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A MODULUS AND AN EXTREMAL FORM OF A FOLIATION

Abstract. We prove that the p-modulus of the foliation is conformal invariant. We study the problem of existing of the extremal form for a foliaton on Riemannian manifold. We also compute the value of p-modulus and the extremal form for k-dimensional foliation given by a submersion.

1. Introduction

The idea of *modulus* is directly connected with the concept of *extremal length* of curves in $\mathbb{R}^2$ introduced by Beurling and Ahlfors [AhBe] in the beginning of 50-ties. In 1957 Fuglede generalized this notion to the modulus of k-dimensional surface families in $\mathbb{R}^n$. It was very usefull tool in the theory of conformal and quasiconformal maps, extremely popular in 60-ties and 70-ties.

Using a geometric characterization Suominen [Su] extended the modulus to the case of an arbitrary differential Riemannian manifold, and in 1979 Krivov [Kr] defined *generalized p-modulus* for a family of k-forms.

The *modulus of a foliation*, introduced by the author in [Bl], connects Fuglede's and Krivov's ideas. We used the fact that the foliation of Riemannian manifold may be defined as a family of surfaces or by a family of forms. The modulus of the foliation is just a modulus in the Krivov's sence of the family of forms characteristic for the foliation. For these forms, by Hodge star, arises the family of dual forms. Both these classes seem to characterize pairs of foliations orthogonal to each other, but this is an open problem yet.

In this paper we prove, that the p-modulus of the foliation is conformal invariant. We study the problem of existing of the extremal form for a foliaton on Riemannian manifold. We also compute the value of p-modulus and find the extremal form for k-dimensional foliation given by a submersion.

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2. Preliminaries

2.1. Foliations

Let $\mathcal{F}$ be a smooth oriented k-dimensional foliation on smooth oriented Riemannian n-manifold M.

DEFINITION 2.1. A *foliated chart* on M is a pair (U, ϕ), where $U \subset M$ is open, $\phi = (\phi_1, \phi_2) : U \to B_\tau \times B_\pitchfork$ is a diffeomorphism and B_τ, $B_\pitchfork$ are open rectangular neighborhoods is $\mathbb{R}^k$ and $\mathbb{R}^l$ respectively.

Sets $P_y = \phi^{-1}(B_\tau \times \{y\})$, $y \in B_\pitchfork$ are called *plaques*, and $S_x = \phi^{-1}(\{x\} \times B_\pitchfork)$, $x \in B_\tau$ — *transversals* of the foliated chart.

If M admits an atlas $\mathcal{U} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in \mathcal{A}}$ of foliated charts and for each $\alpha \in \mathcal{A}$ and each $x \in M$ $L_x \cap U_\alpha$ is a union of plaques, then $\mathcal{U}$ is said to be a *foliated atlas associated to* $\mathcal{F}$.

DEFINITION 2.2. A foliated atlas $\mathcal{U} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in \mathcal{A}}$ is said to be *regular* if

1. for each $\alpha \in \mathcal{A}$, $\overline{U_\alpha}$ is a compact subset of a foliated chart $(W_\alpha, \psi_\alpha) \in \mathcal{U}$ and $\phi_\alpha = \psi_\alpha|_{U_\alpha}$,
2. the cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ is locally finite,
3. if (U_α, ϕ_α) and (U_β, ϕ_β) belong to $\mathcal{U}$, each plaque from U_α meets at most one plaque in U_β.

In [CaCo] one can find that for every foliated manifold $(M, \mathcal{F})$ there exists a regular foliated atlas associated with $\mathcal{F}$.

Moreover, if $\mathcal{U} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in \mathcal{A}}$ is a regular atlas on M, then the map

$$g_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} \colon \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

has the form

$$g_{\alpha\beta}((\phi_\beta)_1, (\phi_\beta)_2) = (\rho_{\alpha\beta}((\phi_\beta)_1, (\phi_\beta)_2), \gamma_{\alpha\beta}((\phi_\beta)_2)),$$

where $\gamma_{\alpha\beta}$ is a diffeomorphism from $(\phi_\beta)_2(U_\alpha \cap U_\beta)$ to $(\phi_\alpha)_2(U_\alpha \cap U_\beta)$. The family $\gamma = \{\gamma_{\alpha,\beta}\}_{\alpha,\beta \in \mathcal{A}}$ satisfies the cocycle condition, that is for every $\alpha, \beta, \delta \in \mathcal{A}$ and for each $x \in U_\alpha \cap U_\beta \cap U_\delta$

$$\gamma_{\alpha,\delta}(x) = \gamma_{\alpha,\beta} \circ \gamma_{\beta,\delta}(x).$$

We will call this family the *holonomy cocycle* of the regular foliated atlas $\mathcal{U}$.

REMARK 2.3 ([CaCo]). The submersion $(\phi_\alpha)_2 : U_\alpha \to \mathbb{R}^l$ let us to identify all transversals $S_{\alpha,x}$ for each $x \in U_\alpha$, so we can use S_α instead of $S_{\alpha,x}$. Therefore, if $\alpha, \beta \in \mathcal{A}$ and $U_\alpha \cap U_\beta \neq \emptyset$, we view $\gamma_{\alpha,\beta}$ as a diffeomorphism from an open subset of S_β onto an open subset of S_α. It is clear that y belongs to the domain of $\gamma_{\alpha,\beta}$ if and only if the plaque $P_y \subset U_\beta$ intersects a unique plaque $P_z \subset U_\alpha$, where $P_z \cap S_\alpha = \{z\}$ and $\gamma_{\alpha,\beta}(y) = z$.

2.2. Modulus

Let$(M,\mathcal{F})$ be a smooth oriented foliated Riemannian n-manifold and $dim\mathcal{F} = k$. We denote by $\mathbb{L}_p^k(M)$ $(p \geq 1)$ the space of measurable and p-integrable k-forms ω on M with norm

$$\|\omega\| = \Big(\int\limits_M |\omega(x)|^p \sigma_M \Big)^{\frac{1}{p}},$$

where σ_M is the volume form of M.

