{j_o}}, (j_0^2(x^{i_o} y^{j_o}))^*\right).$$

By the invariance of $A$ with respect to the change of coordinates $x^{k_o}$ and $x^{i_o}$ only we have

$$\Phi_A\left(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + x^{k_o} dx^{i_o} \otimes \frac{\partial}{\partial y^{j_o}}, (j_0^2(x^{i_o} x^{k_o}))^*\right)$$

$$= \Phi_A\left(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + x^{i_o} dx^{k_o} \otimes \frac{\partial}{\partial y^{j_o}}, (j_0^2(x^{i_o} x^{k_o}))^*\right).$$

Hence

$$\Phi_A\left(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + x^{k_o} dx^{i_o} \otimes \frac{\partial}{\partial y^{j_o}}, (j_0^2(x^{i_o} x^{k_o}))^*\right)$$

$$= \frac{1}{2}\Phi_A\left(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + x^{k_o} dx^{i_o} \otimes \frac{\partial}{\partial y^{j_o}} + x^{i_o} dx^{k_o} \otimes \frac{\partial}{\partial y^{j_o}}, (j_0^2(x^{i_o} x^{k_o}))^*\right).$$

Therefore Lemma 9 follows from Lemma 8. ∎

LEMMA 10. *Suppose that*

$$A\Big(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + df \otimes \frac{\partial}{\partial y^{j_o}}\Big) = B\Big(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + df \otimes \frac{\partial}{\partial y^{j_o}}\Big)$$

*over* $(0,0) \in \mathbb{R}^m \times \mathbb{R}^n$ *for any* $f \in C^\infty(\mathbb{R}^m)$ *and any* $j_o = 1, \dots, n$. *Then* $A = B$.

Proof. From the assumption for $f = x^{k_o} x^{i_o}$ there follows the assumption of Lemma 9. Using Lemma 9 we complete the proof. ■

LEMMA 11. *Suppose that*

$$A\Big(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + dx^1 \otimes \frac{\partial}{\partial y^{j_o}}\Big) = B\Big(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + dx^1 \otimes \frac{\partial}{\partial y^{j_o}}\Big)$$

*over* $(0,0) \in \mathbb{R}^m \times \mathbb{R}^n$ *for any* $j_o = 1, \dots, n$. *Then* $A = B$.

Proof. It remains to verify the assumption of Lemma 10. Because of the regularity of $A$ and $B$ we can assume that $d_0 f \neq 0$. Then $df = dx^1$ near $0 \in \mathbb{R}^m$ modulo a diffeomorphism $\varphi : \mathbb{R}^m \to \mathbb{R}^m$. So Lemma 11 is a consequence of Lemma 10 because of the invariance of $A$ and $B$ with respect to $\varphi$. ■

**We are now in position to prove Proposition 3.** Let $(j_0^2 x^\rho)^* \in T_0^{(2)}\mathbb{R}^m$. We can write

$$A\Big(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i}\Big)((0,0),(j_0^2 x^\rho)^*) = (j^2_{(0,0)} x^\rho)^* + \sum_{|\alpha|+|\beta| \le r,\ |\beta| \neq 0} a_{\alpha\beta} (j^2_{(0,0)}(x^\alpha y^\beta))^*.$$

By the invariance of $A$ with respect to $(x^1, \dots, x^m, ty^1, \dots, ty^n)$ for $t \neq 0$ we deduce that

$$A\Big(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i}\Big)((0,0),(j_0^2 x^\rho)^*) = (j^2_{(0,0)} x^\rho)^*.$$

Using this fact for $A$ and for $B$ instead of $A$ we obtain

$$A\Big(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i}\Big) = B\Big(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i}\Big)$$

over $(0,0)$. Now by the invariance of $A$ and $B$ with respect to $(x^1, \dots, x^m, y^1, \dots, y^{j_o} + x^1, \dots y^n)$ (only $j_o$-position is exceptional) we have

$$A\Big(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + dx^1 \otimes \frac{\partial}{\partial y^{j_o}}\Big) = B\Big(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + dx^1 \otimes \frac{\partial}{\partial y^{j_o}}\Big)$$

over $(0,0) \in \mathbb{R}^m \times \mathbb{R}^n$ for any $j_o = 1, \ldots, n$. By Lemma 11, $A = B$ and the proof of Proposition 3 is complete. ■

Theorem 1 is an immediate consequence of Propositions 2 and 3.

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