DEFINITION 2.4. Let $\mathcal{L} \subset \mathcal{F}$. By $adm(\mathcal{L})$ we denote the family of all k-forms on M such that

1. $\omega \in \mathbb{L}_p^k(M)$,
2. $\int_L \omega \geq 1$ for almost every leaf $L \in \mathcal{L}$,
3. ω is almost everywhere positively defined (i.e. for almost every $x \in M$ and for every orthonormal positive oriented base $e_1, \ldots, e_k \in T_x\mathcal{F}$ holds $\omega(e_1, \ldots, e_k) \geq 0$).

Elements of $adm(\mathcal{L})$ we call *admissible forms* for $\mathcal{L}$.

DEFINITION 2.5. The *p-modulus* of $\mathcal{L}$ we define by

$$mod_p(M,\mathcal{L}) = \inf_{\omega \in adm(\mathcal{L})} \|\omega\|.$$

If there exists an admissible form ω such that $\|\omega\| = mod_p(M,\mathcal{L})$ we call it an *extremal* form for $\mathcal{L}$ and denote by $\omega_0(\mathcal{L})$. If $\mathcal{L} = \mathcal{F}$ we have a *modulus of a foliation* $\mathcal{F}$.

The modulus has some usefull properties [Bl]:

1. it is monotone and countable subadditive, i.e.
$$mod_p(M,\mathcal{L}_1) \leq mod_p(M,\mathcal{L}_2)\,, \quad if \quad \mathcal{L}_1 \subset \mathcal{L}_2\,,$$
and
$$(mod_p(M,\mathcal{L}))^p \leq \sum_{i\in\mathbb{N}} (mod_p(M,\mathcal{L}_i))^p\,, \quad if \quad \mathcal{L} = \bigcup_{i\in\mathbb{N}} \mathcal{L}_i\,,$$
2. if N is an open subset of M and $\mathcal{L} \subset \mathcal{F}$, then
$$mod_p(M, \mathcal{L} \cap \tilde{N}) \leq mod_p(N, \mathcal{L}|_N),$$
where $\mathcal{L}|_N = \{L \cap N, L \in \mathcal{L}\}$ and $\tilde{N}$ is the saturation of N in $\mathcal{F}$,
3. if $\mathcal{L} \subset \mathcal{F}$ and N_1, N_2 are open subsets of M, such that for almost every leaf $L \in \mathcal{L}$ we have $L \cap N_1 \neq \emptyset$ and $L \cap N_2 \neq \emptyset$, then
$$(mod_p(M,\mathcal{L}))^{\frac{p}{1-p}} \geq (mod_p(N_1, \mathcal{L}|_{N_1}))^{\frac{p}{1-p}} + (mod_p(N_2, \mathcal{L}|_{N_2}))^{\frac{p}{1-p}},$$
4. A family $\mathcal{L} \subset \mathcal{F}$ is p-exceptional if and only if there exists an admissible form ω for $\mathcal{L}$ such that $\int_L \omega = \infty$ for almost all leaves $L \in \mathcal{L}$,

5. If the volume of M is finite, then the family of leaves $\mathcal{L} = \{L \in \mathcal{F};\ volL = \infty\}$ is p-exceptional.

2.3. Integrability of forms

In this paper we will need the following version of the Fubini theorem:

THEOREM 2.6 ([Pi]). *Let M and N be Riemannian manifolds of dimension m and n respectively $(m > n)$, $f: M \to N$ a submersion, and J_f— Jacobian of the mapping $df(x)|_{(ker df(x))^\perp}$. If $h: M \to \mathbb{R}$ is an integrable function on M then for almost every $y \in N$ the integral*

$$F(x) = \int_{f^{-1}(y)} h \frac{1}{J_f} \sigma_{f^{-1}(y)}$$

is finite, the function $y \mapsto F(y)$ is measurable, integrable over N and

$$\int_M h\, \sigma_M = \int_N F\, \sigma_N\,.$$

We will need also the following lemma:

LEMMA 2.7. *Let ω be a k-form on M. For every leaf L of a foliation $\mathcal{F}$ an inequality*

$$|\int_L \omega| \le \int_L |\omega|\, \sigma_L$$

holds, provided ω and $|\omega|$ are integrable over L.

Proof. Let $\mathcal{U} = \{U_i\}_{i \in \mathcal{I}}$ be a locally finite cover of M and let $x \in U_{i_0}$ for some $i_0 \in \mathcal{I}$. Let us denote by $e_1, \dots, e_n$ an orthonormal basis of smooth vector fields in U_{i_0}, such as for every $x \in U_{i_0}$ $e_1(x), \dots, e_k(x)$ is a positively oriented basis of $T_x\mathcal{F}$. Consider a k-form

$$\omega = \sum_{i_1 \cdots i_k} \omega_{i_1 \cdots i_k} e^*_{i_1} \wedge \cdots \wedge e^*_{i_k}, \qquad 1 \le i_1 < \cdots < i_k \le n$$

and assume that $supp(\omega) \subset U_{i_0}$. Than we have

$$\int_{L \cap U_{i_0}} \omega = \int_{L \cap U_{i_0}} \omega_{1 \dots k} e^*_1 \wedge \cdots \wedge e^*_k \le \int_{L \cap U_{i_0}} \sqrt{\sum_{i_1 \cdots i_k} \omega^2_{i_1 \cdots i_k}} e^*_1 \wedge \cdots \wedge e^*_k = \int_{L \cap U_{i_0}} |\omega| \sigma_L$$

and consequently

$$\left| \int_{L \cap U_{i_0}} \omega \right| \le \int_{L \cap U_{i_0}} |\omega|\, \sigma_L. \tag{2.1}$$

Now assume, that there is no $i \in \mathcal{I}$, such that $supp(\omega) \subset U_i$. Let us denote by $\{\psi_s\}_{s \in \mathbb{N}}$ a partition of unity subordinate to the cover $\mathcal{U}$ i.e. a family of functions $\psi_s: M \to \mathbb{R}$ with $supp(\psi_s) \subset U_s$, satisfying for any $x \in M$

the two following conditions

$$0 \le \psi(x) \le 1 \quad \text{and} \quad \sum_s \psi_s(x) = 1\,.$$

Then $\omega = \sum_s (\psi_s\omega)$ and, using (2.1), we obtain that for any s

$$\Big|\int_L \psi_s\omega\Big| = \Big|\int_{L\cap U_s} \psi_s\omega\Big| \le \int_{L\cap U_s} |\psi_s\omega| = \int_L |\psi_s\omega|\,.$$

Therefore

$$\Big|\int_L \omega\Big| = \Big|\int_L \sum_{s\in\mathbb{N}} \psi_s\omega\Big| = \Big|\sum_{s\in\mathbb{N}}\int_L \psi_s\omega\Big| \le \sum_{s\in\mathbb{N}}\Big|\int_L \psi_s\omega\Big|\,\sigma_L \le$$

$$\le \sum_{s\in\mathbb{N}}\int_L |\psi_s\omega|\,\sigma_L = \sum_{s\in\mathbb{N}}\int_L |\psi_s|\,|\omega|\,\sigma_L = \int_L \Big(\sum_{s\in\mathbb{N}}|\psi_s|\Big)\,|\omega|\,\sigma_L = \int_L |\omega|\,\sigma_L\,,$$

what completes the proof. ■

3. Results

3.1. Conformal invariantness

Consider two n-dimensional Riemannian manifolds M_1 and M_2 and diffeomorphism $f\colon M_1 \to M_2$. Let $x \in M_1$. Denote by $e_1,\dots,e_n$ and $\tilde e_1,\dots,\tilde e_n$ orthonormal basis of T_xM_1 and $T_{f(x)}M_2$ respectively. Then there exist real numbers $\lambda_1 \ge \dots \ge \lambda_n$ such that $f^*(\tilde e_i^{\,*}) = \lambda_i e_i^*$ and the number

$$K_s(f) = \operatorname*{ess\,sup}_{x\in M} \frac{(\lambda_1\cdots\lambda_n)^{\frac{s}{n}}}{\lambda_{n-s+1}\cdots\lambda_n}$$

is called *s-dilatation* of diffeomorphism f.

If $\lambda_1 = \dots = \lambda_n$ then $K_1(f) = \ldots = K_n(f) = 1$ and f is conformal.

In [Kr] Krivov proved that if $p = n/k$ then for every family $\mathcal{A} \in \mathbb{L}_p^k(M_2)$ hold inequalities

$$K_k(f)^{-1} \inf_{\eta\in\mathcal{A}} \|\eta\| \le \inf_{\omega\in f^*\mathcal{A}} \|\omega\| \le K_{n-k}(f) \inf_{\eta\in\mathcal{A}} \|\eta\|.$$

We will use this fact to prove the following

THEOREM 3.1. *Let $(M_1,\mathcal{F}_1)$, $(M_2,\mathcal{F}_2)$ be n-dimensional foliated Riemannian manifolds with $dim\mathcal{F}_1 = dim\mathcal{F}_2 = k$. If $f\colon M_1 \to M_2$ is a diffeomorphism preserving orientation and foliation then for $p = n/k$ we have*

$$K_k(f)^{-1}\, mod_p(M_2,\mathcal{F}_2) \le mod_p(M_1,\mathcal{F}_1) \le K_{n-k}(f)\, mod_p(M_2,\mathcal{F}_2).$$

Proof. Let $(M_1,\mathcal{F}_1)$, $(M_2,\mathcal{F}_2)$ and $f\colon M_1 \to M_2$ satisfy the assumptions of the theorem. Let $x \in M_1$ and (U,ϕ) be a chart on M_1 around x. Denote by $e_1,\dots,e_n$ and $\tilde e_1,\dots,\tilde e_n$ positive oriented orthonormal bases of smooth vector fields in U and $f(U)$, respectively, such that $e_1,\dots,e_k \in T\mathcal{F}_1|_U$ and $\tilde e_1,\dots,\tilde e_k \in T\mathcal{F}_2|_{f(U)}$. Then $f^*(\tilde e_i^{\,*}) = \lambda_i e_i^*$ for some real numbers $\lambda_1 \ge \dots \ge \lambda_n$. Notice that λ_i are positive, since f preserves orientation.

We will show that

$$f^*(adm(\mathcal{F}_2)) = adm(\mathcal{F}_1). \tag{3.1}$$

Let ω be an admissible k-form for $\mathcal{F}_2$. Then locally

$$\omega|_U = \sum_{i_1\cdots i_k} \omega_{i_1\cdots i_k}\tilde{e}^*_{i_1} \wedge \cdots \wedge \tilde{e}^*_{i_k}, \qquad 1 \le i_1 < \cdots < i_k \le n$$

and

$$\begin{aligned} f^*(\omega|_U) &= f^*\Big(\sum_{i_1\cdots i_k} \omega_{i_1\cdots i_k}\, \tilde{e}^*_{i_1} \wedge \cdots \wedge \tilde{e}^*_{i_k}\Big) \\ &= \sum_{i_1\cdots i_k} (\omega_{i_1\cdots i_k} \circ f)\, f^*(\tilde{e}^*)_{i_1} \wedge \cdots \wedge f^*(\tilde{e}^*)_{i_k} \\ &= \sum_{i_1\cdots i_k} (\omega_{i_1\cdots i_k} \circ f)\, \lambda_{i_1} \cdots \lambda_{i_k}\, \tilde{e}^*_{i_1} \wedge \cdots \wedge \tilde{e}^*_{i_k}. \end{aligned}$$

Hence

$$f^*(\omega)(e_1, \ldots, e_k) = \lambda_1 \cdots \lambda_k\, \omega_{1\ldots k}(f(x)) \ge 0,$$

and $f^*\omega$ is almost everywhere on M_1 positively defined. Since $\omega \in \mathbb{L}^k_p(M_2)$ we have

$$\begin{aligned} |f^*\omega(x)|^{2p} &= \Big(\sum_{i_1\cdots i_k} (\lambda_{i_1} \cdots \lambda_{i_k}\, \omega_{i_1\cdots i_k}(f(x)))^2\Big)^p \\ &\le (\lambda_1)^{2kp}\Big(\sum_{i_1\cdots i_k} (\omega_{i_1\cdots i_k}(f(x)))^2\Big)^p \\ &\le (\lambda_1)^{2kp}|\omega(f(x))|^{2p} < \infty, \end{aligned}$$

whence $f^*\omega \in \mathbb{L}^k_p(M_1)$.

Moreover, because f preserves the foliation and ω is admissible for $\mathcal{F}_2$, for almost every $L \in \mathcal{F}_1$ the relation

$$\int_L f^*\omega = \int_{f(L)} \omega \ge 1$$

holds and consequently $f^*\omega \in adm(\mathcal{F}_1)$.

Hence $f^*(adm(\mathcal{F}_2)) \subset adm(\mathcal{F}_1)$. Analogously we can prove that $adm(\mathcal{F}_1) \in f^*(adm(\mathcal{F}_2))$ and the equality (3.1) holds. Now, applying the Krivov's theorem, mentioned above, to families $adm(\mathcal{F}_1)$ and $adm(\mathcal{F}_2)$ we obtain

$$K_k(f)^{-1}\, mod_p(M_2, \mathcal{F}_2) \le mod_p(M_1, \mathcal{F}_1) \le K_{n-k}(f)\, mod_p(M_2, \mathcal{F}_2)\,,$$

what completes the proof. ∎

From Theorem 3.1 it follows immediately

COROLLARY 3.2. *For $p = n/k$ p-modulus of foliation is a conformal invariant.*

3.2. The modulus of a foliation given by a submersion

Now we are going to find an extremal form and calculate a value of modulus of foliation in a special case.

THEOREM 3.3. *If a foliation $\mathcal{F}$ on M is given by a submersion f with connected levels then*

$$mod_p{}^p(M,\mathcal{F}) = \int\limits_{f(M)} \Big(\int\limits_{L_x} J_f^{\frac{1}{p-1}} \, \sigma_{L_x} \Big)^{1-p} \sigma_{f(M)} \,,$$

and the k-form on M

$$\omega_0(x) = \frac{J_f{}^{\frac{1}{p-1}}}{\int\limits_{L_x} J_f{}^{\frac{1}{p-1}} \, \sigma_{L_x}} \sigma_{L_x}$$

is an p-extremal form for $\mathcal{F}$.

Proof. Let $\mathcal{F}$ be a foliation on M defined by a submersion with connected levels and let $\omega \in adm(\mathcal{F})$. For each $y \in f(M)$ there exists $x \in M$ such that $f^{-1}(y) = L_x$, where L_x is the leaf of $\mathcal{F}$ passing through the point x. According to the Fubini theorem we have

$$\int\limits_M |\omega(x)|^p \, \sigma_M = \int\limits_{f(M)} \Big(\int\limits_{L_x} |\omega(x)|^p \frac{1}{J_f} \, \sigma_{L_x} \Big) \sigma_{f(M)} \,.$$

By admissibility of ω and by Lemma 2.7 for almost every $x \in M$ it holds

$$\Big(\int\limits_{L_x} |\omega(x) \, \sigma_{L_x}| \Big)^p \geq 1 \,.$$

Using the Hölder inequality we obtain

$$\int\limits_{L_x} |\omega(x)|^p \frac{1}{J_f(x)} \, \sigma_{L_x} \cdot \Big(\int\limits_{L_x} J_f{}^{\frac{1}{p-1}}(x) \Big)^{p-1} \geq 1 \,,$$

and consequently

$$mod_p{}^p(M,\mathcal{F}) \geq \int\limits_{f(M)} \Big(\int\limits_{L_x} J_f^{\frac{1}{p-1}}(x) \, \sigma_{L_x} \Big)^{1-p} \sigma_{f(M)}. \tag{3.2}$$

On the other hand the k-form

$$\omega_0(x) = \frac{J_f{}^{\frac{1}{p-1}}(x)}{\int\limits_{L_x} J_f{}^{\frac{1}{p-1}}(x) \, \sigma_{L_x}} \sigma_{L_x}$$

on M is admissible for $\mathcal{F}$. Indeed, ω_0 is almost everywhere positively defined (f is the submersion, so its Jacobian $J_f(x)$ is everywhere positive) and for almost every $x \in M$

$$\int_{L_x} \omega_0 = \int_{L_x} \frac{J_f^{\frac{1}{p-1}}(x)}{\int_{L_x} J_f^{\frac{1}{p-1}}(x)\,\sigma_{L_x}}\,\sigma_{L_x} = 1\,.$$

Moreover

$$\begin{aligned}\|\omega_0\|^p = \int_M |\omega_0(x)|^p\,\sigma_M &= \int_{f(M)} \Big(\int_{L_x} |\omega(x)|^p \frac{1}{J_f(x)}\,\sigma_{L_x}\Big)\,\sigma_{f(M)} \\ &= \int_{f(M)} \left(\int_{L_x} \frac{J_f^{\frac{p}{p-1}}(x)}{(\int_{L_x} J_f^{\frac{1}{p-1}}(x)\,\sigma_{L_x})^p J_f(x)}\,\sigma_{L_x}\right)\sigma_{f(M)} \\ &= \int_{f(M)} \Big(\int_{L_x} J_f^{\frac{1}{p-1}}(x)\,\sigma_{L_x}\Big)^{1-p}\,\sigma_{f(M)}\,,\end{aligned}$$

what means that $\omega_0 \in \mathbb{L}_p^k(M)$ and

$$mod_p{}^p(M,\mathcal{F}) \le \int_{f(M)} \left(\int_{L_x} J_f^{\frac{1}{p-1}}(x)\,\sigma_{L_x}\right)^{1-p} \sigma_{f(M)}\,.$$

The last inequality together with (3.1) completes the proof. ∎

As a simple consequence of the above theorem let us note the following result.

COROLLARY 3.4. *If $\mathcal{F}$ is a codimension one foliation given by a submersion $f\colon M \to (a,b) \subset \mathbb{R}$ with connected levels then*

$$mod_p{}^p(M,\mathcal{F}) = \int_a^b \left(\int_{L_x} \|grad\, f\|^{\frac{1}{p-1}}\,\sigma_{L_x}\right)^{1-p} dt\,,$$

where $f(x) = t$ and the $(n-1)$-form on M

$$\omega_0(x) = \frac{\|grad\, f\|^{\frac{1}{p-1}}}{\int_{L_x} \|grad\, f\|^{\frac{1}{p-1}}\,\sigma_{L_x}}\,\sigma_{L_x}$$

is an p-extremal form for the foliation $\mathcal{F}$.

3.3. Extremal forms

Let $(M,\mathcal{F})$ be an n-dimensional oriented foliated Riemannian manifold and $dim\mathcal{F} = k$.

LEMMA 3.5. *For each almost everywhere positively defined k-form $\omega \in \mathbb{L}_p^k(M)$ the function $f\colon M \to \mathbb{R}$ given by*

$$x \mapsto \int_{L_x} \omega$$

is measurable on M.

Proof. Let $\mathcal{U} = \{(U_\alpha, \phi_\alpha)\}_{\alpha\in\mathcal{A}}$ be a regular atlas of $(M, \mathcal{F})$. Let $(U_0, \phi_0) \in \mathcal{U}$. For each $x \in U_0$ we will denote by:

$C(x,r)$ — a chain $\{P_0, \dots, P_r\}$ with the beginning at x such that $P_0 \in U_0$,

$C_m(x)$ — the set of all chains $C(x,r)$ with $r \le m$,

$P_m(x)$ — the set of all plaques of chains from $C_m(x)$,

and

$B_m(x) = \bigcup_{P\in P_m(x)} P$.

From the regularity of $\mathcal{U}$ it follows that $C_m(x)$ and $P_m(x)$ are finite.

Let $\omega \in \mathbb{L}_p^k(M)$ be a continuous k-form defined in U_0 by

$$\omega(x) = \sum_{i_1\cdots i_k} \omega_{i_1\cdots i_k}(x) dx_{i_1} \wedge \dots \wedge dx_{i_k}, \qquad 1 \le i_1 < \dots < i_k \le n,$$

where $(dx_1, \dots, dx_n)$ is a basis of T_x^*M dual to the basis $\frac{\partial}{\partial\phi_1}, \dots, \frac{\partial}{\partial\phi_n}$ of T_xM, such that $(dx_1, \dots, dx_k)$ span $T_x\mathcal{F}$.

We are going to prove that the real function f_m defined on M by

$$x \mapsto \int_{B_m(x)} \omega$$

is measurable for any $m \in \mathbb{N}$. For this purpose we will show that for each $t \in \mathbb{R}$ the set $A_t = \{x; f_m(x) > t\}$ is open in M.

Let $m \in \mathbb{N}$ and $t \in \mathbb{R}$. Take $x \in U_0$ such that $f_m(x) > t$. According to Remark 2.3 for any chain $C(x,r) \in C_m(x)$ there exists a diffeomorphism

$$h^C = \gamma^C_{r,r-1} \circ \gamma^C_{r-1,r-2} \circ \dots \circ \gamma^C_{1,0}, \qquad \gamma^C_{i,i-1} \in \gamma,\ i \in \mathbb{N}$$

from an open (in S_0) neighborhood $Dom(h^C)$ of x to an open subset of the transversal S_r. Moreover, for every chain $C(x,r) = \{P_0, P_1, \dots, P_r\}$ and each $y \in Dom(h^C)$ there exists a unique chain $C(y,r) = \{Q_0, Q_1, \dots, Q_r\}$ such that $P_i \subset U_i$ and $Q_i \subset U_i$ for $i = 0, \dots, r$. From now on we will call the chain $C(y,r)$ *suitable* for the chain $C(x,r)$ and the plaque Q_i *suitable* for the plaque P_i, $i = 0, \dots, r$.

Denote by $D = \bigcap_{C\in C_m(x)} Dom(h^C)$. Then for every $y \in D$ the set $B_m(y)$ crosses all domains of charts containing plaques from $B_m(x)$ and consequently $\overline{\overline{P_m(x)}} \le \overline{\overline{P_m(y)}}$.

Let $y \in D$. Consider suitable plaques $P_i \in B(x,m)$ and $Q_i \in B(y,m)$ contained in (U_i, ϕ_i) for some $0 \le i \le r$. From the regularity of $\mathcal{U}$ and monotone continuity of ω on $\overline{U_i}$ it follows that for every $\epsilon_1 > 0$ there exists $\delta_1 > 0$ such that if $|\phi_{i2}(P_i) - \phi_{i2}(Q_i)| < \delta_1$ then for each $z \in \phi_{i1}(U_i)$ the inequality

$$|\omega \circ \phi_i^{-1}(z, \phi_{i2}(P_i)) - \omega \circ \phi_i^{-1}(z, \phi_{i2}(Q_i))| < \epsilon_1 \tag{3.3}$$

holds. Moreover

$$\Big|\int_{P_i}\omega-\int_{Q_i}\omega\Big| = |\int_{P_i}\omega_{1\cdots k}\,dx_1\cdots dx_k - \int_{Q_i}\omega_{1\cdots k}\,dx_1\cdots dx_k|$$
$$= \Big|\int_{\phi_{i1}(U_i)\times\phi_{i2}(P_i)}\omega_{1\cdots k}\circ\phi_i^{-1}(z,\phi_{i2}(P_i))\,dx_1\cdots dx_k$$
$$-\int_{\phi_{i1}(U_i)\times\phi_{i2}(Q_i)}\omega_{1\cdots k}\circ\phi_i^{-1}(z,\phi_{i2}(Q))\,dx_1\cdots dx_k\Big|$$
$$\le \int_{\phi_{i1}(U_i)\times\phi_{i2}(P_i)}|\omega_{1\cdots k}\circ\phi_i^{-1}(z,\phi_{i2}(P_i))-\omega_{1\cdots k}$$
$$\circ\,\phi_i^{-1}(z,\phi_{i2}(Q_i))|\,dx_1\cdots dx_k$$
$$< \int_{\phi_{i1}(U_i)\times\phi_{i2}(P_i)}\epsilon_1 = \epsilon_1\cdot\mu_k(B_r).$$

Now consider two pairs (P_i,Q_i), (P_j,Q_j) for some $0\le i,j\le r$ of suitable plaques $P_i,P_j\in B(x,m)$ and $Q_i,Q_j\in B(y,m)$ contained in domains of (U_i,ϕ_i), (U_j,ϕ_j), respectively, and such that $P_i\cap P_j\neq\emptyset$ and $Q_i\cap Q_j\neq\emptyset$. Since $\mathcal{U}$ is regular then for every $\epsilon_2>0$ there exists $\delta_2>0$ such that

$$\mu_k(\phi_{i1}(P_i\cap P_j)\setminus\phi_{i1}(Q_i\cap Q_j))<\epsilon_2 \tag{3.4}$$

whenever $|\phi_{i2}(P_i\cap P_j)-\phi_{i2}(Q_i\cap Q_j)|<\delta_2$.

Whence, for each $\epsilon>0$, there exists $\delta>0$ such that if $|\phi_{i2}(P_i\cap P_j)-\phi_{i2}(Q_i\cap Q_j)|<\delta$ then

$$\Big|\int_{P_i\cap P_j}\omega-\int_{Q_i\cap Q_j}\omega\Big|<\epsilon.$$

Indeed, for every $\epsilon>0$ we can take $\epsilon_1=\frac{\epsilon}{2\mu_k(\phi_1(U_i))}$ and $\epsilon_2=\frac{\epsilon}{4\sup_{U_i\cap U_j}|\omega|}$. Then there exist δ_1, $\delta_2>0$ such that inequalities (3.3) and (3.4) hold. Hence if $|\phi_{i2}(P_i\cap P_j)-\phi_{i2}(Q_i\cap Q_j)|<\min\{\delta_1,\delta_2\}$ then

$$|\int_{P_i\cap P_j}\omega-\int_{Q_i\cap Q_j}\omega| = \Big|\int_{\phi_{i1}(P_i\cap P_j)\times\phi_{i2}(P_i\cap P_j)}\omega_{1\cdots k}\circ\phi_i^{-1}(z,\phi_{i2}(P_i\cap P_j))\,dx_1\cdots dx_k +$$
$$-\int_{\phi_{i1}(Q_i\cap Q_j)\times\phi_{i2}(Q_i\cap Q_j)}\omega_{1\cdots k}\circ\phi_i^{-1}(z,\phi_{i2}(Q_i\cap Q_j))\,dx_1\cdots dx_k\Big|$$
$$\le \Big|\int_{(\phi_{i1}(P_i\cap P_j)\cap\phi_{i1}(Q_i\cap Q_j))\times\phi_{i2}(P_i\cap P_j)}(\omega_{1\cdots k}\circ\phi_i^{-1}(z,\phi_{i2}(P_i\cap P_j))$$
$$-\,\omega_{1\cdots k}\circ\phi_i^{-1}(z,\phi_{i2}(Q_i\cap Q_j)))\,dx_1\cdots dx_k\Big|$$
$$+\Big|\int_{(\phi_{i1}(P_i\cap P_j)\setminus\phi_{i1}(Q_i\cap Q_j))\times\phi_{i2}(P_i\cap P_j)}\omega_{1\cdots k}\circ\phi_i^{-1}(z,\phi_{i2}(P_i\cap P_j))\,dx_1\cdots dx_k\Big|$$

$$+ \left| \int_{(\phi_{i1}(Q_i\cap Q_j)\setminus\phi_{i1}(P_i\cap P_j))\times\phi_{i2}(P_i\cap P_j)} \omega_{1\dots k}\circ\phi_i^{-1}(z,\phi_{i2}(Q_i\cap Q_j))dx_1\cdots dx_k \right|$$
$$< \epsilon_1\cdot\mu_k(\phi_{i1}(P_i\cap P_j)\cap\phi_{i1}(Q_i\cap Q_j)) + 2\epsilon_2\cdot \sup_{U_i\cap U_j}|\omega| < \epsilon.$$

One can show that analogous inequality holds for every finite number $r \in \mathbb{N}$ of pairs (P_i, Q_i), $i = 1,\dots,r$ of suitable plaques $P_i \in B(x,m)$ and $Q_i \in B(y,m)$ contained in domains of (U_i,ϕ_i), respectively, and such that $P_1\cap\cdots\cap P_r \neq \emptyset$ and $Q_1\cap\cdots\cap Q_r \neq \emptyset$.

Now let $\lambda = f_m(x) - t$ and $s = \overline{\overline{P_m(x)}}$. From above abbreviations it follows that choosing D smaller, if necessary, we can assume that for any $y \in D$ hold inequalities

$$|\int_{P_i}\omega - \int_{Q_i}\omega| < \frac{\lambda}{2^s} \quad \text{for} \quad i = 1,\dots,s$$

and

$$\left|\int_{\bigcap_{j\le r} P_{i_j}}\omega - \int_{\bigcap_{j\le r} Q_{i_j}}\omega\right| < \frac{\lambda}{2^s} \quad \text{for} \quad i_j = 1,\dots,s; \quad r \le s,$$

where $Q_i \in P_m(y)$ are suitable plaques to $P_i \in P_m(x)$ for $i = 1,\dots,s$.

Since ω is almost everywhere positively defined we have

$$\int_{B_m(y)}\omega \ge \int_{\bigcup_{i=1}^s Q_i}\omega = \sum_{i=1}^s\int_{Q_i}\omega + \cdots + (-1)^{k+1}\sum_{\substack{i_j<i_l\\ i_1,\dots,i_k=1}}^s \int_{\bigcap_{i=1}^k Q_{i_j}}\omega + \cdots + (-1)^{s+1}\int_{\bigcap_{i=1}^s Q_i}\omega$$
$$> \sum_{i=1}^s\left(\int_{P_i}\omega - \frac{\lambda}{2^s}\right) + \cdots + (-1)^{k+1}\sum_{\substack{i_j<i_l\\ i_1,\dots,i_k=1}}^s\left(\int_{\bigcap_{j=1}^k P_{i_j}}\omega + (-1)^k\frac{\lambda}{2^s}\right) + \cdots$$
$$+(-1)^{s+1}\left(\int_{\bigcap_{i=1}^s P_i}\omega + (-1)^s\frac{\lambda}{2^s}\right) = \int_{B_m(x)}\omega - \frac{\lambda}{2^s}\left(\binom{s}{1} + \cdots + \binom{s}{s}\right)$$
$$> \int_{B_m(x)}\omega - \lambda = t.$$

Therefore x is contained in A_t together with the open neighborhood $D\times P_0$ and, consequently, A_t is open in M.

Since for every $x \in M$

$$\lim_{m\to\infty}\int_{B_m(x)}\omega = \int_{L_x}\omega$$

then the function $f : M \to \mathbb{R}$ defined by $x \mapsto \int_{L_x} \omega$ is measurable on M as the limit of the sequence $(f_m)_{m\in\mathbb{N}}$ of measurable functions.

Now let $\omega \in \mathbb{L}_p^k(M)$. The set of all continuous forms is dense in $\mathbb{L}_p^k(M)$, so there exists a sequence $(\omega_n)_{n\in\mathbb{N}}$ of continuous forms convergent to ω. Since all functions $x \mapsto \int_{L_x} \omega_n$ are measurable on M we obtain that the function $f : M \to \mathbb{R}$ defined by

$$x \mapsto \int_{L_x} \omega$$

is also measurable on M. ■

THEOREM 3.6. *If M has finite volume and $\mathcal{F}$ is given by a submersion with connected level sets then $adm(\mathcal{F})$ is closed in $\mathbb{L}_p^k(M)$.*

Proof. Let $f : M \to B \subset \mathbb{R}$ be a submersion. Consider a convergent sequence $(\omega_m)_{m\in\mathbb{N}}$ of admissible k-forms for $\mathcal{F}$. Denote its limit by ω and suppose that $\omega \notin adm(\mathcal{F})$. By completness of $\mathbb{L}_p^k(M)$, ω belongs to $\mathbb{L}_p^k(M)$. Moreover, $(\omega_m)_{m\in\mathbb{N}}$ includes a subsequence convergent almost everywhere to $\omega(x)$. It means, in particular, that ω is almost everywhere on M positively defined. Therefore ω is not admissible for $\mathcal{F}$ if the set

$$A = \Big\{L \in \mathcal{F} : \int_L \omega < 1\Big\}$$

is of nonzero measure. By Lemma 3.5 the real function $h : x \mapsto \int_{L_x} \omega$ is measurable on M, so $A = h^{-1}(-\infty, 1)$ is also measurable on M and, consequently, $\mu(A) > 0$.

Since level sets of f are connected we have

$$f(A) = \Big\{z \in B : \int_{f^{-1}(z)} \omega < 1\Big\}.$$

This set, by the Fubini theorem, is measurable as preimage of $(-\infty, 1)$ by measurable function $B \ni z \mapsto \int_{f^{-1}(z)} \omega \in \mathbb{R}$ and $\mu_B(f(A)) > 0$.

Hence

$$\int_A \omega \wedge f^*\sigma_B = \int_{f(A)} \Big(\int_{f^{-1}(z)} \omega \Big) \sigma_B < \int_{f(A)} 1 \, \sigma_B = \mu_B(f(A))$$

and for each $m \in \mathbb{N}$ we have

$$\int_A \omega_m \wedge f^*\sigma_B = \int_{f(A)} \Big(\int_{f^{-1}(z)} \omega_m \Big) \sigma_B \geq \int_{f(A)} 1 \, \sigma_B = \mu_B(f(A)).$$

Denote by

$$\epsilon = \mu_B(f(A)) - \int_{f(A)} \Big(\int_{f^{-1}(z)} \omega \Big) \sigma_B.$$

Then for all $m \in \mathbb{N}$ we obtain

$$\begin{aligned} 0 < \epsilon = \mu_B(f(A)) - \int_{f(A)} \Big(\int_{f^{-1}(z)} \omega \Big) \sigma_B \\ \leq \int_{f(A)} \Big(\int_{f^{-1}(z)} \omega_m \Big) \sigma_B - \int_{f(A)} \Big(\int_{f^{-1}(z)} \omega \Big) \sigma_B \\ = \int_{f(A)} \Big(\int_{f^{-1}(z)} \omega_m - \omega \Big) \sigma_B = \Big| \int_{f(A)} \Big(\int_{f^{-1}(z)} \omega_m - \omega \Big) \sigma_B \Big| . \end{aligned}$$

By Fubini theorem and Hölder inequality we have

$$\begin{aligned} \epsilon < \Big| \int_A (\omega_m - \omega) \wedge f^*\sigma_B \Big| \leq \int_A |(\omega_m - \omega) \wedge f^*\sigma_B| \sigma_M \\ \leq \int_M |(\omega_m - \omega) \wedge f^*\sigma_B| \sigma_M \\ \leq \int_M |\omega_m - \omega| \cdot |f^*\sigma_B| \, \sigma_M \\ \leq \Big(\int_M |\omega_m - \omega|^p \, \sigma_M \Big)^{\frac{1}{p}} \cdot \Big(\int_M |f^*\sigma_B|^{\frac{p}{p-1}} \, \sigma_M \Big)^{\frac{p-1}{p}} . \end{aligned}$$

Since M has the finite volume then for $c = \big(\int_M |f^*\sigma_B|^{\frac{p}{p-1}} \, \sigma_M \big)^{\frac{p-1}{p}}$ we obtain an inequality

$$\|\omega_m - \omega\| \geq \epsilon c^{-1},$$

what is a contrary to the assumptions. ∎

Krivov in [Kr] proved that a family $\mathcal{A}$ of k-forms has a unique p-extremal form if $\mathcal{A}$ is closed and convex in $\mathbb{L}_p^k(M)$. Since the convexity of the family $adm\mathcal{F}$ is obvious, by the above theorem it follows directly

COROLLARY 3.7. *If $(M, \mathcal{F})$ satisfies assumptions of Theorem 3.6 then there exists a unique p-extremal form for the foliation $\mathcal{F}$.*

In last two theorems we describe some properties of an extremal form.

THEOREM 3.8. *If ω_0 is an extremal form of a family $\mathcal{L} \subset \mathcal{F}$ then for almost all leaves $L \in \mathcal{L}$ we have*

$$\int_L \omega_0 = 1 .$$

Proof. Let $\mathcal{L}$ be a family of leaves of the foliation $\mathcal{F}$ and let ω_0 be an p-extremal form for $\mathcal{L}$. Suppose that the set

$$A = \{x \in M; L_x \in \mathcal{L} \text{ and } \int_{L_x} \omega_0 > 1\}$$

has nonzero measure. By Lemma 3.5 A is measurable, so $\mu(A) > 0$. Consider a cover $\{C_m\}_{m \in \mathbb{N}}$ of the set $f(A)$ defined for each $m \in \mathbb{N}$ in the following way

$$C_m = \left(1 + \frac{1}{2m}, 1 + \frac{1}{m}\right] \cup (m+1, m+2] .$$

Since the measure of A is positive there exists $m_0 \in \mathbb{N}$ such that the set $f^{-1}(C_{m_0})$ has also positive measure. Moreover the measure of $B = A \cap f^{-1}((C_{m_0})$ is positive, too. Now consider the form

$$\omega(x) = \begin{cases} \frac{2m_0}{2m_0+1}\omega_0 & x \in B, \\ \omega_0 & x \in M \setminus B. \end{cases}$$

Since ω_0 is admissible for $\mathcal{L}$ then the form ω is almost everywhere on M positively defined and for almost all leaves $L \in \mathcal{L} \setminus \tilde{B}$ we have

$$\int_L \omega = \int_L \omega_0 \geq 1.$$

By the definition of ω it follows that for $L \in \mathcal{L} \cap \tilde{B}$ we have

$$\int_L \omega = \frac{2m_0}{2m_0+1} \int_L \omega_0 > \frac{2m_0}{2m_0+1}\left(1 + \frac{1}{2m_0}\right) = 1 ,$$

so $\int_L \omega \geq 1$ for almost all leaves $L \in \mathcal{L}$. Moreover,

$$\begin{aligned} \|\omega\|^p &= \int_M |\omega(x)|^p \, \sigma_M = \int_B \left|\frac{2m_0}{2m_0+1}\omega_0(x)\right|^p \sigma_M + \int_{M\setminus B} |\omega_0(x)|^p \, \sigma_M = \\ &= \left(\frac{2m_0}{2m_0+1}\right)^p \int_B |\omega_0(x)|^p \, \sigma_M + \int_{M\setminus B} |\omega_0(x)|^p \, \sigma_M < \|\omega_0\|^p \end{aligned}$$

what means that $\omega \in \mathbb{L}_p^k(M)$ and, simultanously, contradicts with the assumption of extremality of ω_0. ∎

THEOREM 3.9. *If an extremal form ω_0 of a foliation $\mathcal{F}$ is continuous then for all vector fields $X_1, \ldots, X_{n-k} \in T\mathcal{F}$ we have*

$$*\omega_0(X_1, \ldots, X_{n-k}) = 0.$$

That means, in particular, that for each leaf $L \in \mathcal{F}$ there exists a real function f_L on M such that

$$\omega_0|_L = f_L \cdot \sigma_L,$$

where σ_L is the volume form of L.

Proof. Let ω_0 be an extremal k-form for $\mathcal{F}$. Suppose that there exists $x_0 \in M$ and vector fields $X_1, \ldots, X_{n-k} \in T\mathcal{F}$ such that

$$*\omega_0(X_1, \ldots, X_{n-k})(x_0) \neq 0 . \tag{3.5}$$

Consider a chart (U, ϕ) around x_0 and denote by $e_1, \ldots, e_n$ an orthonormal positively oriented basis of smooth vector fields on U such that for any $x \in U$

vectors $e_1(x),\ldots,e_k(x)$ span $T_x\mathcal{F}$. Then

$$\omega=\sum_{i_1\cdots i_k}\omega_{i_1\cdots i_k}e^*_{i_1}\wedge\cdots\wedge e^*_{i_k},\qquad 1\le i_1<\cdots<i_k\le n$$

and

$$*\omega=\sum_{i_1\cdots i_k}\omega_{i_1\cdots i_k}\cdot\operatorname{sgn}\sigma(i_1,\ldots,i_k,j_1,\ldots,j_{n-k})e^*_{j_1}\wedge\cdots\wedge e^*_{j_{n-k}},$$

where $j_1,\ldots,j_{n-k}\in\{1,\ldots,n\}\setminus\{i_1,\ldots,i_k\}$, $1\le j_1<\cdots<j_{n-k}\le n$ and $\sigma(i_1,\ldots,i_k,j_1,\ldots,j_{n-k})$ is a permutation of $(i_1,\ldots,i_k,j_1,\ldots,j_{n-k})$. By continuity of ω, choosing U smaller, if necessary, we can assume that the condition (3.5) holds in whole U. Since $X_1,\ldots,X_{n-k}\in T\mathcal{F}$ then for every $x\in U$ we have

$$(\omega_{1\cdots k}\,e^*_{k+1}\wedge\cdots\wedge e^*_n)(X_1,\ldots,X_{n-k})(x)=0\,.$$

This condition together with (3.5) gives the existence of a multiindex $i_1\cdots i_k\neq 1\cdots k$ such that $\omega_{i_1\cdots i_k}(x)\neq 0$.

Now let W be an open subset of U such that $\overline{W}\subset U$. Consider a continous function $f\colon M\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}1 & \text{dla } x\in M\setminus U,\\ \frac{1}{2} & \text{dla } x\in\overline{W}.\end{cases}$$

Then

$$\omega=f(\omega_0-\omega_{1\cdots k}e^*_1\wedge\cdots\wedge e^*_k)+\omega_{1\cdots k}e^*_1\wedge\cdots\wedge e^*_k$$

is continous, positively defined k-form on M identical with ω_0 on $M\setminus U$. Since

$$\omega=\omega_{1\cdots k}e^*_1\wedge\cdots\wedge e^*_k+\sum_{\substack{i_1\cdots i_k\neq 1\cdots k\\ i_1\cdots i_k}}f\cdot\omega_{i_1\cdots i_k}e^*_{i_1}\wedge\cdots\wedge e^*_{i_k}$$

we have

$$\int_L\omega=\int_{L\setminus U}\omega+\int_{L\cap U}\omega=\int_{L\setminus U}\omega_0+\int_{L\cap U}\omega_{1\cdots k}e^*_1\wedge\cdots\wedge e^*_k=\int_L\omega_0$$

for any leaf L of $\mathcal{F}$. Moreover, for all $x\in U$ the inequality

$$\begin{aligned}|\omega(x)|^2&=\omega^2_{1\cdots k}(x)+\sum_{\substack{i_1\cdots i_k\neq 1\cdots k\\ i_1\cdots i_k}}(f\cdot\omega_{i_1\cdots i_k}(x))^2\\ &<\omega^2_{1\cdots k}(x)+\sum_{\substack{i_1\cdots i_k\neq 1\cdots k\\ i_1\cdots i_k}}\omega^2_{i_1\cdots i_k}(x)=|\omega_0(x)|^2\quad\text{holds.}\end{aligned}$$

Therefore ω is admissible for $\mathcal{F}$ and

$$\|\omega\|^p = \int_M |\omega(x)|^p \, \sigma_M = \int_{M\setminus U} |\omega(x)|^p \, \sigma_M + \int_U |\omega(x)|^p \, \sigma_M$$

$$< \int_{M\setminus U} |\omega_0(x)|^p \, \sigma_M + \int_U |\omega_0(x)|^p \, \sigma_M < \|\omega_0\|^p,$$

what contradicts with extremality of ω_0. ∎

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